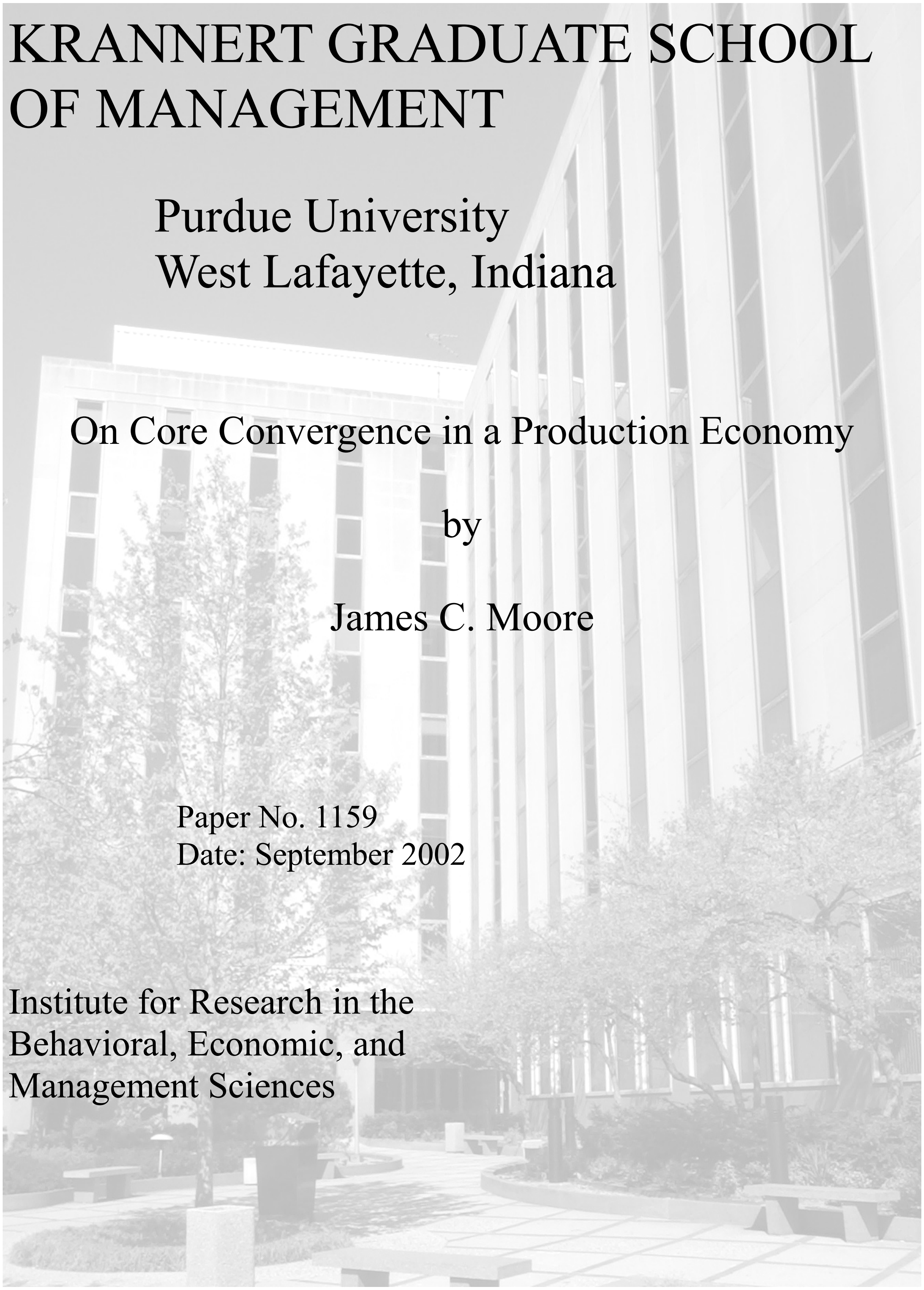

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On Core Convergence in a Production Economy

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1 Introduction

There are a number of articles which have been written to generalize Debreu and Scarf's original article [1963] on core convergence by allowing for production. To the best of my knowledge, however, all such articles, in showing that the replicated core was a Walrasian allocation, have assumed that each individual in the economy could, on the basis of his or her own production capabilities, produce a positive (or greater than minimal) amount of each commodity in the commodity space. The primary purpose of the present article is to weaken this assumption. I have also been able to provide a simpler proof of the main convergence theorem than that developed by Nikaido [1968] in his proof of a similar (but less general) result. Finally, I have dispensed with the assumption that (strict) preferences are negatively transitive; thus allowing for the intransitivity of indifference.

