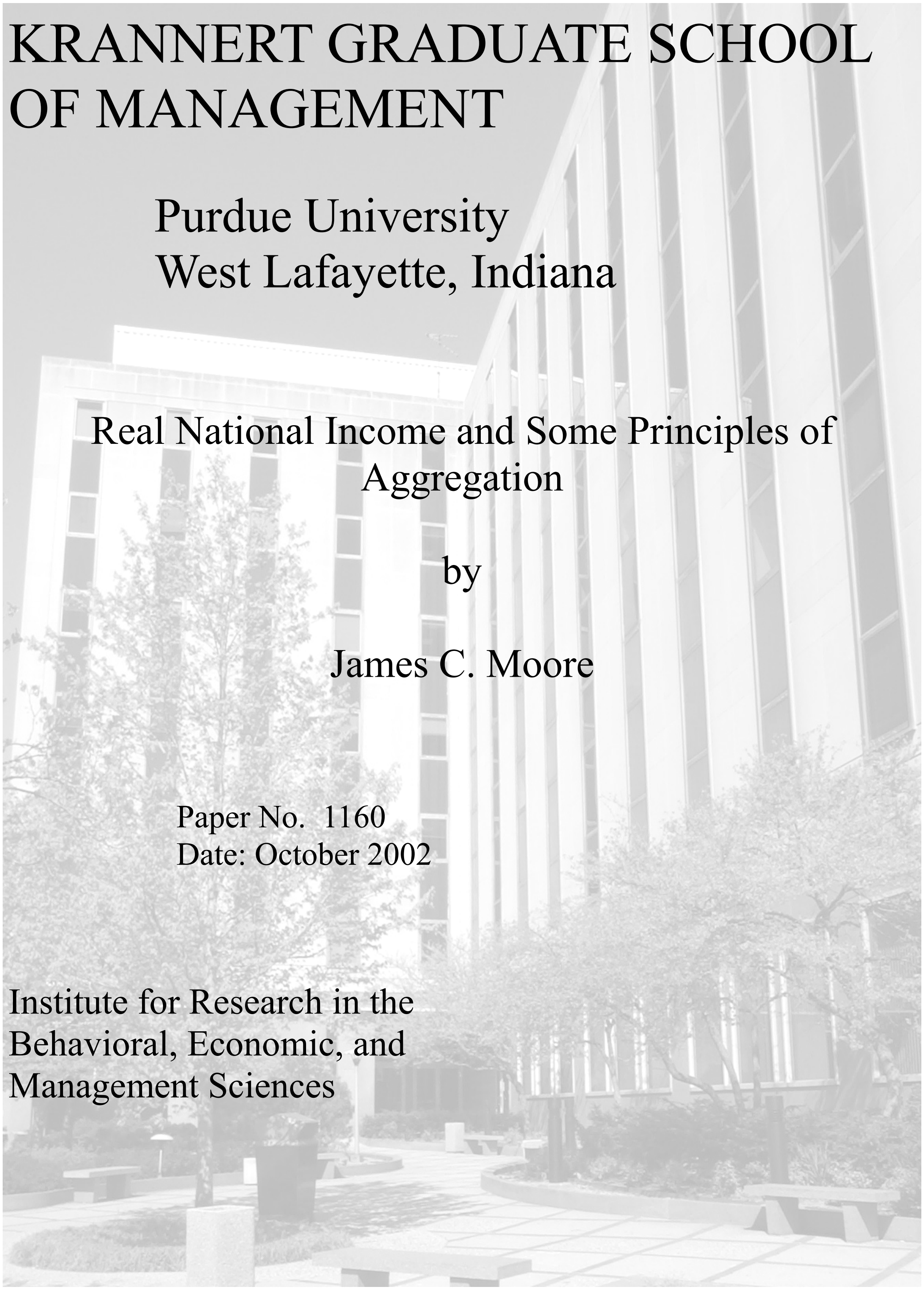

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Real National Income and Some Principles of Aggregation

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Abstract

In this paper, the beginnings of a new approach to the theory of aggregation are developed. The basic idea is that aggregation should involve two things: (a) data over which social preferences are defined should be mapped into a smaller-dimensional space, and (b) there should exist an ordering on that lower-dimensional space such that an improvement in this criterion implies an improvement for any of a class of Social Welfare Functions and of a class of individual preference relations defined over the original space. Results are developed which show that this conception of aggregation can yield meaningful results; particularly with respect to comparisons of 'real national income' in two situations, for a given economy.

1 Introduction.

My intent in this paper is to take a new look at the basic idea of 'aggregation' in economics. Prior treatments of aggregation issues in economics have emphasized the relationships between algebraic operations on a space of economic variables (usually a 'commodity space,' or a price-income space), or the properties of functions defined upon such a space; as compared to corresponding algebraic operations, or the properties of functions on a smaller-dimensional representation of the space. Thus, for example, suppose the commodity demands of the m consumers in an economy can be expressed by the individual demand functions, $h_i \colon \Omega \to \mathbb{R}^n_+$, for $i = 1, \ldots, m$, where:

$$\Omega = \{(\boldsymbol{p}, w_i) \in \mathbb{R}^{n+1}_+ \mid \boldsymbol{p} \gg \mathbf{0} \ \&\ w_i \geq 0\}.$$

If we define the set $\boldsymbol{\Omega}$ by:

$$\boldsymbol{\Omega} = \{(\boldsymbol{p}, \boldsymbol{w}) \in \mathbb{R}^{n+m}_+ \mid \boldsymbol{p} \gg \mathbf{0} \ \&\ \boldsymbol{w} \in \mathbb{R}^m_+\},$$

we can consider the collective demands of the consumers as being represented by the function $\boldsymbol{h} \colon \boldsymbol{\Omega} \to \mathbb{R}^{mn}_+$ defined by:

$$\boldsymbol{h}(\boldsymbol{p}, \boldsymbol{w}) = \big(h_1(\boldsymbol{p}, w_1), h_2(\boldsymbol{p}, w_2), \ldots, h_m(\boldsymbol{p}, w_m)\big).$$

It is much more convenient, however, particularly from the standpoint of empirical estimation, to consider the function $h \colon \boldsymbol{\Omega} \to \mathbb{R}^n_+$ defined by:

$$h(\boldsymbol{p}, \boldsymbol{w}) = \sum\nolimits_{i=1}^{m} h_i(\boldsymbol{p}, w_i).$$

Several questions have then received much attention in the literature. For example, 'what properties of the functions h_i are inherited by the function h? and 'under what conditions does there exists a function $\overline{h} \colon \Omega \to \mathbb{R}^n_+$ satisfying, for all $(\boldsymbol{p}, \boldsymbol{w}) \in \boldsymbol{\Omega}$:

$$\overline{h}\Big(\boldsymbol{p}, \sum\nolimits_{i=1}^{m} w_i\Big) = h(\boldsymbol{p}, \boldsymbol{w})?'$$

In this context it seems to me that a more fundamental issue can be set out as follows. Suppose one 'cares about' the values of the function $\boldsymbol{h}(\cdot)$; a state of affairs which we will characterize as meaning that there exists a (social welfare) ordering, $\succ$, defined on $\mathbb{R}^{mn}_+$ such that the situation in which the price-wealth pair $(\boldsymbol{p}, \boldsymbol{w})$ prevails is considered as being better than that in which $(\boldsymbol{p}', \boldsymbol{w}')$ prevails if and only if $\boldsymbol{h}(\boldsymbol{p}, \boldsymbol{w}) \succ \boldsymbol{h}(\boldsymbol{p}', \boldsymbol{w}')$. The aggregation problem which then seems to me to be fundamental is: under what conditions can we derive a mapping, $f \colon \mathbb{R}^{mn}_+ \to \mathbb{R}^q$, with $q < mn$, and a binary relation, P, on $\mathbb{R}^q$ such that:[1]

$$f\big[\boldsymbol{h}(\boldsymbol{p}, \boldsymbol{w})\big] P f\big[\boldsymbol{h}(\boldsymbol{p}', \boldsymbol{w}')\big] \Rightarrow \boldsymbol{h}(\boldsymbol{p}, \boldsymbol{w}) \succ \boldsymbol{h}(\boldsymbol{p}', \boldsymbol{w}')?$$

[1]The requirement:

$$f\big[\boldsymbol{h}(\boldsymbol{p}, \boldsymbol{w})\big] P f\big[\boldsymbol{h}(\boldsymbol{p}', \boldsymbol{w}')\big] \iff \boldsymbol{h}(\boldsymbol{p}, \boldsymbol{w}) \succ \boldsymbol{h}(\boldsymbol{p}', \boldsymbol{w}')$$

seems to me to be impossibly stringent; and the criterion:

$$\boldsymbol{h}(\boldsymbol{p}, \boldsymbol{w}) \succ \boldsymbol{h}(\boldsymbol{p}', \boldsymbol{w}') \Rightarrow f\big[\boldsymbol{h}(\boldsymbol{p}, \boldsymbol{w})\big] P f\big[\boldsymbol{h}(\boldsymbol{p}', \boldsymbol{w}')\big]$$

does not seem to be as useful for policy evaluation as that in the text.

Of course, a solution to this problem will be of interest to a policy-maker only if he or she believes the social ordering, $\succ$, to be the appropriate criterion for improvement. Consequently, it seems appropriate to expand upon this criterion to seek such a function, f, such that for each $\succ$ in some class of social preference orderings there exists a binary relation on $\mathbb{R}^q$, $Q_\succ$, such that:

$$f[\boldsymbol{h}(\boldsymbol{p},\boldsymbol{w})]Q_\succ f[\boldsymbol{h}(\boldsymbol{p}',\boldsymbol{w}')] \Rightarrow \boldsymbol{h}(\boldsymbol{p},\boldsymbol{w}) \succ \boldsymbol{h}(\boldsymbol{p}',\boldsymbol{w}'). \tag{1}$$

Roughly speaking, this is the approach which is taken in this paper; however, while the basic approach taken here could be applied to a number of different situations, in this paper we will concentrate on the application of this idea to comparisons of the type indicated in equation (1).[2]

In the next section, we set out the assumptions about individual preferences to be used throughout this paper. In Section 3, we set out a somewhat new approach to the idea of measuring utility, and in Section 4 we specify the types of social preference relations to be investigated; with particular emphasis on a somewhat novel way of looking at 'social welfare functions.' In Section 5, we develop the idea of 'indirect social preference relations,' as well as the notion of a 'cost-of-living indicator' for individual consumers. In Section 6, we develop the two new normative criteria which are being presented in this paper. While some results are presented for the first criterion, I believe that the second of these criteria is much more promising. This second criterion is based upon the idea that what is wanted is data that a policy-maker can evaluate in specific ways; and several results are developed in Section 7 demonstrating that specific data (functions of prices and income) can be used to make valid welfare inferences by policy-makers with fairly general types of social preference relations. In Section 8 we present a more thorough and leisurely discussion and evaluation of the ideas presented in the body of the paper than is done in this introduction, as well as presenting some suggestions for feasible applications and future extensions of the analysis.

2 The Framework.

We will deal with classes of economies, each of which will involve m consumers, and n commodities. We will suppose throughout that each consumer has a (strict) preference relation P_i, which will be taken to be an element of a class, $\mathcal{P}^0$, of asymmetric and negatively transitive[3] binary relations on $\mathbb{R}^n_+$. We will often suppose that the admissible preference space, $\mathcal{P}$, satisfies further assumptions as well, but these will be specified where needed.

Given a preference relation, $P_i \in \mathcal{P}^0$, we will denote the negation of P_i by 'G_i,' and the 'indifference relation' induced by P_i by 'I_i;' that is:

$$\boldsymbol{x}_i G_i \boldsymbol{x}'_i \iff \neg \boldsymbol{x}'_i P_i \boldsymbol{x}_i,$$

[2] In our two papers, [5, 7], John Chipman and I showed that what was then the conventional justification (the generalized 'compensation principle' criterion developed in Samuelson [18], and slightly refined in Chipman and Moore [4]) for the notion of 'real national income' was valid if, *and only if,* consumers had identical homothetic preference relations. Hopefully it is obvious that this result is not directly pertinent to the present investigation.

[3] That is, for each $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$ in $\mathbb{R}^n_+$ we have that if $\boldsymbol{x}P_i\boldsymbol{y}$, then either $\boldsymbol{x}P_i\boldsymbol{z}$ or $\boldsymbol{z}P_i\boldsymbol{y}$.

and:

$$\boldsymbol{x}_i I_i \boldsymbol{x}_i' \iff [\neg \boldsymbol{x}_i P_i \boldsymbol{x}_i' \ \& \ \neg \boldsymbol{x}_i' P_i \boldsymbol{x}_i].$$

Under the assumptions being maintained here, G_i will always be a weak order (total, reflexive, and transitive), and I_i will be an equivalence relation (reflexive, symmetric, and transitive).

Formally, we define the **space of preference relations**, $\boldsymbol{\mathcal{P}}^{\mathbf{0}}$, by:

$$\boldsymbol{\mathcal{P}}^{\mathbf{0}} = \prod_{i=1}^{m} \mathcal{P}_i = (\mathcal{P}^0)^m,$$

where:

$$\mathcal{P}_i = \mathcal{P}^0 \quad \text{for } i = 1, \ldots, m;$$

and $\mathcal{P}^0$ is the family of all asymmetric and negatively transitive binary relations on $\mathbb{R}^n_+$. In the simplified framework to be considered here, an economy is simply a specification of m (strict) preference relations, each of which is required to be an element of $\mathcal{P}^0$. In other words, an economy is completely specified by an m-tuple, $\boldsymbol{P} = (P_1, \ldots P_m) \in \boldsymbol{\mathcal{P}}^{\mathbf{0}}$; which, in keeping with the social choice literature, we will call a **preference profile**. We specify a class of economies by a choice of a subset of $\boldsymbol{\mathcal{P}}^{\mathbf{0}}$, which we will call the 'admissible preference space.' More precisely, when we say that $\boldsymbol{\mathcal{P}}$ is an **admissible preference space**, it is to be understood that $\boldsymbol{\mathcal{P}}$ is of the form:

$$\boldsymbol{\mathcal{P}} = \prod_{i=1}^{m} \mathcal{P}_i,$$

with $\mathcal{P}_i = \mathcal{P} \subseteq \mathcal{P}^0$, for $i = 1, \ldots, m$.

Since each consumer's consumption set is equal to $\mathbb{R}^n_+$, we will be considering throughout a fixed **allocation space**, $\boldsymbol{X}$, given by:

$$\boldsymbol{X} = \prod_{i=1}^{m} X_i = \mathbb{R}^{mn}_+,$$

where $X_i = \mathbb{R}^n_+$, for each i, and we will use the generic notation:

$$\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m), \boldsymbol{x}' = (\boldsymbol{x}_1', \ldots, \boldsymbol{x}_m'),$$

and so on, to denote elements of $\boldsymbol{X}$; where, for example, $\boldsymbol{x}_i \in \mathbb{R}^n_+$ is of the form:

$$\boldsymbol{x}_i = (x_{i1}, \ldots, x_{in}).$$

We will also be concerned with the (unanimity) Pareto ordering concept. Given $\boldsymbol{P} \in \boldsymbol{\mathcal{P}}^{\mathbf{0}}$, we will (hopefully without risk of confusion) use the same notation to define the binary relation $\boldsymbol{P}$ on $\boldsymbol{X} = \mathbb{R}^{mn}_+$ by:

$$\boldsymbol{x}\boldsymbol{P}\boldsymbol{x}' \iff \boldsymbol{x}_i P_i \boldsymbol{x}_i' \text{ for } i = 1, \ldots, m.$$

3 Utility Functions.

We will make use of a somewhat unorthodox treatment of utility functions, as follows. Letting $\mathcal{U}$ be defined by:

$$\mathcal{U} = \{f \mid f\colon \mathbb{R}^n_+ \to \mathbb{R}_+\},$$

we will say that a family of preference relations, $\boldsymbol{\mathcal{P}}$, **is representable** iff there exists a (at least one) function, $\mu\colon \mathcal{P} \to \mathcal{U}$, satisfying the following condition: for each $P \in \mathcal{P}$, $f = \mu(P)$ satisfies:

$$(\forall \boldsymbol{x}, \boldsymbol{x}' \in \mathbb{R}^n_+)\colon f(\boldsymbol{x}) > f(\boldsymbol{x}') \iff \boldsymbol{x}P\boldsymbol{x}'; \tag{2}$$

in which case we shall say that $\boldsymbol{\mu}$ **is a measurement function for** $\boldsymbol{\mathcal{P}}$.

3.1 Definition. We denote by '$\mathcal{P}^c$' the family of all asymmetric and negatively transitive binary relations on $\mathbb{R}^n_+$ which are also:

a. continuous,
b. increasing,[4] and
c. strictly convex.[5]

We then let '$\boldsymbol{\mathcal{P}}^c$' denote the collection of all m-tuples of elements of $\mathcal{P}^c$; that is:

$$\boldsymbol{\mathcal{P}}^c = (\mathcal{P}^c)^m.$$

The following example is a variation and extension of the original utility representation result due to Wold [20], and will be utilized a great deal in the remainder of this paper.

3.2 Example. We define a function $\varphi\colon \mathcal{P}^c \times \mathbb{R}^n_{++} \to \mathcal{U}$ in the following way. Given $\mathcal{P} = \mathcal{P}^c$, and $\boldsymbol{x}^* \in \mathbb{R}^n_{++}$,[6] we define $u = \varphi(P, \boldsymbol{x}^*)$ as follows. For $\boldsymbol{x} \in \mathbb{R}^n_+$, there exists a unique value of θ satisfying:

$$\boldsymbol{x}\, I\, \theta\boldsymbol{x}^*, \tag{3}$$

where I is the indifference relation for P, and we let $u(\boldsymbol{x}) = \theta$. In other words, $u(\boldsymbol{x}) = \varphi(P, \boldsymbol{x}^*)(\boldsymbol{x})$ is that unique real number satisfying:

$$\boldsymbol{x}\, I\, [u(\boldsymbol{x})\boldsymbol{x}^*], \tag{4}$$

It can then be shown that $u = \varphi(P, \boldsymbol{x}^*)$ is a continuous function satisfying (2), above; that is, it represents P on $\mathbb{R}^n_+$. $\square$

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[4]That is, if $\boldsymbol{x}, \boldsymbol{x}' \in \mathbb{R}^n_+$ are such that $\boldsymbol{x} \gg \boldsymbol{x}'$, then $\boldsymbol{x}P\boldsymbol{x}'$.

[5]That is, denoting the negation of P_i by 'G_i:' if $\boldsymbol{x}, \boldsymbol{x}^* \in \mathbb{R}^n_+$ are such that $\boldsymbol{x}G_i\boldsymbol{x}^*$ and $\boldsymbol{x} \neq \boldsymbol{x}^*$, and if $0 < \theta < 1$, then:

$$\left[\theta\boldsymbol{x} + (1-\theta)\boldsymbol{x}^*\right] P_i \boldsymbol{x}^*.$$

Strict convexity will not really be needed in the vast majority of our work, but making use of it greatly simplifies many of our definitions and proofs.

[6]Where '$\mathbb{R}^n_{++}$' denotes the set of strictly positive elements of $\mathbb{R}^n$; that is:

$$\mathbb{R}^n_{++} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} \gg \boldsymbol{0}\}.$$

While the above example sets out only one type of measurement function for $\mathcal{P}^c$,[7] we will make a great deal of use of this one type of measurement *In fact, in the remainder of this paper, when we say that* $\mu\colon \mathcal{P} \to \mathcal{U}$ *is a measurement function for* $\mathcal{P} \subseteq \mathcal{P}^c$, *we will mean that* μ *is defined as in Example 3.2; that is, there exists* $\boldsymbol{x}^* \in \mathbb{R}^n_{++}$ *such that* $\mu = \varphi(\cdot, \boldsymbol{x}^*)$. In other words, if we speak of μ^* and $\mu^\dagger$ as being two different measurement functions for $\mathcal{P}$, we will mean that there exist $\boldsymbol{x}^*, \boldsymbol{x}^\dagger \in \mathbb{R}^n_{++}$ such that for each $P \in \mathcal{P}$:

$$\mu^*(P) = \varphi(P, \boldsymbol{x}^*) \ \& \ \mu^\dagger(P) = \varphi(P, \boldsymbol{x}^\dagger),$$

where φ is defined in Example 3.2. That is, μ^* and $\mu^\dagger$ will be obtained by the same process, but may use different 'units of measure' ($\boldsymbol{x}^*$ versus $\boldsymbol{x}^\dagger$ in this example). Rather surprisingly, we lose very little generality in our main results in this paper by confining our attention to this one type of measurement, as we will see.

In this paper, our attention will often be focused upon a special class of preference relations, those satisfying the following condition.

3.3 Definition. Let P be a binary relation on a cone, X. We will say that P is **homothetic** iff, for all $\boldsymbol{x}, \boldsymbol{x}' \in X$, and all $\theta \in \mathbb{R}_{++}$, we have:

$$\boldsymbol{x} P \boldsymbol{x}' \Rightarrow \theta \boldsymbol{x} P \theta \boldsymbol{x}'.$$

3.4 Definition. We denote by '$\mathcal{P}^h$' the subset of $\mathcal{P}^c$ consisting of all elements of $\mathcal{P}^c$ which are also homothetic; that is, $\mathcal{P}^h$ is the family of all binary relations on $\mathbb{R}^n_+$ which are asymmetric, negatively transitive, continuous, increasing, and homothetic. We use '$\boldsymbol{\mathcal{P}}^h$' to denote the collection of all m-tuples of elements of $\mathcal{P}^h$.

We then have the following.

3.5 Proposition. *If* $\mu\colon \mathcal{P}^h \to \mathcal{U}$ *is a measurement function for* $\mathcal{P}^h$, *then, given any* $P \in \mathcal{P}^h$, *the function* $u^* = \mu(P)$ *[satisfies (2), above, and] is concave, continuous, increasing, and positively homogeneous of degree one.*

Proof. The proof of the properties other than concavity can proceed by a straightforward modification of the proof of Theorem 2 of Chipman and Moore [4, pp. 57-8]. The fact that u^* is concave is an immediate consequence of Corollary 5.101, p. 334, of Moore [13].[8] Details will be left to 'the interested reader.' □

We also note the following.

3.6 Proposition. *Let* $P \in \mathcal{P}^h$, *and let* $u\colon \mathbb{R}^n_+ \to \mathbb{R}_+$ *be any function representing* P *which is positively homogeneous of degree one. Then there exists* $\boldsymbol{x}^* \in \mathbb{R}^n_{++}$ *such that* $u = \mu(P; \boldsymbol{x}^*)$, *where* $\mu\colon \mathcal{P}^c \to \mathcal{U}$ *is defined from* $\boldsymbol{x}^*$ *as in Example 3.2.*

[7]The 'money metric' is another example which one could use, under more or less the same assumptions. See Weymark [19].

[8]Which in turn is a variant of a result by Berge [3, Theorem 3, p. 208].

Proof. Since P is increasing and $u(\cdot)$ is positively homogeneous of degree one, there exists $\boldsymbol{x}^* \in \mathbb{R}^n_{++}$ such that $u(\boldsymbol{x}^*) = 1$, and we let $u^* = \varphi(P; \boldsymbol{x}^*)$; where $\varphi(\cdot)$ is from Example 3.2. We then note that, for an arbitrary $\boldsymbol{x} \in \mathbb{R}^n_+$, we have, since $\boldsymbol{x} I[u^*(\boldsymbol{x})\boldsymbol{x}^*]$ and $u(\cdot)$ represents P:

$$u(\boldsymbol{x}) = u[u^*(\boldsymbol{x})\boldsymbol{x}^*].$$

However, since $u(\cdot)$ is positively homogeneous of degree one, we then have:

$$u(\boldsymbol{x}) = u^*(\boldsymbol{x})u(\boldsymbol{x}^*) = u^*(\boldsymbol{x}),$$

and our result follows. □

From the above we see that we lose very little generality in the homothetic case by dealing only with utility functions defined by a measurement function μ.

3.7 Proposition. *If $P \in \mathcal{P}^h$, and $u \colon \mathbb{R}^n_+ \to \mathbb{R}_+$ and $u^* \colon \mathbb{R}^n_+ \to \mathbb{R}_+$ are any two functions representing P which are also positively homogeneous of degree one, then there exists $a \in \mathbb{R}_{++}$ such that for all $\boldsymbol{x} \in \mathbb{R}^n_+$, we have $u(\boldsymbol{x}) = au^*(\boldsymbol{x})$.*

Proof. Let u and u^* satisfy the stated hypotheses. Making use of Proposition 3.6, we let $\boldsymbol{x}^* \in \mathbb{R}^n_{++}$ be such that $u^* = \varphi(P; \boldsymbol{x}^*)$, and define $a = u(\boldsymbol{x}^*)$. As in the proof of Proposition 3.6, we note that for arbitrary $\boldsymbol{x} \in \mathbb{R}^n_+$, we must have $\boldsymbol{x} I[u^*(\boldsymbol{x})\boldsymbol{x}^*]$; and thus, since u is positively homogeneous of degree one and represents P, we then conclude that:

$$u(\boldsymbol{x}) = u[u^*(\boldsymbol{x})\boldsymbol{x}^*] = u^*(\boldsymbol{x})u(\boldsymbol{x}^*) = au^*(\boldsymbol{x}). \quad \square$$

4 Social Preference Relations.

Given an admissible preference space, $\boldsymbol{\mathcal{P}}$, we will refer to any asymmetric order,[9] $\succ$, on $\boldsymbol{X} \times \boldsymbol{\mathcal{P}}$ as a **social preference relation for $\boldsymbol{\mathcal{P}}$**; and we will denote the family of all such asymmetric orders by '$\mathcal{S}(\boldsymbol{\mathcal{P}})$.' It may seem a bit fanciful to speak of a social ordering on $\boldsymbol{X} \times \boldsymbol{\mathcal{P}}$; however, given $\succ \in \mathcal{S}(\boldsymbol{\mathcal{P}})$, we will attach normative significance only to the **conditional social preference relation for $\boldsymbol{P}$ induced on $\boldsymbol{X}$ by $\succ, \succ_{\boldsymbol{P}}$**, defined by:

$$\boldsymbol{x} \succ_{\boldsymbol{P}} \boldsymbol{x}^* \iff (\boldsymbol{x}, \boldsymbol{P}) \succ (\boldsymbol{x}^*, \boldsymbol{P}).$$

Letting '$\mathcal{G}$' denote the collection of all asymmetric orders on $\boldsymbol{X} = \mathbb{R}^{mn}_+$, we note that if $\succ$ is an element of $\mathcal{S}(\boldsymbol{\mathcal{P}})$, then, for each $\boldsymbol{P} \in \boldsymbol{\mathcal{P}}$, $\succ_{\boldsymbol{P}}$ is an element of $\mathcal{G}$.

The concept of a social preference function is defined here as follows.

4.1 Definition. Given an admissible preference space, $\boldsymbol{\mathcal{P}}$, we will say that any function $\omega \colon \boldsymbol{\mathcal{P}} \to \mathcal{G}$ is a (**consumer sovereign-type**) **social preference function for $\boldsymbol{\mathcal{P}}$**.

Clearly, any $\succ \in \mathcal{S}(\boldsymbol{\mathcal{P}})$ defines a social preference function for $\boldsymbol{\mathcal{P}}$ as:

$$(\forall \boldsymbol{P} \in \boldsymbol{\mathcal{P}}) \colon \omega(\boldsymbol{P}) = \succ_{\boldsymbol{P}};$$

[9] That is, a binary relation which is asymmetric and transitive.

that is, the social preference function, $\omega(\boldsymbol{P})$ is simply the conditional social preference relation for $\boldsymbol{P}$ induced on X by $\succ$. Making use of this observation, it can be said that the whole body of what has been called the 'New Welfare Economics' has been developed around the assumption that the following postulate is 'universally acceptable.'

4.2 Definition. Given an admissible preference space, $\mathcal{P}$, we will say that a social preference relation, $\succ \in \mathcal{S}(\mathcal{P})$, satisfies the **Bergson consumer sovereignty principle** iff the conditional social preference relation for $\succ$ satisfies the **Pareto principle**: that is, given any $\boldsymbol{P} \in \mathcal{P}$, we have:

$$(\forall \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X})\colon \boldsymbol{x}\boldsymbol{P}\boldsymbol{y} \Rightarrow \boldsymbol{x} \succ_{\boldsymbol{P}} \boldsymbol{y}.$$

We shall denote the subset of $\mathcal{S}(\mathcal{P})$ consisting of all social preference relations satisfying the Bergson consumer sovereignty principle by '$\mathcal{B}(\mathcal{P})$.'