The basic notational framework for the analysis is presented in Section 2. Section 3 gathers together some fairly familiar and standard definitions regarding the core of a production economy, and our new results are presented in Section 4.

2 Notational Framework.

We will be dealing with a private ownership economy, $\mathcal{E} = (\langle X_i, P_i\rangle, \langle Y_k\rangle, \langle \boldsymbol{r}_i\rangle, [s_{ik}])$, where X_i $(i = 1, \ldots, m)$ and Y_k $(k = 1, \ldots, \ell)$ are non-empty subsets of $\mathbb{R}^n$; P_i is an irreflexive binary relation on X_i and $\boldsymbol{r}_i \in \mathbb{R}^n$, for each $i \in M$; while $s_{ik} \in \mathbb{R}_+$ for $i = 1, \ldots, m, k = 1, \ldots, \ell$ and for each $k \in L$:

$$\sum\nolimits_{i \in M} s_{ik} = 1,$$

where we define $M = \{1, \ldots, m\}$ and $L = \{1, \ldots, \ell\}$. In such an economy, we will often be concerned with the allocation of consumption bundles to the consumers; denoting such an allocation by, for example, '$\langle \boldsymbol{x}_i\rangle_{i \in M}$.'

2.1. Definitions. If $\mathcal{E} = (\langle X_i, P_i\rangle, \langle Y_k\rangle, \langle \boldsymbol{r}_i\rangle, [s_{ik}])$ is a private ownership economy, we will say that $\langle \boldsymbol{x}_i\rangle_{i \in M} \in \mathbb{R}^{mn}$ is an **attainable consumption allocation for $\mathcal{E}$** iff:

$$\boldsymbol{x}_i \in X_i \quad \text{for } i = 1, \ldots, m;$$

and there exist $\boldsymbol{y}_k \in Y_k$ $(k = 1, \ldots, \ell)$ such that:

$$\sum\nolimits_{i \in M} \boldsymbol{x}_i = \sum\nolimits_{i \in M} \boldsymbol{r}_i + \sum\nolimits_{k \in L} \boldsymbol{y}_k. \tag{1}$$

In other words, $\langle \boldsymbol{x}_i\rangle_{i \in M}$ is an attainable consumption allocation for $\mathcal{E}$ iff there exists $\langle \boldsymbol{y}_k\rangle_{k \in L}$ such that $\big(\langle \boldsymbol{x}_i\rangle_{i \in M}, \langle \boldsymbol{y}_k\rangle_{k \in L}\big) \in A(\mathcal{E})$; where we define:

$$A(\mathcal{E}) = \Big\{\big(\langle \boldsymbol{x}_i\rangle_{i \in M}, \langle \boldsymbol{y}_k\rangle_{k \in L}\big) \in \mathbb{R}^{mn \times \ell n} \mid \sum\nolimits_{i \in M} \boldsymbol{x}_i = \sum\nolimits_{i \in M} \boldsymbol{r}_i + \sum\nolimits_{k \in L} \boldsymbol{y}_k\Big\}. \tag{2}$$

We will denote the set of all attainable consumption allocations for $\mathcal{E}$ by '$\boldsymbol{X}^*(\mathcal{E})$,' or simply by '$\boldsymbol{X}^*$.' We denote the aggregate consumption set for $\mathcal{E}$ by '$X^*(\mathcal{E})$' [or simply 'X^*'], thus:

$$X^*(\mathcal{E}) = \big\{\boldsymbol{x} \in \mathbb{R}^n \mid \big(\exists \langle \boldsymbol{x}_i\rangle_{i \in M} \in \boldsymbol{X}^*(\mathcal{E})\big) \colon \boldsymbol{x} = \sum\nolimits_{i=1}^m \boldsymbol{x}_i\big\}$$

Further bits of notation are the following. We define the sets Π_k and Π by:

$$\Pi_k = \{\boldsymbol{p} \in \mathbb{R}^n \mid (\exists \boldsymbol{y}^* \in Y_k)(\forall \boldsymbol{y} \in Y_k) \colon \boldsymbol{p} \cdot \boldsymbol{y}^* \geq \boldsymbol{p} \cdot \boldsymbol{y}\} \quad \text{for } k = 1, \ldots, \ell;$$

and:

$$\Pi = \bigcap\nolimits_{k \in L} \Pi_k,$$

respectively, and the functions $\pi_k \colon \Pi_k \to \mathbb{R}$ by:

$$\pi_k(\boldsymbol{p}) = \max_{\boldsymbol{y} \in Y_k} \boldsymbol{p} \cdot \boldsymbol{y} \quad \text{for } k \in L.$$

We will also need a couple of fairly standard definitions, and a 'well-known' proposition, as follows.