In words, a social preference relation, $\succ \in \mathcal{S}(\mathcal{P})$, satisfies the Bergson consumer sovereignty principle iff, given any preference profile, $\boldsymbol{P} \in \mathcal{P}$, we have that whenever each of the consumers prefers his or her bundle in allocation $\boldsymbol{x}$ to that received in the allocation $\boldsymbol{x}'$, then the social preference relation ($\succ_{\boldsymbol{P}}$) ranks $\boldsymbol{x}$ as better than $\boldsymbol{x}'$

Now, the next question is, "what does it mean to say that the Bergson consumer sovereignty postulate is 'universally acceptable'?" In this paper, we will take this to mean the following. Suppose there are q policy decision-makers in society (with $q \geq 1$),[10] and that the k^{th} such decision-maker bases his or her decisions (or acts as if he or she bases his or her decisions) on an ordering, $\succ_k \in \mathcal{S}(\mathcal{P})$. We will say that the **Bergson consumer sovereignty postulate is universally acceptable (given $\mathcal{P}$)** iff:

$$\succ_k \in \mathcal{B}(\mathcal{P}) \quad \text{for } k = 1, \ldots, q.$$

In fact, we will be particularly interested in the following sub-class of the Bergson family.

4.3 Definition. Given an admissible preference space, $\mathcal{P}$, we will say that a social preference relation, $\succ \in \mathcal{S}(\mathcal{P})$, is a **Pareto-Bergson-Samuelson (PBS) Social Preference Relation for $\mathcal{P}$** iff there exists a measurement function for $\mathcal{P}$, μ, and an increasing function $F\colon \mathbb{R}^m_+ \to \mathbb{R}$ such that for all $\boldsymbol{P} \in \mathcal{P}$, if we define $u_i = \mu(P_i)$ for $i = 1, \ldots, m$; we have:

$$(\forall \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X})\colon \boldsymbol{x} \succ_{\boldsymbol{P}} \boldsymbol{y} \iff F[\boldsymbol{u}(\boldsymbol{x})] > F[\boldsymbol{u}(\boldsymbol{y})],$$

where we define:[11]

$$\boldsymbol{u}(\boldsymbol{x}) = \big(u_1(\boldsymbol{x}_1), \ldots, u_m(\boldsymbol{x}_m)\big) \text{ and } \boldsymbol{u}(\boldsymbol{y}) = \big(u_1(\boldsymbol{y}_1), \ldots, u_m(\boldsymbol{y}_m)\big).$$

The composite function, $W = F \cdot \boldsymbol{u}$ will be called a **PBS Welfare Function**, or simply a **PBS function.**

In dealing with PBS functions, we will refer to the function μ as the **measurement function**, and to F as the **aggregator function**. We will denote the family of all PBS

[10] This set might, in a democracy, be taken to coincide with the set of consumers.

[11] Thus, for purists, $W(\boldsymbol{x}, \boldsymbol{P}) = F\big[\mu(P_1)(\boldsymbol{x}_1), \ldots, \mu(P_m)(\boldsymbol{x}_m)\big]$.

social preference relations by '$\mathcal{B}^*(\mathcal{P})$;' and we will call this set the '**PBS family for $\mathcal{P}$**.' Since each such social preference relation has the property that it can be represented (more correctly, the conditional social preference relation it induces can be represented) by a composite function which is determined by a measurement function, μ, and an aggregator function, F, we will speak of $\succ \in \mathcal{B}^*(\mathcal{P})$ as being **determined by** (μ, F).[12] Notice, however, that we are not requiring $W = F \cdot \boldsymbol{u}$ to represent $\succ$ on $\boldsymbol{X} \times \mathcal{P}$; that is, while such a function induces an ordering on $\boldsymbol{X} \times \mathcal{P}$, we are ascribing normative significance only to the conditional ordering it induces on $\boldsymbol{X}$, given $\boldsymbol{P}$.

4.4 Examples. Consider the aggregator function $F\colon \mathbb{R}^m_+ \to \mathbb{R}_+$ defined by:

$$F(\boldsymbol{u}) = \sum\nolimits_{i=1}^{m} u_i. \tag{5}$$

The function F defines a Bergson-Samuelson Social Welfare Function when paired with any measurement function, $\mu\colon \mathcal{P}^c \to \mathcal{U}$; as does the aggregator function F^* defined by:

$$F^*(\boldsymbol{u}) = \prod\nolimits_{i=1}^{m} (u_i)^{a_i}, \tag{6}$$

where $a_i \in \mathbb{R}_{++}$, for $i = 1, \ldots, m$, and $\sum_{i=1}^{m} a_i = 1$. We shall refer to the first of these two examples as the **utilitarian aggregator function**, and any function of the type in equation (6) as an **Eisenberg-type aggregator function**.

A final example of interest is the **Rawlsian aggregator function**, defined by:

$$F(\boldsymbol{u}) = \min\{u_1, \ldots, u_m\}. \quad \square \tag{7}$$

In connection with the examples just presented, it should be noted that, using the method of Example 3.2, one obtains a significantly different measurement function for each different value of $\boldsymbol{x}^* \in \mathbb{R}^n_{++}$. This in turn means that, for a given aggregator function, F, one may obtain very different social preference functions if one combines one measurement function, μ^*, defined from $\boldsymbol{x}^*$, than one does from $\mu^\dagger$, say, defined from $\boldsymbol{x}^\dagger \in \mathbb{R}^n_{++}$. *However, it follows from Propositions 3.6 and 3.7 that if the admissible preference space is $\mathcal{P}^h$, a PBS function of the Eisenberg form induces a social preference ordering which is independent of the measurement function, μ, with which it is paired.*

In the remainder of this paper, we will pursue the implications of the assumption that our 'clientele' as economists consists of q decision-makers, the k^{th} of whom resolves his or her opinion regarding alternative economic policies on the basis of a social preference relation, $\succ_k \in \mathcal{B}(\mathcal{P})$. The basic thesis is that if this assumption is appropriate, then the function of economic models, forecasts, and policy analysis should be to provide information pertinent to such social preference relations. Our concern in the next three sections will be to apply this approach to the problem of comparing 'real national income' at two different equilibria. Before proceeding, however, let's pause to note an important fact: it will quickly become apparent that it makes no difference whatever how many policy decision-makers there are in a given society (as long as there is more than one);

[12]This approach was introduced in [12] and [14]; although in [14] it was discussed rather in passing.

the key issue being to what subset of $\mathcal{B}(\mathcal{P})$ do their social preference relations belong. Consequently, hereafter we shall simply suppose that the social preference relations of the decision-makers are all elements of some subset of $\mathcal{B}(\mathcal{P})$, which subset we shall generally denote by '$\mathcal{B}^d$.' Correspondingly, we will drop the subscript k when dealing with a social preference relation, and simply speak of an arbitrary element, $\succ$, of $\mathcal{B}^d$.

5 Indirect Preferences: Individual and Social.

We will narrow the focus of our investigation of 'real national income comparisons' to concentrate upon the fundamentals of the problem examined in [5] and [7]; namely, we will look at the problem of comparing situations which are competitive equilibria from the standpoint of the consumers in the economy; which we define as follows.

5.1 Definition. Let $\boldsymbol{P} \in \mathcal{P}^0$. We will say that a pair $(\boldsymbol{x}^*, \boldsymbol{p}^*) \in \mathbb{R}_+^{mn} \times \mathbb{R}_{++}^n$ is a **(consumers') competitive equilibrium for $\boldsymbol{P}$** iff, for each i ($i = 1, \ldots, m$), the following condition holds:

$$(\forall \boldsymbol{x}_i \in \mathbb{R}_+^n) \colon \boldsymbol{x}_i P_i \boldsymbol{x}_i^* \Rightarrow \boldsymbol{p}^* \cdot \boldsymbol{x}_i > \boldsymbol{p}^* \cdot \boldsymbol{x}_i^*.$$

We then denote by '$C(\boldsymbol{P})$' the subset of $\mathbb{R}_+^{mn} \times \mathbb{R}_{++}^n$ consisting of all competitive equilibria for $\boldsymbol{P}$, $(\boldsymbol{x}^*, \boldsymbol{p}^*)$, which satisfy $\boldsymbol{x}_i^* \neq \boldsymbol{0}$ for $i = 1, \ldots, m$.

Now, suppose a policy-maker orders allocations according to some social preference relation, $\succ \in \mathcal{B}(\mathcal{P})$. Clearly, given $\boldsymbol{P} \in \mathcal{P}$, the conditional social preference relation, $\succ_{\boldsymbol{P}}$, induces an ordering, $Q_{\boldsymbol{P}}$ on $C(\boldsymbol{P})$ defined by:

$$(\boldsymbol{x}, \boldsymbol{p}) Q_{\boldsymbol{P}} (\boldsymbol{x}', \boldsymbol{p}') \iff \boldsymbol{x} \succ_{\boldsymbol{P}} \boldsymbol{x}'.$$

Consequently, it might appear that if we, as economists, could calculate/predict the equilibrium position, $(\boldsymbol{x}, \boldsymbol{p})$, resulting from some policy measure, and if the policy-maker knows the status quo equilibrium, $(\boldsymbol{x}', \boldsymbol{p}')$, then our policy-maker could easily evaluate the desirability of the policy measure. Unfortunately, this appearance is completely misleading in that, in principle, the policy-maker also needs to know *each* consumer's preference relation in order to make such a determination, since his conditional social preference relation depends upon $\boldsymbol{P}$. In any case, the task of estimating/predicting not only $(\boldsymbol{x}, \boldsymbol{p})$ and $(\boldsymbol{x}', \boldsymbol{p}')$, but each preference relation, P_i as well, appears to be totally beyond our capability both now and in the forseeable future. Consequently, we will be interested in developing a means for avoiding such staggering informational requirements. Our approach will make use of indirect preferences, both individual and social.

Given any relation $P_i \in \mathcal{P}^c$, P_i induces an **indirect preference relation**, P_i^*, on:

$$\Omega \stackrel{\text{def}}{=} \mathbb{R}_{++}^n \times \mathbb{R}_+$$

by:

$$(\boldsymbol{p}, w) P_i^* (\boldsymbol{p}', w') \iff h(\boldsymbol{p}, w; P_i) P_i h(\boldsymbol{p}', w'; P_i);$$

where '$h(\cdot;P_i)$' denotes **the demand function determined by $\boldsymbol{P_i}$**.[13] We say that a function $v_i\colon \Omega \to \mathbb{R}$ is an **indirect utility function corresponding to $\boldsymbol{P_i}$** iff v_i represents P_i^* on Ω.

It is not only obvious but well-known that if u_i is a utility function representing P_i, and if $h(\cdot;P_i)$ is the demand function determined by P_i, then the composite function $v_i\colon \Omega \to \mathbb{R}_+$ defined by:

$$v_i(\boldsymbol{p},w) = u_i[h(\boldsymbol{p},w;P_i)] \quad \text{for } (\boldsymbol{p},w) \in \Omega,$$

is an indirect utility function representing P_i^* on Ω.[14]

A related notion is that of a cost-of-living indicator, which we define as follows.

5.2 Definition. If $\mathcal{P}$ is a subset of $\mathcal{P}^a$, we shall say that a function $C\colon \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \times \mathcal{P} \to \mathbb{R}_{++}$ is a **cost-of-living indicator for $\mathcal{P}$** iff, for all $P_i \in \mathcal{P}$, and for all $(\boldsymbol{p},w), (\boldsymbol{p}',w') \in \Omega$, we have:

$$\boldsymbol{p} = \boldsymbol{p}' \Rightarrow C_i(\boldsymbol{p}',\boldsymbol{p};P_i) = 1, \tag{8}$$

and:

$$\frac{w}{C_i(\boldsymbol{p}',\boldsymbol{p};P_i)} > w' \Rightarrow (\boldsymbol{p},w)P_i^*(\boldsymbol{p}',w'), \tag{9}$$

where P_i^* is the indirect preference relation induced by P_i. In the special case in which $\mathcal{P} = \{P_i\}$, for some $P_i \in \mathcal{P}^0$, we will drop the third argument in $C(\cdot)$, and speak of a **cost-of-living indicator for $\boldsymbol{P_i}$**.

5.3 Example. Given $\mathcal{P} = \mathcal{P}^h$, let $\boldsymbol{x}^* \in \mathbb{R}^n_{++}$ be an arbitrary (fixed) vector, and define $u_i^* = \varphi(P_i;\boldsymbol{x}^*)$ as in Example 3.2. If we then define the function $v^*\colon \Omega \times \mathcal{P}^h \to \mathbb{R}_+$ by:

$$v^*(\boldsymbol{p},w;P_i) = u_i^*[h(\boldsymbol{p},w;P_i)] \quad \text{for } (\boldsymbol{p},w;P_i) \in \Omega \times \mathcal{P}^h,$$

it is clear that, for each $P_i \in \mathcal{P}^h$, $v^*(\cdot;P_i)$ is an indirect utility function corresponding to P_i. Moreover, it follows from Proposition 3.5 that u_i^* is positively homogeneous of degree one in $\boldsymbol{x}_i$; while $h(\boldsymbol{p},\cdot;P_i)$ is positively homogeneous of degree one in w. Consequently, $v^*(\boldsymbol{p},\cdot;P_i)$ is also positively homogeneous of degree one in w, and thus it is easy to show that the function $C^*\colon \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \times \mathcal{P}^h \to \mathbb{R}_{++}$ defined by:

$$C^*(\boldsymbol{p}',\boldsymbol{p};P_i) = \frac{v^*(\boldsymbol{p}',1;P_i)}{v^*(\boldsymbol{p},1;P_i)},$$

is a cost-of-living indicator for $\mathcal{P}^h$. In fact, in this case, for any $P_i \in \mathcal{P}^h$, and for all $(\boldsymbol{p},w), (\boldsymbol{p}',w') \in \Omega$:

$$\frac{w}{C(\boldsymbol{p}',\boldsymbol{p};P_i)} > w' \iff (\boldsymbol{p},w)P_i^*(\boldsymbol{p}',w').$$

[13] So that $h\colon \Omega \times \mathcal{P}^c \to \mathbb{R}^n_+$.

[14] There exist indirect utility functions which are not obtainable in this way, however. For example, the function:

$$v(\boldsymbol{p},w) = w/p_1,$$

is an indirect utility function corresponding to the lexicographic order on $\mathbb{R}^2_+$. See [6, p.74].

The observations of the above paragraph can be expanded upon as follows. If we define $\gamma^*: \mathbb{R}^n_{++} \times \mathcal{P}^h \to \mathbb{R}_{++}$ by:

$$\gamma^*(\boldsymbol{p}; P_i) = 1/v^*(\boldsymbol{p}, 1; P_i) = 1/u_i^*[h(\boldsymbol{p}, 1; P_i)],$$

then it is clear that, for any $P_i \in \mathcal{P}^h$, $\gamma^*(\cdot; P_i)$ is positively homogeneous of degree one in $\boldsymbol{p}$. Moreover, we can express the indirect utility function defined in this example as:

$$v^*(\boldsymbol{p}, w_i; P_i) = \frac{w_i}{\gamma^*(\boldsymbol{p}; P_i)}. \quad \square$$

5.4 Definition. If $P_i \in \mathcal{P}^h$, and $\gamma_i: \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ is such that the function $v_i: \Omega \to \mathbb{R}_+$ defined by:

$$v_i(\boldsymbol{p}, w_i) = \frac{w_i}{\gamma_i(\boldsymbol{p})},$$

is an indirect utility function for P_i, then we shall say that γ_i is an **income deflator function for $\boldsymbol{P_i}$**.

An argument similar to the proof of Proposition 3.7 can be used to establish the following.

5.5 Proposition. *If $P_i \in \mathcal{P}^h$, while $\gamma_i: \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ and $\gamma_i^*: \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ are any two income deflator functions for P_i, then there exists $a \in \mathbb{R}_{++}$ such that for all $\boldsymbol{p} \in \mathbb{R}^n_{++}$, we have:*

$$\gamma_i^*(\boldsymbol{p}) = a\gamma_i(\boldsymbol{p}).$$

For future reference, we note the following relationships between cost-of-living indicators and income deflator functions.

5.6 Proposition. *If $P_i \in \mathcal{P}^h$, $\gamma_i: \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ is any income deflator function for P_i, and $C_i: \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ is any cost-of-living indicator for P_i, we have:*

$$\left(\forall (\boldsymbol{p}^1, \boldsymbol{p}^2) \in \mathbb{R}^n_{++} \times \mathbb{R}^n_{++}\right): C_i(\boldsymbol{p}^1, \boldsymbol{p}^2) \geq \frac{\gamma_i(\boldsymbol{p}^2)}{\gamma_i(\boldsymbol{p}^1)}.$$

Conversely, if γ_i is any income deflator function for P_i, then the function $C_i^*: \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ defined by:

$$C_i^*(\boldsymbol{p}^1, \boldsymbol{p}^2) = \frac{\gamma_i(\boldsymbol{p}^2)}{\gamma_i(\boldsymbol{p}^1)}, \tag{10}$$

is a cost-of-living indicator for P_i.

Proof. Suppose that C_i and γ_i are a cost-of-living indicator and an income deflator function for P_i, respectively, but that for some $(\boldsymbol{p}^1, \boldsymbol{p}^2) \in \mathbb{R}^n_{++} \times \mathbb{R}^n_{++}$, we have:

$$C_i(\boldsymbol{p}^1, \boldsymbol{p}^2) < \frac{\gamma_i(\boldsymbol{p}^2)}{\gamma_i(\boldsymbol{p}^1)}$$

Then there exists $\bar{w}_i^2$ satisfying:

$$C_i(\boldsymbol{p}^1, \boldsymbol{p}^2) < \bar{w}_i^2 < \frac{\gamma_i(\boldsymbol{p}^2)}{\gamma_i(\boldsymbol{p}^1)}. \tag{11}$$

But then, since C_i is a cost-of-living indicator for P_i, it follows from the left-hand inequality in (11) that:

$$(\boldsymbol{p}^2, \bar{w}_i^2) P_i^* (\boldsymbol{p}^1, 1);$$

whereas from the right-hand inequality in (11), we have:

$$v_i(\boldsymbol{p}^2, \bar{w}_i^2) = \frac{\bar{w}_i^2}{\gamma_i(\boldsymbol{p}^2)} < v_i(\boldsymbol{p}^1, 1) = \frac{1}{\gamma_i(\boldsymbol{p}^1)}.$$

The fact that the function defined in (10) is a cost-of-living indicator for P_i is an easy consequence of the definitions, and the proof will be left to the interested reader. □

It is an immediate consequence of Proposition 5.5 that the function C_i^* defined in (10), above, is uniquely determined by P_i, in the sense that if γ_i and $\gamma_i^\dagger$ are any two income deflator functions for P_i, then for all $(\boldsymbol{p}^1, \boldsymbol{p}^2) \in \mathbb{R}^n_{++} \times \mathbb{R}^n_{++}$ we will have:

$$C_i^*(\boldsymbol{p}^1, \boldsymbol{p}^2) = \frac{\gamma_i(\boldsymbol{p}^2)}{\gamma_i(\boldsymbol{p}^1)} = \frac{\gamma_i^\dagger(\boldsymbol{p}^2)}{\gamma_i^\dagger(\boldsymbol{p}^1)} = C^*(\boldsymbol{p}^1, \boldsymbol{p}^2; P_i) = \frac{\gamma^*(\boldsymbol{p}^2; P_i)}{\gamma^*(\boldsymbol{p}^1; P_i)};$$

where $C^*(\cdot)$ and $\gamma^*(\cdot)$ are as defined in Example 5.3. We will hereafter refer to the functions γ^* and C^* as the **income deflator** and **cost-of-living indicator functions determined by $\boldsymbol{\mu}$**, respectively. From Proposition 5.6, we can also see that, for $P_i \in \mathcal{P}^h$, the function $C^*(\,\cdot\,; P_i)$ is the *best* cost-of-living indicator for P_i, in the sense that its value is always the greatest lower bound among cost-of-living indicators for P_i.

In our next result, we establish properties of a very practical cost-of-living indicator for the homothetic case. In it, we make use of the following definition/notation. For $P_i \in \mathcal{P}^h$, the demand function corresponding to P_i can be written in the form:

$$h(\boldsymbol{p}, w_i; P_i) = g(\boldsymbol{p}, P_i) \cdot w_i, \tag{12}$$

where $g \colon \mathbb{R}^n_{++} \times \mathcal{P}^h \to \mathbb{R}^n_{+}$; and, for each $P_i \in \mathcal{P}^h$, $g(\cdot, P_i)$ is positively homogeneous of degree minus one in $\boldsymbol{p}$, and satisfies:

$$(\forall \boldsymbol{p} \in \mathbb{R}^n_{++}) \colon \boldsymbol{p} \cdot g(\boldsymbol{p}, P_i) = 1. \tag{13}$$

5.7 Proposition. *If we define $L \colon \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \times \mathcal{P}^h \to \mathbb{R}_{++}$ by:*

$$L(\boldsymbol{p}^1, \boldsymbol{p}^2; P_i) = \boldsymbol{p}^2 \cdot g(\boldsymbol{p}^1; P_i) = \frac{\boldsymbol{p}^2 \cdot h(\boldsymbol{p}^1, w^1; P_i)}{w_i^1}, \tag{14}$$

then L is a cost-of-living indicator for $\mathcal{P}^h$.

Proof. Let $(\boldsymbol{p}^t, w^t) \in \Omega$, for $t = 1, 2$. If $P_i \in \mathcal{P}^h$, and:

$$w_i^2 / L(\boldsymbol{p}^1, \boldsymbol{p}^2; P_i) > w_i^1,$$

then:

$$w_i^2 > \boldsymbol{p}^2 \cdot g(\boldsymbol{p}^1; P_i) w_i^1 = \boldsymbol{p}^2 \cdot h(\boldsymbol{p}^1, w_i^1; P_i),$$

and it follows that $(\boldsymbol{p}^2, w^2) P_i^* (\boldsymbol{p}^1, w^1)$. Furthermore, it follows at once from (13) that $L(\boldsymbol{p}^1, \boldsymbol{p}^2; P_i) = 1$ if $\boldsymbol{p}^1 = \boldsymbol{p}^2$. $\square$

Now, given an admissible preference space, $\boldsymbol{\mathcal{P}} \subseteq \boldsymbol{\mathcal{P}}^c$ and $\boldsymbol{P} \in \boldsymbol{\mathcal{P}}$, any $\succ \in \mathcal{S}(\boldsymbol{\mathcal{P}})$ induces an **indirect (conditional) social preference relation**, $\succ_{\boldsymbol{P}}^*$ on:

$$\boldsymbol{\Omega} \stackrel{\text{def}}{=} \mathbb{R}_{++}^n \times \mathbb{R}_{++}^m.$$

defined by:

$$(\boldsymbol{p}'', \boldsymbol{w}'') \succ_{\boldsymbol{P}}^* (\boldsymbol{p}', \boldsymbol{w}') \iff \boldsymbol{h}(\boldsymbol{p}'', \boldsymbol{w}''; \boldsymbol{P}) \succ_{\boldsymbol{P}} \boldsymbol{h}(\boldsymbol{p}', \boldsymbol{w}'; \boldsymbol{P});$$

where we define $\boldsymbol{h} \colon \boldsymbol{\Omega} \times \boldsymbol{\mathcal{P}}^c \to \mathbb{R}_+^{mn}$ by:

$$\boldsymbol{h}(\boldsymbol{p}, \boldsymbol{w}; \boldsymbol{P}) = \big(h(\boldsymbol{p}, w_1; P_1), \dots, h(\boldsymbol{p}, w_m; P_m)\big).$$

Moreover, if $\succ \in \mathcal{B}^*(\boldsymbol{\mathcal{P}})$, we can define an indirect social welfare function to represent $\succ_{\boldsymbol{P}}^*$, as follows. If $\succ$ is determined by (μ, F), and we define $\boldsymbol{v} \colon \boldsymbol{\Omega} \times \boldsymbol{\mathcal{P}} \to \mathbb{R}_+^m$ by:

$$\boldsymbol{v}(\boldsymbol{p}, \boldsymbol{w}; \boldsymbol{P}) = \big(u_1[h(\boldsymbol{p}, w_1; P_1)], \dots, u_m[h(\boldsymbol{p}, w_m; P_m)]\big) = \big(v(\boldsymbol{p}, w_1; P_1), \dots, v(\boldsymbol{p}, w_m; P_m)\big),$$

where $u_i = \mu(P_i)$ and $v = u_i \circ h$, for $i = 1, \dots, m$; then it is easily seen that $F \circ \boldsymbol{v}$ represents the indirect conditional social preference relation, $\succ_{\boldsymbol{P}}^*$; that is, for all $\boldsymbol{P} \in \boldsymbol{\mathcal{P}}$, and all $(\boldsymbol{p}, \boldsymbol{w}), (\boldsymbol{p}', \boldsymbol{w}') \in \boldsymbol{\Omega}$:

$$(\boldsymbol{p}, \boldsymbol{w}) \succ_{\boldsymbol{P}}^* (\boldsymbol{p}', \boldsymbol{w}') \iff F\big[\boldsymbol{v}(\boldsymbol{p}, \boldsymbol{w}; \boldsymbol{P})\big] > F\big[\boldsymbol{v}(\boldsymbol{p}', \boldsymbol{w}'; \boldsymbol{P})\big].$$

We shall refer to the composite function, $F \circ \boldsymbol{v}$, as the **indirect social welfare function for $\succ$**.[15]

6 Welfare Indicators and Welfare Content.

In the remainder of this paper, we will make use of the fact that for $\boldsymbol{P} \in \boldsymbol{\mathcal{P}}^c$ there is a one-to-one correspondence between competitive equilibria, $(\boldsymbol{x}, \boldsymbol{p}) \in C(\boldsymbol{P})$, and points $(\boldsymbol{p}, \boldsymbol{w}) \in \boldsymbol{\Omega}$.[16] In dealing with points/observations $(\boldsymbol{p}, \boldsymbol{w}) \in \boldsymbol{\Omega}$, we will always assume that

[15] There are, of course, other functions which represent $\succ_{\boldsymbol{P}}^*$, but this function is uniquely determined by the pair (μ, F).

[16] If $(\boldsymbol{x}^*, \boldsymbol{p}^*) \in C(\boldsymbol{P})$, the corresponding point in $\boldsymbol{\Omega}$ is given by $(\boldsymbol{p}^*, \boldsymbol{w}^*)$, where:

$$w_i^* = \boldsymbol{p}^* \cdot \boldsymbol{x}_i^*, \quad \text{for } i = 1, \dots, m.$$

If $(\boldsymbol{p}', \boldsymbol{w}') \in \boldsymbol{\Omega}$, the corresponding equilibrium is $(\boldsymbol{x}', \boldsymbol{p}')$, where:

$$\boldsymbol{x}_i' = h(\boldsymbol{p}', w_i'; P_i) \quad \text{for } i = 1, \dots, m.$$

$(\boldsymbol{p}, \boldsymbol{w})$ corresponds to (is generated by) $(\boldsymbol{x}, \boldsymbol{p}) \in C(\boldsymbol{P})$, where:

$$\boldsymbol{x}_i = h(\boldsymbol{p}, w_i; P_i) \quad \text{for } i = 1, \ldots, m.$$

In the material to follow, we define the set $\boldsymbol{D}$ by:

$$\boldsymbol{D} = \boldsymbol{\Omega} \times \boldsymbol{\Omega}.$$

The first of the two normative criteria to be investigated can now be set out as follows. We seek a triple, (f_1, f_2, Q), where:

$$f_t \colon \boldsymbol{D} \to \mathbb{R}^s_+ \quad \text{for } t = 1, 2, \tag{15}$$

and where $s < mn$, and Q is an asymmetric order on $\mathbb{R}^s_+$, and where, for $\succ \in \mathcal{S}(\mathcal{P})$, $\boldsymbol{P} \in \mathcal{P}$ and for every $\big((\boldsymbol{p}, \boldsymbol{w}), (\boldsymbol{p}^*, \boldsymbol{w}^*)\big) \in \boldsymbol{D}^*$:

$$f_2[(\boldsymbol{p}, \boldsymbol{w}), (\boldsymbol{p}^*, \boldsymbol{w}^*)] \, Q \, f_1[(\boldsymbol{p}, \boldsymbol{w}), (\boldsymbol{p}^*, \boldsymbol{w}^*)] \Rightarrow (\boldsymbol{p}^*, \boldsymbol{w}^*) \succ^*_{\boldsymbol{P}} (\boldsymbol{p}, \boldsymbol{w}), \tag{16}$$

While this is the basis for the approach to be taken, we need to modify and extend our framework in two directions. First, we need to take into account the fact that neither the policy-maker nor the economist doing the analysis will generally know the consumers' preference profile, $\boldsymbol{P}$. Secondly, our goal is to obtain a criterion that is valid for each social preference relation, $\succ$, in some interesting subset of $\mathcal{S}(\mathcal{P})$. We begin by modifying our definitions to take account of the unknown preference profile aspect, as follows.

We require that the pair of functions, (f_1, f_2) have the domain $\boldsymbol{D}^* \times \mathcal{P}$. That is, suppose $\mathcal{P}$ is an admissible preference space, that $\succ \in \mathcal{S}(\mathcal{P})$, and $\boldsymbol{D}^* \subseteq \boldsymbol{D}$, that:

$$f_t \colon \boldsymbol{D}^* \times \mathcal{P} \to \mathbb{R}^s_+ \quad \text{for } t = 1, 2, \tag{17}$$

and that:

$$Q \text{ is an asymmetric order on } \mathbb{R}^s_+. \tag{18}$$

We will say that **(f_1, f_2, Q) provides a welfare criterion for $\succ$ on $\boldsymbol{D}^* \times \mathcal{P}$** iff, for every $\boldsymbol{P} \in \mathcal{P}$ and for every $\big((\boldsymbol{p}, \boldsymbol{w}), (\boldsymbol{p}^*, \boldsymbol{w}^*)\big) \in \boldsymbol{D}^*$, we have:

$$f_2[(\boldsymbol{p}, \boldsymbol{w}), (\boldsymbol{p}^*, \boldsymbol{w}^*); \boldsymbol{P}] \, Q \, f_1[(\boldsymbol{p}, \boldsymbol{w}), (\boldsymbol{p}^*, \boldsymbol{w}^*); \boldsymbol{P}] \Rightarrow (\boldsymbol{p}^*, \boldsymbol{w}^*) \succ^*_{\boldsymbol{P}} (\boldsymbol{p}, \boldsymbol{w}) \tag{19}$$

Of course, requiring that the functions f_t depend upon both price-wealth pairs *and* the preference profile, $\boldsymbol{P}$, may appear to defeat the whole idea of trying to develop a criterion which can be applied in the absence of full knowledge of $\boldsymbol{P}$. This will, however, be less of a problem than it appears, as we shall see. We expand this notion to require that the triple provides a welfare criterion for each of a class of social preference relations, as follows.

6.1 Definition. Let $\mathcal{P}$ be an admissible preference space, let $\mathcal{B}^d \subseteq \mathcal{S}(\mathcal{P})$, let $\boldsymbol{D}^* \subseteq \boldsymbol{D}$, and suppose the triple (f_1, f_2, Q) satisfies equations (17) and (18), above. We shall say that **(f_1, f_2, Q) provides a welfare criterion for $\mathcal{B}^d$ on $\boldsymbol{D}^* \times \mathcal{P}$** iff (f_1, f_2, Q) provides a welfare criterion on $\boldsymbol{D}^* \times \mathcal{P}$ for each $\succ \in \mathcal{B}^d$.

The following example should help to illustrate and explicate the definition, as well as showing how and why it is that we may need to make our functions dependent upon the preference profile as well as the pair of price-wealth pairs.

6.2 Example. Let $\mathcal{P} = \mathcal{P}^c, s = m$, and define the pair of functions, (f_1, f_2), where $f_t \colon \boldsymbol{D} \to \mathbb{R}^m_+$, for $t = 1, 2$, by:

$$f_1\big[(\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2); \boldsymbol{P}\big] = \big(\boldsymbol{p}^2 \cdot h(\boldsymbol{p}^1, w^1_1; P_1), \dots, \boldsymbol{p}^2 \cdot h(\boldsymbol{p}^1, w^1_m; P_m)\big)$$

and:

$$f_2\big[(\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2); \boldsymbol{P}\big] = \boldsymbol{w}^2 = (w^2_1, \dots, w^2_m),$$

respectively. If we take Q to be $\gg$, the strict inequality relation on $\mathbb{R}^m$, then we note that if:

$$f_2\big[(\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2); \boldsymbol{P}\big] Q f_1\big[(\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2); \boldsymbol{P}\big],$$

it is easily seen that:

$$\boldsymbol{h}(\boldsymbol{p}^2, \boldsymbol{w}^2; \boldsymbol{P}) \boldsymbol{P}\, \boldsymbol{h}(\boldsymbol{p}^1, \boldsymbol{w}^1; \boldsymbol{P}),$$

and thus, for any $\succ \in \mathcal{B}(\mathcal{P}^c)$, $(\boldsymbol{p}^2, \boldsymbol{w}^2) \succ^*_{\boldsymbol{P}} (\boldsymbol{p}^1, \boldsymbol{w}^1)$, where $\succ^*_{\boldsymbol{P}}$ is the indirect social preference relation induced by $\succ_{\boldsymbol{P}}$. Consequently, we see that, in terms of the definition just stated, $(f_1, f_2, \gg)$ provides a welfare criterion for $\mathcal{B}(\mathcal{P}^c)$ on $\boldsymbol{D} \times \mathcal{P}^c$. $\square$

In this last example we have developed a triple (f_1, f_2, Q) which provides a welfare criterion for $\mathcal{B}(\mathcal{P}^c)$, and where $s = m$. Unfortunately, the functions in the above example leave too many pairs of points in $\boldsymbol{\Omega}$ which are noncomparable. In fact, the triple (f_1, f_2, Q) only allows comparisons to be made if $(\boldsymbol{p}^2, \boldsymbol{w}^2)$ is strictly Pareto superior to $(\boldsymbol{p}^1, \boldsymbol{w}^1)$; and then only if one could demonstrate such dominance by a revealed preference argument. This example rather typifies the results I have obtained thus far in striving to find such welfare criteria; only under special conditions do they only allow comparisons to be made if $(\boldsymbol{p}^2, \boldsymbol{w}^2)$ is not strictly Pareto superior to $(\boldsymbol{p}^1, \boldsymbol{w}^1)$.[17] Consequently, in the remainder of this paper we will investigate a somewhat weaker requirement and a simplified approach.

The essence of the weakening of the welfare criterion idea which we will pursue is that we will allow the asymmetric order, Q, to be different for different social preference relations; in effect, we will take the point of view that the economist will provide the data to the policy-makers, perhaps along with advice about how they might evaluate the data, but the economist will not generally attempt to make statements as to the desirability of a change from the standpoint of *every* social preference relation.

Insofar as the simplification of the approach is concerned, we will take as our guide the 'cost-of-living indicator' for individuals developed in the previous section; adapting Definition 6.1 as follows.

6.3 Definition. Suppose $\mathcal{P}$ is an admissible preference space, that $\succ \in \mathcal{S}(\mathcal{P})$ and $\boldsymbol{D}^* \subseteq \boldsymbol{D}$, and suppose that $f \colon \boldsymbol{\Omega} \to \mathbb{R}^s_+$, where $s < mn$, that $\Theta \colon \boldsymbol{D}^* \times \mathcal{P} \to \mathbb{R}_{++}$, and that Q is an

[17] However, see Proposition 7.4, in the next section.

asymmetric order on $\mathbb{R}^s_+$.. We will say that (f, Θ, Q) **provides a welfare criterion for $\succ$ on $\boldsymbol{D}^* \times \mathcal{P}$** iff, for every $\big((\boldsymbol{p}, \boldsymbol{w}), (\boldsymbol{p}^*, \boldsymbol{w}^*)\big) \in \boldsymbol{D}^*$, and for every $\boldsymbol{P} \in \mathcal{P}$, we have:

$$\left(\frac{f(\boldsymbol{p}^2, \boldsymbol{w}^2)}{\Theta\big[(\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2); \boldsymbol{P}\big]}\right) Q f(\boldsymbol{p}^1, \boldsymbol{w}^1) \Rightarrow (\boldsymbol{p}^2, \boldsymbol{w}^2) \succ^*_{\boldsymbol{P}} (\boldsymbol{p}^1, \boldsymbol{w}^1) \tag{20}$$

where $\succ^*_{\boldsymbol{P}}$ is the indirect social preference relation induced on $\boldsymbol{\Omega}$ by $\succ_{\boldsymbol{P}}$.

The requirement which we will be investigating can now be presented as follows.

6.4 Definition. Let $\mathcal{P}$ be an admissible preference space, let $\mathcal{B}^d \subseteq \mathcal{S}(\mathcal{P})$, and let $\boldsymbol{D}^* \subseteq \boldsymbol{D}$. We shall say that a pair **(f, Θ) has welfare content for $\mathcal{B}^d$ on $\boldsymbol{D}^* \times \mathcal{P}$** iff for each $\succ \in \mathcal{B}^d$, there exists an asymmetric order, Q, on $\mathbb{R}^s_+$ such that (f, Θ, Q) provides a welfare criterion for $\succ$ on $\boldsymbol{D}^* \times \mathcal{P}$.

Notice that in the above definition, the relation Q may be different for each $\succ \in \mathcal{B}^d$. On the other hand, for a given $\succ \in \mathcal{B}^d$, we require Q to be the same for each $\boldsymbol{P} \in \mathcal{P}$. The idea behind the definition is this: we suppose that, in trying to evaluate data provided him or her by the pair (f, Θ), a policy-maker may know that the true m-tuple of preferences, $\boldsymbol{P}$, is an element of some set $\mathcal{P} \subseteq \mathcal{P}^0$, but will generally not know exactly what $\boldsymbol{P}$ is.

Following the precedent established in the definition of cost-of-living indicators, we will also generally require that Θ satisfies the condition: for all $\Big(\big((\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2)\big), \boldsymbol{P}\Big) \in \boldsymbol{D}^*$:

$$\boldsymbol{p}^1 = \boldsymbol{p}^2 \Rightarrow \Theta\big[\big((\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2)\big); \boldsymbol{P}\big] = 1. \tag{21}$$

Because of this, and since we will often be dealing with domains, $\boldsymbol{D}^*$, satisfying:

$$\Big(\forall \big((\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2)\big) \in \boldsymbol{D}^*\Big) : \boldsymbol{p}^1 = \boldsymbol{p}^2, \tag{22}$$

we will also be interested in the following condition.

6.5 Definition. Let $\mathcal{P}$ be an admissible preference space, let $\succ$ be an element of $\mathcal{S}(\mathcal{P})$, and let $\boldsymbol{D}^* \subseteq \boldsymbol{D}$. We shall say that a function $f \colon \boldsymbol{\Omega} \to \mathbb{R}^s_+$ **has welfare content for $\succ$ on $\boldsymbol{D}^* \times \mathcal{P}$** iff there exists an asymmetric order, Q, on $\mathbb{R}^s_+$ such that for all $\Big(\big((\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2)\big), \boldsymbol{P}\Big) \in \boldsymbol{D}^* \times \mathcal{P}$:

$$f(\boldsymbol{p}^2, \boldsymbol{w}^2) Q f(\boldsymbol{p}^1, \boldsymbol{w}^1) \Rightarrow (\boldsymbol{p}^2, \boldsymbol{w}^2) \succ^*_{\boldsymbol{P}} (\boldsymbol{p}^1, \boldsymbol{w}^1). \tag{23}$$

If $\mathcal{B}^d \subseteq \mathcal{S}(\mathcal{P})$, we shall say that **$f$ has welfare content for $\mathcal{B}^d$ on $\boldsymbol{D}^* \times \mathcal{P}$** iff f has welfare content on $D^* \times \mathcal{P}$ for each $\succ \in \mathcal{B}^d$.[18]

An example may help to illustrate the concepts just presented.

[18]Notice that this is equivalent to the statement that (f, Θ) has welfare content for $\mathcal{B}^d$ on $\boldsymbol{D}^* \times \mathcal{P}$ if Θ and $\boldsymbol{D}^*$ satisfy (21) and (22), respectively.

6.6 Example. Let $\mathcal{B}^d$ be the set of all $\succ \in \mathcal{B}^*(\mathcal{P}^h)$ determined by a pair (μ, F), where F is positively homogeneous of degree one, let $\mathcal{P}$ be defined by:

$$\mathcal{P} = \{\boldsymbol{P} \in \mathcal{P}^h \mid P_1 = P_2 = \cdots = P_m\};$$

and notice that, given the definition of $\mathcal{P}$, it follows from Proposition 3.7 that if an aggregator function, F, satisfies the present assumptions, then the social preference relation $\succ \in \mathcal{B}^d$ determined by (μ, F) is actually independent of μ. Consequently, in the context of this example we will speak of $\succ$ as being determined by an aggregator function, F.

Now, let $s = m$, define $\boldsymbol{D}^* \subseteq \boldsymbol{D}$ by:

$$\boldsymbol{D}^* = \Big\{\big((\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2)\big) \in \boldsymbol{D} \mid \boldsymbol{p}^1 = \boldsymbol{p}^2\Big\},$$

and define $f\colon \Omega \to \mathbb{R}^m_+$ by $f(\boldsymbol{p}, \boldsymbol{w}) = \boldsymbol{w}$. We can then show that f has welfare content for $\mathcal{B}^d$ on $\boldsymbol{D}^* \times \mathcal{P}$, as follows.

Let $\succ \in \mathcal{B}^d$ be given; so that $\succ$ is determined by an aggregator function F which is positively homogeneous of degree one and increasing in $\boldsymbol{u}$, and define $Q_\succ$ on $\mathbb{R}^s$ by:

$$\boldsymbol{w}^2 Q_\succ \boldsymbol{w}^1 \iff F(\boldsymbol{w}^2) > F(\boldsymbol{w}^1).$$

Now, let $\mu\colon \mathcal{P}^c \to \mathcal{U}$ be any measurement function of the form set out in Example 3.2, and let $\gamma\colon \mathbb{R}^n_{++} \times \mathcal{P}^h \to \mathbb{R}_{++}$ be the income deflator function determined by μ. Then the indirect utility function determined by μ is given by:

$$v(\boldsymbol{p}, w_i; P_i) = w_i / \gamma(\boldsymbol{p}; P_i);$$

and, for $\boldsymbol{P} \in \mathcal{P}^h$, the indirect social preference relation induced by $\succ$ can be represented by the function:

$$F\big[\boldsymbol{v}(\boldsymbol{p}, \boldsymbol{w}; \boldsymbol{P})\big] = F\Big[\frac{w_1}{\gamma(\boldsymbol{p}; P_1)}, \ldots, \frac{w_m}{\gamma(\boldsymbol{p}; P_m)}\Big].$$

However, for $\boldsymbol{P} \in \mathcal{P}$, we have $\gamma(\boldsymbol{p}; P_i) = \gamma(\boldsymbol{p}; P_m)$ for $i = 1, \ldots, m$, and, by definition of $\boldsymbol{D}^*$, we have $\boldsymbol{p}^1 = \boldsymbol{p}^2$. Therefore, since F is positively homogeneous of degree one, it follows that:

$$F\big[\boldsymbol{v}(\boldsymbol{p}^t, \boldsymbol{w}^t; \boldsymbol{P})\big] = \frac{1}{\gamma(\boldsymbol{p}^1; P_1)} F(w_1^t, \ldots, w_m^t) \equiv \frac{F(\boldsymbol{w}^t)}{\gamma(\boldsymbol{p}^1; P_1)} \quad \text{for } t = 1, 2.$$

Consequently, for $\big((\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2)\big) \in \boldsymbol{d}^*$ and $\boldsymbol{P} \in \mathcal{P}$:

$$(\boldsymbol{p}^2, \boldsymbol{w}^2) \succ^*_{\boldsymbol{P}} (\boldsymbol{p}^1, \boldsymbol{w}^1) \iff F(\boldsymbol{w}^2) > F(\boldsymbol{w}^1) \iff f(\boldsymbol{p}^2, \boldsymbol{w}^2) Q_\succ f(\boldsymbol{p}^1, \boldsymbol{w}^1);$$

and we see that (f, Θ) has welfare content for $\mathcal{B}^d$ on $\boldsymbol{D}^* \times \mathcal{P}$.

We can use this same example to illustrate the significance of the difference between-providing welfare content and providing a welfare criterion, as we have defined these terms here. In fact, despite the especially simple structure of $\boldsymbol{D}^* \times \mathcal{P}$ in this case, under the present assumptions the only ordering, Q, of $\mathbb{R}^m$ such that (f, Q) provides a welfare criterion for $\mathcal{B}^d$ on $\boldsymbol{D}^* \times \mathcal{P}$ is the strictly greater than relation, $\gg$, on $\mathbb{R}^m$. That this is so follows readily once we note the fact that the aggregator functions $F_i(\boldsymbol{u}) = u_i$ $(i = 1, \ldots, m)$ all satisfy the assumptions of this example. $\square$

6.7 Definition. Let $\boldsymbol{P} \in \mathcal{P}^c$. We define the binary relation, $\gg_{\boldsymbol{P}}$ on $\boldsymbol{\Omega}$ by:

$$(\boldsymbol{p}^2, \boldsymbol{w}^2) \gg_{\boldsymbol{P}} (\boldsymbol{p}^1, \boldsymbol{w}^1),$$

if, and only if:

$$w_i^2 \geq \boldsymbol{p}^2 \cdot h(\boldsymbol{p}^1, w_i^1; P_i) \quad \text{for } i = 1, \ldots, m,$$

and, for some $j \in \{1, \ldots, m\}$:

$$w_j^2 > \boldsymbol{p}^2 \cdot h(\boldsymbol{p}^1, w_j^1; P_j).$$

We shall refer to the relation $\gg_{\boldsymbol{P}}$ as the **revealed preference criterion for $\boldsymbol{P}$**.

If a pair of functions, (f, Θ) only allows comparisons to be made which agree with the revealed preference criterion, then obviously our marginal gain in using it rather than the revealed preference criterion itself is nil. This is the motivation behind the next definition.

6.8 Definitions. Let $\mathcal{P}$ be an admissible preference space, let $\succ \in \mathcal{S}(\mathcal{P})$, and let $\boldsymbol{D}^* \subseteq \boldsymbol{D}$. We shall say that **$(f, \Theta)$ has significant welfare content for $\succ$ on $\boldsymbol{D}^* \times \mathcal{P}$** iff, there exists an asymmetric order, Q, on $\mathbb{R}^s_+$ such that (f, Θ, Q) provides a welfare criterion for $\succ$ on $\boldsymbol{D}^* \times \mathcal{P}$; and, in addition, for each $\boldsymbol{P} \in \mathcal{P}$, there exists $\big((\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2)\big) \in \boldsymbol{D}^*$ such that:

$$\neg\big[(\boldsymbol{p}^2, \boldsymbol{w}^2) \gg_{\boldsymbol{P}} (\boldsymbol{p}^1, \boldsymbol{w}^1)\big] \ \& \ \left(\frac{f(\boldsymbol{p}^2, \boldsymbol{w}^2)}{\Theta\big[(\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2); \boldsymbol{P}\big]}\right) Q f(\boldsymbol{p}^1, \boldsymbol{w}^1). \tag{24}$$

Given a set $\mathcal{B}^d \subseteq \mathcal{S}(\mathcal{P})$, we then say that **$(f, \Theta)$ has significant welfare content for $\mathcal{B}^d$ on $\boldsymbol{D}^* \times \mathcal{P}$**, iff (f, Θ) has significant welfare content on $\boldsymbol{D}^* \times \mathcal{P}$, for each $\succ \in \mathcal{B}^d$.

We will also be interested in investigating the issue of whether a function $f \colon \boldsymbol{\Omega} \to \mathbb{R}^s_+$ has significant welfare content, where this is defined by obvious analogy with the definition just given. The definitions just given have some rather surprising implications, as is shown by the following example.

6.9 Example. Let $s = m = n = 2$, and let $\mathcal{P}$ consist of all those preference relations which can be represented by a (Leontief) utility function of the form:

$$u_i(\boldsymbol{x}_i) = \min\left\{\frac{x_{i1}}{a_i}, \frac{x_{i2}}{b_i}\right\},$$

where $a_i, b_i > 0$. Next, let $\succ$ be that element of $\mathcal{B}^*(\mathcal{P})$ determined by the pair (μ, F); where μ is defined as in Example 3.2, taking $\boldsymbol{x}^* = (1, 1)$,[19] and F is of the utilitarian form:

$$F(\boldsymbol{u}) = u_1 + u_2.$$

[19] Actually, if we take $\boldsymbol{x}^* = (1, 1)$, the representation u^* has either the form $u(\boldsymbol{x}_i) = \min\{x_{i1}, x_{i2}/b_i\}$ or $u(\boldsymbol{x}_i) = \min\{x_{i1}/a_i, x_{i2}\}$. However, the present representation is easier to use, and doesn't misrepresent any of the principles involved.

We then define $f \colon \boldsymbol{\Omega} \to \mathbb{R}^2_+$ by:

$$f(\boldsymbol{p}, \boldsymbol{w}) = \boldsymbol{w},$$

and $\boldsymbol{D}^*$ by:

$$\boldsymbol{D}^* = \{((\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2)) \in \boldsymbol{D} \mid \boldsymbol{p}^1 = \boldsymbol{p}^2\}.$$

While it may appear that we can now define Q on $\mathbb{R}^2_+$ by:

$$\boldsymbol{w}^2 Q \boldsymbol{w}^1 \iff w_1^2 + w_2^2 > w_1^1 + w_2^1,$$

to obtain a relation Q such that for all $((\boldsymbol{p}, \boldsymbol{w}^1), (\boldsymbol{p}, \boldsymbol{w}^2)) \in \boldsymbol{D}^*$, and all $\boldsymbol{P} \in \mathcal{P}$:

$$\boldsymbol{w}^2 Q \boldsymbol{w}^1 \Rightarrow (\boldsymbol{p}, \boldsymbol{w}^2) \succ^*_{\boldsymbol{P}} (\boldsymbol{p}, \boldsymbol{w}^1),$$

such is not the case. In fact, we will show that if $\boldsymbol{w}^1$ and $\boldsymbol{w}^2$ are elements of $\mathbb{R}^2_+$ such that $\boldsymbol{w}^2 \not\geq \boldsymbol{w}^1$, then there exist $\boldsymbol{P}^* \in \mathcal{P}$ and $\boldsymbol{p}^* \in \mathbb{R}^2_{++}$ such that:

$$(\boldsymbol{p}^*, \boldsymbol{w}^1) \succ^*_{\boldsymbol{P}^*} (\boldsymbol{p}^*, \boldsymbol{w}^2).$$

Accordingly, suppose $\boldsymbol{w}^2 \not\geq \boldsymbol{w}^1$. Then by symmetry, we may assume without loss of generality that $w_2^2 < w_2^1$. If also $w_1^1 \geq w_1^2$, then $\boldsymbol{w}^1 > \boldsymbol{w}^2$, and our conclusion is immediate. Suppose, therefore, that:

$$w_1^2 > w_1^1 \text{ and } w_2^2 < w_2^1.$$

We can then define $c > 0$ by:

$$c = \frac{w_1^2 - w_1^1}{w_2^1 - w_2^2},$$

and consider the preferences $\boldsymbol{P}^*$ defined by:

$$(a_1, b_1) = (1, 4c) \text{ and } (a_2, b_2) = (1/c, 1);$$

that is:

$$u_1(\boldsymbol{x}_1) = \min\left\{x_{11}, \frac{x_{12}}{4c}\right\} \text{ and } u_2(\boldsymbol{x}_2) = \min\{c \cdot x_{21}, x_{22}\}.$$

One can then easily show that income deflator functions for these two preference relations are given by:

$$\gamma_1(\boldsymbol{p}) = p_1 + 4c \cdot p_2 \text{ and } \gamma_2(\boldsymbol{p}) = p_1/c + p_2,$$

respectively. Consequently, if we let $\boldsymbol{p}^*$ be given by:

$$\boldsymbol{p}^* = (1, 1/2c),$$

we have:

$$\frac{w_2^1 - w_2^2}{\gamma_2(\boldsymbol{p}^*)} = \frac{w_2^1 - w_2^2}{1/c + 1/2c} = \frac{2c(w_2^1 - w_2^2)}{3} = \frac{2(w_1^2 - w_1^1)}{3} = \frac{2(w_1^2 - w_1^1)}{\gamma_1(\boldsymbol{p}^*)}.$$

Therefore,

$$(\boldsymbol{p}^*, \boldsymbol{w}^1) \succ^*_{\boldsymbol{P}^*} (\boldsymbol{p}^*, \boldsymbol{w}^2).$$

It is of interest to note that in this example we have, for every $\boldsymbol{P} \in \mathcal{P}$, and every $((\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2)) \in \boldsymbol{D}^*$:

$$(\boldsymbol{p}^2, \boldsymbol{w}^2) \gg_{\boldsymbol{P}} (\boldsymbol{p}^1, \boldsymbol{w}^1) \iff \boldsymbol{w}^2 > \boldsymbol{w}^1.$$

Thus we have just shown that the function, f, defined in this example does *not* have significant welfare content for $\succ$ on $\boldsymbol{D}^*$. □

7 'Welfare Content' for the Eisenberg Case.

In this section, unless otherwise stated, we will be dealing with the issue of determining whether a pair (f, Θ) *has welfare content for* $\mathcal{B}^e(\mathfrak{P}^h)$ *on* $\boldsymbol{D}$*, where '*$\mathcal{B}^e(\mathfrak{P}^h)$*' refers to those PBS social welfare functions of the Eisenberg form.* Thus we will deal with social preference relations $\succ$, of the PBS form determined by a pair (μ, F), where F takes the form:

$$F(\boldsymbol{u}) = \prod_{i=1}^{m} (u_i)^{a_i}, \tag{25}$$

with:

$$a_i > 0 \ \text{ for } i = 1, \ldots, m, \ \text{ and } \ \sum_{i=1}^{m} a_i = 1. \tag{26}$$

Moreover, given $P_i \in \mathcal{P}^h$, we will let γ_i^* be the income deflator function for P_i determined by μ.

Our principal result in this paper is the following.

7.1 Theorem. *Let* $C \colon \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \times \mathcal{P}^h \to \mathbb{R}_{++}$ *be a cost-of-living indicator for* $\mathcal{P}^h$*, and let* $\Theta \colon \boldsymbol{D} \times \mathfrak{P}^h \to \mathbb{R}_{++}$ *be any function satisfying [equation (21) of Section 6, and]:*

$$\Theta[(\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2); \boldsymbol{P}] \geq C(\boldsymbol{p}^1, \boldsymbol{p}^2; P_i) \ \ \textit{for } i = 1, \ldots, m, \tag{27}$$

for all $\big((\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2), \boldsymbol{P}\big) \in \boldsymbol{D} \times \mathfrak{P}^h$*. If we define the function* $f \colon \boldsymbol{\Omega} \to \mathbb{R}^m_+$ *by:*

$$f(\boldsymbol{p}, \boldsymbol{w}) = \boldsymbol{w}, \tag{28}$$

then (f, Θ) *has significant welfare content for* $\mathcal{B}^e(\mathfrak{P}^h)$ *on* $\boldsymbol{D} \times \mathfrak{P}^h$*.*

Proof. Let $\succ \in \mathcal{B}^e(\mathfrak{P}^h)$ be determined by a pair (μ, F), where the aggregator function, F satisfies (25) and (26), and define the relation Q_a on $\mathbb{R}^m_{++}$ by:

$$\boldsymbol{z}^2 Q_a \boldsymbol{z}^1 \iff F(\boldsymbol{z}^2) = \prod_{i=1}^{m} (z_i^2)^{a_i} > F(\boldsymbol{z}^1) = \prod_{i=1}^{m} (z_i^1)^{a_i}. \tag{29}$$

Suppose $\big((\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2)\big) \in \boldsymbol{D}$ and $\boldsymbol{P} \in \mathfrak{P}^h$ are such that:

$$\left(\frac{f(\boldsymbol{p}^2, \boldsymbol{w}^2)}{\Theta\big[(\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2); \boldsymbol{P}\big]}\right) Q_a f(\boldsymbol{p}^1, \boldsymbol{w}^1),$$

where f is defined in (28). Then, by definition of Q_a and f, we have:

$$\Big(1/\Theta[(\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2); \boldsymbol{P}]\Big) \prod_{i=1}^{m} (w_i^2)^{a_i} > \prod_{i=1}^{m} (w_i^1)^{a_i}. \tag{30}$$

Now let $i \in \{1, \ldots, m\}$ be arbitrary. Using (27) and Proposition 5.6 in turn, we have:

$$\frac{w_i^2}{\Theta[(\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2); \boldsymbol{P}]} \leq \frac{w_i^2}{C(\boldsymbol{p}^1, \boldsymbol{p}^2; P_i)} \leq \frac{w_i^2}{\gamma^*(\boldsymbol{p}^2; P_i)/\gamma^*(\boldsymbol{p}^1; P_i)}; \tag{31}$$

where γ^* is the income deflator function defined by μ. Therefore, since F is increasing:

$$\prod_{i=1}^{m} \left(\frac{w_i^2}{[\gamma^*(\boldsymbol{p}^2; P_i)/\gamma^*(\boldsymbol{p}^1; P_i)]} \right)^{a_i} \geq \left(1/\Theta[(\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2); \boldsymbol{P}]\right) \prod_{i=1}^{m} (w_i^2)^{a_i} \tag{32}$$

Combining (30) and (32), we have:

$$\prod_{i=1}^{m} \left(\frac{w_i^2}{[\gamma^*(\boldsymbol{p}^2; P_i)/\gamma^*(\boldsymbol{p}^1; P_i)]} \right)^{a_i} = \left[\frac{\prod_{i=1}^{m} \gamma^*(\boldsymbol{p}^1; P_i)^{a_i}}{\prod_{i=1}^{m} \gamma^*(\boldsymbol{p}^2; P_i)^{a_i}} \right] \prod_{i=1}^{m} (w_i^2)^{a_i} > \prod_{i=1}^{m} (w_i^1)^{a_i}.$$

Therefore:

$$F[\boldsymbol{v}(\boldsymbol{p}^2, \boldsymbol{w}^2; \boldsymbol{P})] = \prod_{i=1}^{m} \left(\frac{w_i^2}{\gamma^*(\boldsymbol{p}^2; P_i)} \right)^{a_i} > \prod_{i=1}^{m} \left(\frac{w_i^1}{\gamma^*(\boldsymbol{p}^1; P_i)} \right)^{a_i} = F[\boldsymbol{v}(\boldsymbol{p}^1, \boldsymbol{w}_i^1; \boldsymbol{P})];$$

and it follows that $(\boldsymbol{p}^2, \boldsymbol{w}^2) \succ_{\boldsymbol{P}}^* (\boldsymbol{p}^1, \boldsymbol{w}^1)$. We conclude, therefore, that (f, Θ) has welfare content for $\mathcal{B}^e(\mathfrak{P}^h)$ on $\boldsymbol{D} \times \mathfrak{P}^h$.

Next we note that if $\big((\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2)\big) \in \boldsymbol{D}$ satisfies $\boldsymbol{p}^1 = \boldsymbol{p}^2$, then we have:

$$(\boldsymbol{p}^2, \boldsymbol{w}^1) \gtrdot_{\boldsymbol{P}} (\boldsymbol{p}^1, \boldsymbol{w}^1) \iff \boldsymbol{w}^2 > \boldsymbol{w}^1,$$

for any $\boldsymbol{P} \in \mathfrak{P}^h$. Since for each aggregator function, F, of the Eisenberg form, there obviously exist $\boldsymbol{w}^1, \boldsymbol{w}^2 \in \mathbb{R}_{++}^m$ such that:

$$\boldsymbol{w}^2 \not> \boldsymbol{w}^1 \text{ and } F(\boldsymbol{w}^2) > F(\boldsymbol{w}^1),$$

and, since $\Theta(\cdot)$ satisfies (21), it follows that (f, Θ) has significant welfare content for $\mathcal{B}^e(\mathfrak{P}^h)$ on $\boldsymbol{D} \times \mathfrak{P}^h$. $\square$

We can interpret the above result as follows. Suppose $C(\cdot)$ is *any* cost-of-living indicator for $\mathfrak{P}$, that Θ is any function satisfying equations (21) and (27), and that 'the economist' provides any decision-maker whose social preference function is of the form being considered here with the initial vector of incomes, $\boldsymbol{w}^1$, and the new vector of incomes deflated by Θ, which vector we will denote by '$\boldsymbol{\omega}^2$.' Given the assumptions of this section, if our decision-maker finds that $F(\boldsymbol{\omega}^2) > F(\boldsymbol{w}^1)$, then our decision-maker can conclude that, given his or her social preferences, the state $(\boldsymbol{p}^2, \boldsymbol{w}^2)$ is better than $(\boldsymbol{p}^1, \boldsymbol{w}^1)$. In the figure on the next page we illustrate a case in which (f, Θ) has significant welfare content for a social preference relation representable by a PBS social welfare function of the Eisenberg form. Notice that, while neither $\boldsymbol{w}^2$ nor $(1/\Theta)\boldsymbol{w}^2$ is greater than $\boldsymbol{w}^1$ in the vector inequality sense, the decision-maker can determine that $(\boldsymbol{p}^2, \boldsymbol{w}^2) \succ_{\boldsymbol{P}}^* (\boldsymbol{p}^1, \boldsymbol{w}^1)$.

In trying to apply Theorem 7.1, the difficult estimation problem revolves around obtaining a function Θ, satisfying (27). One approach to obtaining such a function is the following. Let C be the income deflator cost-of-living indicator function for $\mathfrak{P}^h$ defined by any measurement function, μ. If we then define $\Theta \colon \mathbb{R}_{++}^n \times \mathbb{R}_{++}^n \times \mathfrak{P}^h \to \mathbb{R}_{++}$ by:

$$\Theta\big[(\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2); \boldsymbol{P}\big] = \max_i C(\boldsymbol{p}^1, \boldsymbol{p}^2; P_i), \tag{33}$$

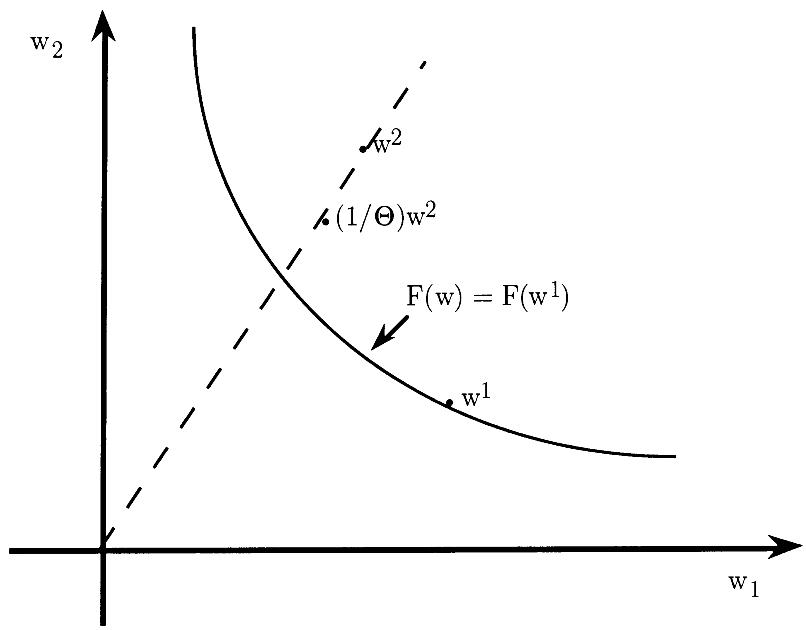

Figure 1: (f, Θ) Has Significant Welfare Content.

it is easy to see Proposition 5.7 that Θ satisfies (21) and (27), above. Moreover, it follows from the discussion following Proposition 5.6 that the function Θ defined in (33) is the *best* function satisfying (27), in the sense that it is the greatest lower bound among the functions satisfying (27). However, the function Θ defined in our next result is probably of greater interest. The result itself is a more or less immediate consequence of Theorem 7.1 and Proposition 5.7.

7.2 Corollary. *If we define* $\Theta\colon \boldsymbol{D} \times \mathcal{P}^h \to \mathbb{R}_{++}$ *by:*

$$\Theta[(\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2); \boldsymbol{P}] = \max_i L(\boldsymbol{p}^1, \boldsymbol{p}^2; P_i) = \max_i \boldsymbol{p}^2 \cdot g(\boldsymbol{p}^1; P_i), \tag{34}$$

and the function $f\colon \boldsymbol{\Omega} \to \mathbb{R}^m_+$ *as in Theorem 7.1, then* (f, Θ) *has significant welfare content for* $\mathcal{B}^e(\mathcal{P}^h)$ *on* $\boldsymbol{D} \times \mathcal{P}^h$.

The above corollary appears to be of particular interest in that in practice one would be fairly comfortable in taking a finite sample of the values of $L(\boldsymbol{p}^1, \boldsymbol{p}^2; P_i)$, and letting Θ be defined as the maximum value in the sample. Another corollary of Theorem 7.1 which seems to me to be of interest is the following, the proof of which is immediate.

7.3 Corollary. *Let* $\mathcal{P} = \mathcal{P}^h$*, and define* $\boldsymbol{D}^*$ *by:*

$$\boldsymbol{D}^* = \{((\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2)) \in \boldsymbol{D} \mid \boldsymbol{p}^1 = \boldsymbol{p}^2\}.$$

If we define $f\colon \boldsymbol{\Omega} \to \mathbb{R}^m_+$ *as in (28), above, then* f *has significant welfare content for* $\mathcal{B}^e(\mathcal{P}^h)$ *on* $\boldsymbol{D}^* \times \mathcal{P}^h$.

Thus, if prices remain constant, only the vector of individual incomes in the two situations needs to be known by the policy-makers, if their social orderings are of the Eisenberg form; and assuming that the individual preference relations are homothetic, or at least homothetic in the appropriate neighborhoods. Of course, one would still like to be able to contemplate the acquistion and transmission of less information than the full vector of individual incomes; and in some special cases (or under some further assumptions) this may be possible, as we will see shortly.

In the proposition below, we make use of the following definitions: for $(\boldsymbol{p}^t, \boldsymbol{w}^t) \in \boldsymbol{\Omega}$, define w^t and $\boldsymbol{\delta}^t$ by:

$$w^t = \sum_{i=1}^m w_i^t, \tag{35}$$

and $\boldsymbol{\delta}^t = (\delta_1^t, \ldots, \delta_m^t)$, respectively, where:

$$\delta_i^t = \frac{w_i^t}{w^t} \quad \text{for } i = 1, \ldots, m. \tag{36}$$

Once again we deal with the Eisenberg class of PBS welfare functions, defined over m-tuples of homothetic preference relations, $\mathcal{B}^e(\mathcal{P}^h)$.

7.4 Proposition. *Suppose $C(\cdot)$, and $\Theta(\cdot)$ are as in Theorem 7.1, define $\sigma \colon \boldsymbol{\Omega} \to \mathbb{R}_+$ by:*

$$\sigma(\boldsymbol{p}, \boldsymbol{w}) = w \stackrel{def}{=} \sum_{i=1}^m w_i,$$

and let $\boldsymbol{D}^* = \{((\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2)) \in \boldsymbol{D} \mid \boldsymbol{\delta}^1 = \boldsymbol{\delta}^2\}$. Then $(\sigma, \Theta, >)$ provides a welfare criterion for $\mathcal{B}^e(\mathcal{P}^h)$ on $\boldsymbol{D}^* \times \mathcal{P}^h$.

Proof. Suppose $\boldsymbol{P} \in \mathcal{P}^h$, and that $((\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2)) \in \boldsymbol{D}^*$ is such that:

$$\frac{\sigma(\boldsymbol{p}^2, \boldsymbol{w}^2)}{\Theta\big[\big((\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2)\big); \boldsymbol{P}\big]} \equiv \frac{w^2}{\Theta\big[\big((\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2)\big); \boldsymbol{P}\big]} > \sigma(\boldsymbol{p}^1, \boldsymbol{w}^1) \equiv w^1. \tag{37}$$

Now let $\succ \in \mathcal{B}^e(\mathcal{P}^h)$; so that $\succ$ is determined by a pair (μ, F), where F takes the form indicated in (25) and (26), and again let γ^* be the income deflator function determined by μ. It then follows as in the proof of Theorem 7.1 [see equation (31)], that:

$$\frac{w_i^2}{\Theta[(\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2); \boldsymbol{P}]} \le \frac{w_i^2}{C(\boldsymbol{p}^1, \boldsymbol{p}^2; P_i)} \le \frac{w_i^2}{\gamma^*(\boldsymbol{p}^2; P_i)/\gamma^*(\boldsymbol{p}^1; P_i)}.$$

Consequently, since $F(\cdot)$ is increasing, and using (37), we have:

$$\begin{aligned}\prod_{i=1}^m \left(\frac{w_i^2}{[\gamma^*(\boldsymbol{p}^2; P_i)/\gamma^*(\boldsymbol{p}^1; P_i)]}\right)^{a_i} &\ge \Big(1/\Theta[(\boldsymbol{p}^1, \boldsymbol{w}^1), (\boldsymbol{p}^2, \boldsymbol{w}^2); \boldsymbol{P}]\Big) \prod_{i=1}^m (w_i^2)^{a_i} \\ &> \left(\frac{w^1}{w^2}\right) \prod_{i=1}^m (w_i^2)^{a_i} = w^1 \prod \left(\frac{w_i^2}{w^2}\right)^{a_i} = w^1 \prod_{i=1}^m (\delta_i^2)^{a_i}.\end{aligned}$$

However, since $\boldsymbol{\delta}^2 = \boldsymbol{\delta}^1$, it then follows that:

$$\prod_{i=1}^m \left(\frac{w_i^2}{[\gamma^*(\boldsymbol{p}^2; P_i)/\gamma^*(\boldsymbol{p}^1; P_i)]}\right)^{a_i} > w^1 \prod_{i=1}^m (\delta_i^1)^{a_i} = \prod_{i=1}^m (w_i^1)^{a_i}.$$

Therefore:

$$F\Big(\frac{w_1^2}{\gamma^*(\boldsymbol{p}^2;P_1)},\ldots,\frac{w_m^2}{\gamma^*(\boldsymbol{p}^2;P_m)}\Big)\equiv\prod\nolimits_{i=1}^m\Big(\frac{w_i^2}{\gamma^*(\boldsymbol{p}^2;P_i)}\Big)^{a_i}$$
$$>\prod\nolimits_{i=1}^m\Big(\frac{w_i^1}{\gamma^*(\boldsymbol{p}^1;P_i)}\Big)^{a_i}\equiv F\Big(\frac{w_1^1}{\gamma^*(\boldsymbol{p}^1;P_1)},\ldots,\frac{w_m^1}{\gamma^*(\boldsymbol{p}^1;P_m)}\Big).\quad\square$$

It is interesting to combine the domain $\boldsymbol{D}^*$ used in this last proposition with the assumptions regarding $\mathfrak{P}$ and $\mathcal{B}^d$ which were used in Example 6.6. While the resulting example is very special, I believe that it provides some interesting fresh perspectives on the the evaluation of 'real national income.'

7.5 Example. In this example, we will consider social preference relations of the PBS form determined by a pair (μ, F), where F is positively homogeneous of degree one. We will also let $\mathfrak{P}$ be defined by:

$$\mathfrak{P}=\{\boldsymbol{P}\in\mathfrak{P}^h\mid P_1=P_2=\cdots=P_m\}.$$

Given a preference profile, $\boldsymbol{P}\in\mathfrak{P}$, we know that the income deflator function, $\gamma(\cdot)$ determined by μ for the (common) preference relation, P, characterizing the profile can be used to obtain an indirect utility function, v, representing P^* as:

$$v(\boldsymbol{p},w_i)=w_i/\gamma(\boldsymbol{p}).$$

Thus, the indirect social welfare function in this case takes the form:

$$F\Big[\frac{w_1}{\gamma(\boldsymbol{p})},\ldots,\frac{w_m}{\gamma(\boldsymbol{p})}\Big]=\Big(\frac{1}{\gamma(\boldsymbol{p})}\Big)F(w_1,\ldots,w_m).$$

Consequently, we see that, given any $\big((\boldsymbol{p}^1,\boldsymbol{w}^1),(\boldsymbol{p}^2,\boldsymbol{w}^2)\big)\in\boldsymbol{D}$:

$$(\boldsymbol{p}^2,\boldsymbol{w}^2)\succ^*_{\boldsymbol{P}}(\boldsymbol{p}^1,\boldsymbol{w}^1)\iff[1/\gamma(\boldsymbol{p}^2)]F(w_1^2,\ldots,w_m^2)>[1/\gamma(\boldsymbol{p}^1)]F(w_1^1,\ldots,w_m^1);\tag{38}$$

and it follows that in this special case, a measure of 'real national income' for the decision-maker is $F(\boldsymbol{w})$ deflated by γ. In particular, then, in the utilitarian case, a valid measure of 'real national income' at $(\boldsymbol{p},\boldsymbol{w})\in\Omega$ is given by:

$$y^u=[1/\gamma(\boldsymbol{p})]\sum\nolimits_{i=1}^m w_i.$$

Similarly, in the Rawlsian case, one can use:

$$y^R=[1/\gamma(\boldsymbol{p})]\min_i w_i,$$

while in the Eisenberg casewe can take:

$$y^e=[1/\gamma(\boldsymbol{p})]\prod\nolimits_{i=1}^m(w_i)^{a_i}.$$

In fact, we can generalize these observations as follows: if the aggregator function, F, has the form:

$$F(\boldsymbol{u}) = \Big[\sum\nolimits_{i=1}^{m} a_i u_i^{-\rho}\Big]^{-1/\rho}, \tag{39}$$

with:

$$-1 \leq \rho \leq +\infty \;\text{ and }\; a_i > 0 \text{ for } i = 1, \ldots, m, \tag{40}$$

a measure of 'real national income' is given by:

$$y = [1/\gamma(\boldsymbol{p})]\Big[\sum\nolimits_{i=1}^{m} a_i w_i^{-\rho}\Big]^{-1/\rho}. \tag{41}$$

The real point, however, is that even given the assumptions of this example the appropriate ideal measure of 'real national income' depends upon the form of the aggregator function specific to the decision-maker.

It is interesting also to combine the assumptions of this example with the approach of Proposition 7.4. Given an aggregator function of the form being considered here, and given $\boldsymbol{P} \in \mathfrak{P}$, the indirect social welfare function can be written as:

$$F\big[\boldsymbol{v}(\boldsymbol{p}, \boldsymbol{w})\big] = F\Big(\frac{w_1}{\gamma(\boldsymbol{p})}, \ldots, \frac{w_m}{\gamma(\boldsymbol{p})}\Big) = \Big(\frac{w}{\gamma(\boldsymbol{p})}\Big) F(\delta_1, \ldots, \delta_m), \tag{42}$$

where w and $\boldsymbol{\delta}$ are defined in equations (35) and (36), above. When written in this form, and for decision-makers whose social welfare function satisfies the conditions being considered here, we can see that the indirect social welfare function can be expressed as the product of a proxy for 'real national income,' $w/\gamma(\boldsymbol{p})$, and a function which depends solely upon the distribution of income. This approach becomes of greater interest in the Eisenberg case, however, as we will see shortly. In the meantime, it is worthwhile in closing this example to note that the expression in equation (42) suggests a more pivotal role for the summation of the w_i's (and, correspondingly, for the utilitarian measure of 'real national income') than is deserved. The vector $\boldsymbol{\delta}$ defined in (36) is essentially the vector of ratios of individual incomes to average income in the economy (as measured by the arithmetic mean). We could equally well consider the ratio of individual i's income to geometric mean income, defining:

$$\eta_i = \frac{w_i}{\prod_{k=1}^{m}(w_k)^{1/m}}, \tag{43}$$

for example, and letting '$\boldsymbol{\eta}$' denote the vector whose i^{th} coordinate is η_i, we can write:

$$F\big[\boldsymbol{v}(\boldsymbol{p}, \boldsymbol{w})\big] = F\Big(\frac{w_1}{\gamma(\boldsymbol{p})}, \ldots, \frac{w_m}{\gamma(\boldsymbol{p})}\Big) = \Big(\frac{\prod_{i=1}^{m}(w_i)^{1/m}}{\gamma(\boldsymbol{p})}\Big) F(\boldsymbol{\eta}) \quad \square \tag{44}$$

8 Income Distribution and Welfare Judgments.

It should be noted that in the results of the previous section, neither the economist predicting the value of the function pairs (f, Θ) utilized in those results, nor the decision-maker evaluating the information provided by these function pairs needs to know the individual utility functions; nor does the economist need to know specifically what the decision-makers' social preference relations are in computing the value of the two functions. Moreover, even in the context of Theorem 7.1, the pair (f, Θ) defined in (28) and:

$$\Theta(\boldsymbol{p}^1, \boldsymbol{p}^2; \boldsymbol{P}) = \max_i \boldsymbol{p}^2 \cdot g(\boldsymbol{p}^1; P_i), \tag{45}$$

has significant welfare content for all $\succ \in \mathcal{B}^e(\mathfrak{P}^h)$; and requires the transmission of only $2m$ variables, rather than the original $2(m+1)n$ variables. Of course, in practice, even the prediction and transmission of the $2m$ variables involved in this procedure is a very ambitious task; however, we probably can predict the income shares of broadly-defined groups, as opposed to individual income shares. If it is reasonable to suppose that our decision-makers have social preference relations which satisfy appropriate separability assumptions, then this may suffice.

Moreover, for decision-makers whose social preference relation is of the Eisenberg PBS form, we can construct a particularly interesting scenario. Suppose that individual preferences can be assumed to be homothetic throughout a neighborhood of the status quo. Then we can write the indirect social welfare function as:

$$F\big[\boldsymbol{v}(\boldsymbol{p}, \boldsymbol{w})\big] = F\Big[\frac{w_1}{\gamma(\boldsymbol{p}; P_1)}, \ldots, \frac{w_m}{\gamma(\boldsymbol{p}; P_m)}\Big] = \Big(\frac{w}{\prod_{k=1}^m \gamma(\boldsymbol{p}; P_k)^{a_k}}\Big) \prod_{i=1}^m (\delta_i)^{a_i}, \tag{46}$$

so that the indirect social welfare function can be expressed as the product of a proxy measure of 'real national income,' $w / \prod_{i=1}^m \gamma(\boldsymbol{p}; P_i)^{a_i}$, with a measure of the desirability of the distribution of income, $\prod_{i=1}^m (\delta_i)^{a_i}$. Thus if we as economists predict that a given policy change will result in a movement from the status quo, , $(\boldsymbol{p}^1, \boldsymbol{w}^1) \in \boldsymbol{\Omega}$, to a second point $(\boldsymbol{p}^2, \boldsymbol{w}^2) \in \boldsymbol{\Omega}$, and we find that:

$$\frac{w^2}{\Theta[\boldsymbol{p}^1, \boldsymbol{p}^2; \boldsymbol{P}]} > w^1, \tag{47}$$

where $\Theta(\cdot)$ satisfies the conditions in Theorem 7.1, we can assure any such policy-maker that, if they believe that the policy change will not adversely affect the distribution of income, then the change is desirable from their point of view.

To this we can add a further observation of interest. Suppose that 'the economist' transmits the following information to the decision maker:

$$C(\boldsymbol{p}^1, \boldsymbol{p}^2; P_i) \text{ for } i = 1, \ldots, m, \text{ and } \boldsymbol{w}^t \text{ for } t = 1, 2.$$

By Proposition 5.6 we have:

$$\frac{1}{C(\boldsymbol{p}^1, \boldsymbol{p}^2; P_i)} \leq \frac{\gamma(\boldsymbol{p}^1; P_i)}{\gamma(\boldsymbol{p}^2; P_i)},$$

for each i; and thus it follows that:

$$\frac{1}{\prod_{i=1}^m C(\boldsymbol{p}^1, \boldsymbol{p}^2; P_i)^{a_i}} \leq \prod_{i=1}^m \Big[\frac{\gamma(\boldsymbol{p}^1; P_i)}{\gamma(\boldsymbol{p}^2; P_i)}\Big]^{a_i}.$$

Consequently, if:

$$\left[\frac{w^2}{\prod_{i=1}^{m} C(\boldsymbol{p}^1, \boldsymbol{p}^2; P_i)^{a_i}}\right] \prod_{i=1}^{m} (\delta_i^2)^{a_i} > w^1 \cdot \prod_{i=1}^{m} (\delta_i^1)^{a_i}, \tag{48}$$

then it is easily seen that $(\boldsymbol{p}^2, \boldsymbol{w}^2) \succ_{\boldsymbol{P}}^{*} (\boldsymbol{p}^1, \boldsymbol{w}^1)$.

Of course, it is rather unrealistic to imagine communicating this amount of information to policy-makers, and then expecting them to do the calculations necessary to utilize the information. Thus in our results of the previous section, we have made use of a function Θ satisfying equations (21) and (27), rather than the vector $\big(C(\boldsymbol{p}^1, \boldsymbol{p}^2; P_i), \ldots, C(\boldsymbol{p}^1, \boldsymbol{p}^2; P_m)\big)$. However, even if we take:

$$\Theta(\boldsymbol{p}^1, \boldsymbol{p}^2; \boldsymbol{P}) = \max_i C(\boldsymbol{p}^1, \boldsymbol{p}^2; P_i),$$

the comparison of $F\big[\boldsymbol{w}^2/\Theta(\boldsymbol{p}^1, \boldsymbol{p}^2; \boldsymbol{P})\big]$ and $F(\boldsymbol{w}^1)$ will typically result in a less powerful test than the comparison in equation (48); since, except in the case of identical preferences, it will generally be the case that:

$$\frac{1}{\prod_{i=1}^{m} C(\boldsymbol{p}^1, \boldsymbol{p}^2; P_i)^{a_i}} > \frac{1}{\max_i C(\boldsymbol{p}^1, \boldsymbol{p}^2; P_i)}.$$

This discussion suggests another point, however. If one has a social preference relation of the Eisenberg PBS form, it is difficult to see why one would use individual weights other than:

$$a_i = 1/m \quad \text{for } i = 1, \ldots, m; \tag{49}$$

at least not if the subscripts are assigned anonymously. Given this and the other assumptions of the above paragraph, if we were to find that (47) holds, and in addition:

$$\prod_{i=1}^{m} w_i^2 \geq \prod_{i=1}^{m} w_i^1, \tag{50}$$

then one could assure the policy-maker that the policy change is a desirable one without the necessity of the policy-maker's undertaking any computations for him or herself! We can also construct a better cost-of-living deflator than is envisioned in the requirements placed upon the function Θ in 7.1; a practical and more sensitive function is given by:

$$L(\boldsymbol{p}^1, \boldsymbol{p}^2; \boldsymbol{P}) \stackrel{\text{def}}{=} \prod_{i=1}^{m} L(\boldsymbol{p}^1, \boldsymbol{p}^2; P_i)^{1/m}, \tag{51}$$

where, as before:[20]

$$L(\boldsymbol{p}^1, \boldsymbol{p}^2; P_i) = \boldsymbol{p}^2 \cdot g(\boldsymbol{p}^1; P_i) \quad \text{for } i = 1, \ldots, m.$$

The analytic framework developed here can be extended to the consideration of several recent streams of literature. First, and most obviously, the results and examples in this

[20]Once again, of course, one would in practice substitute a finite sample of values of $L(\boldsymbol{p}^1, \boldsymbol{p}^2; P_i)$.

paper have potential applications to the literature on inequality (of income) measures, and the measurement of poverty.[21]

Another area of potential application is to the theory of index numbers. Here a few comments are probably in order. In the context of Theorem 7.1, suppose $(\boldsymbol{p}^0, \boldsymbol{w}^0)$ is the status quo equilibrium, that $C^*(\cdot)$ is the cost-of-living indicator determined by μ, and define:

$$\widehat{\Theta}(\boldsymbol{p}^t, \boldsymbol{P}) = \max_i C^*(\boldsymbol{p}^0, \boldsymbol{p}^t; P_i),$$

and $f \colon \boldsymbol{\Omega} \times \mathcal{P} \to \mathbb{R}^m$ by:

$$f(\boldsymbol{p}^t, \boldsymbol{w}^t; \boldsymbol{P}) = \left[1/\widehat{\Theta}(\boldsymbol{p}^t, \boldsymbol{P})\right] \boldsymbol{w}^t.$$

Then it follows readily from the results of Section 7 that, if $\succ \in \mathcal{B}^e(\mathcal{P}^h)$ is determined by (μ, F), and we define $Q_\succ$ on $\mathbb{R}^m_+$ by:

$$\boldsymbol{\omega}^2 Q_\succ \boldsymbol{\omega}^1 \iff F(\boldsymbol{\omega}^2) > F(\boldsymbol{\omega}^1);$$

then, for all $(\boldsymbol{p}, \boldsymbol{w}; \boldsymbol{P}) \in \boldsymbol{\Omega} \times \mathcal{P}^h$:

$$f(\boldsymbol{p}, \boldsymbol{w}; \boldsymbol{P}) Q_\succ f(\boldsymbol{p}^0, \boldsymbol{w}^0; \boldsymbol{P}) \Rightarrow (\boldsymbol{p}, \boldsymbol{w}) \succ^*_{\boldsymbol{P}} (\boldsymbol{p}^0, \boldsymbol{w}^0).$$

However, if Theorem 7.1 is used as our justification, we can only compare the image of price-wealth pairs to a fixed base-level equilibrium. In particular, if $(\boldsymbol{p}^t, \boldsymbol{w}^t; \boldsymbol{P})$ (for $t = 1, 2$) are such that:

$$f(\boldsymbol{p}^2, \boldsymbol{w}^2; \boldsymbol{P}) Q_\succ f(\boldsymbol{p}^1, \boldsymbol{w}^1; \boldsymbol{P}),$$

but $(\boldsymbol{p}^t, \boldsymbol{w}^t) \neq (\boldsymbol{p}^0, \boldsymbol{w}^0)$ for $t = 1, 2$, then we do not necessarily have:

$$(\boldsymbol{p}^2, \boldsymbol{w}^2) \succ^*_{\boldsymbol{P}} (\boldsymbol{p}^1, \boldsymbol{w}^1).$$

This is related to the index number problem of a change in base, and is also related to the question of whether $\widehat{\Theta}(\boldsymbol{p}^t, \boldsymbol{P})$ is a 'superlative index number' ([9]);[22] but, again, a more complete discussion of these relationships will have to await a later work.

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[21] See, for example, [1], [2], [8], [11], and [24].

[22] See also Diewert [9, 10], and Pollak [15, 16, 17].

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