2.2. Definitions. If P_i is an irreflexive binary relation on the non-empty set $X_i \subseteq \mathbb{R}^n$, we shall say that P_i is:

1. **weakly convex** iff X_i is a convex set, and, for each $\boldsymbol{x}_i^* \in X_i$, the set:

$$P_i \boldsymbol{x}_i^* = \{\boldsymbol{x}_i \in X_i \mid \boldsymbol{x}_i P_i \boldsymbol{x}_i^*\},$$

is a convex set.

2. **lower semi-continuous** iff, for each $\boldsymbol{x}_i^* \in X_i$ and each $\boldsymbol{x}_i' \in P_i \boldsymbol{x}_i^*$, there exists a neighborhood of $\boldsymbol{x}_i'$, $N(\boldsymbol{x}_i')$, such that:

$$\big(\forall \boldsymbol{x}_i \in N(\boldsymbol{x}_i')\big) \colon \boldsymbol{x}_i P_i \boldsymbol{x}_i^*.$$

2.3. Proposition. *If P_i is a lower semi-continuous binary relation on a convex set, $X_i \subseteq \mathbb{R}^n$, and $\boldsymbol{x}_i^* \in X_i$ and $\boldsymbol{p}^* \in \mathbb{R}^n$ satisfy:*

$$(\forall \boldsymbol{x}_i \in X_i) \colon \boldsymbol{x}_i P_i \boldsymbol{x}_i^* \Rightarrow \boldsymbol{p}^* \cdot \boldsymbol{x}_i \geq \boldsymbol{p}^* \cdot \boldsymbol{x}_i^*,$$

and:

$$\boldsymbol{p}^* \cdot \boldsymbol{x}_i^* > \min \boldsymbol{p}^* \cdot \boldsymbol{X}_i \overset{def}{=} \min_{\boldsymbol{x}_i \in X_i} \boldsymbol{p}^* \cdot \boldsymbol{x}_i,$$

then:

$$(\forall \boldsymbol{x}_i \in X_i) \colon \boldsymbol{x}_i P_i \boldsymbol{x}_i^* \Rightarrow \boldsymbol{p}^* \cdot \boldsymbol{x}_i > \boldsymbol{p}^* \cdot \boldsymbol{x}_i^*.$$

We will be considering possible actions of coalitions of consumers, where a coalition of consumers can be identified with a subset, S, of M; the collection of all such coalitions, that is, the collection of all non-empty subsets of M, will be denoted by '$\mathcal{S}$.'

In order to define the production possibilities available to a coalition, $S \subseteq M$, we begin by defining the sets Z_{ik}, for $(i, k) \in M \times L$, by:

$$Z_{ik} = s_{ik} Y_k \overset{\text{def}}{=} \{\boldsymbol{z} \in \mathbb{R}^n \mid (\exists \boldsymbol{y}_k \in Y_k) \colon \boldsymbol{z} = s_{ik} \boldsymbol{y}_k\}. \tag{3}$$

We then define:

$$\Pi_{ik} = \{\boldsymbol{p} \in \mathbb{R}^n \mid (\exists \boldsymbol{z}^* \in Z_{ik})(\forall \boldsymbol{z} \in Z_{ik})\colon \boldsymbol{p}\cdot\boldsymbol{z}^* \geq \boldsymbol{p}\cdot\boldsymbol{z}\},$$

and $\widehat{\pi}_{ik}\colon \Pi_{ik} \to \mathbb{R}$ by:

$$\widehat{\pi}_{ik}(\boldsymbol{p}) = \max_{\boldsymbol{z}\in Z_{ik}} \boldsymbol{p}\cdot\boldsymbol{z}.$$

Finally, we define the i^{th} consumer's production set, Z_i, as:

$$Z_i = \sum_{k=1}^{\ell} Z_{ik} = \sum_{k=1}^{\ell} s_{ik} Y_k. \tag{4}$$

With these definitions, it is easy to prove the following.

2.4. Proposition. *Let $\mathcal{E} = (\langle X_i, P_i\rangle, \langle Y_k\rangle, \langle \boldsymbol{r}_i\rangle, [s_{ik}])$ be an economy, and $\boldsymbol{p}^* \in \mathbb{R}^n$. Then:*

1. *if $\boldsymbol{y}_k^*$ maximizes $\boldsymbol{p}^*\cdot\boldsymbol{y}$ on Y_k, then $\boldsymbol{z}_{ik}^* \stackrel{def}{=} s_{ik}\boldsymbol{y}_k^*$ maximizes $\boldsymbol{p}^*\cdot\boldsymbol{z}$ on Z_{ik}; and:*
2. *if $s_{ik} > 0$, and $\boldsymbol{z}_{ik} \in Z_{ik}$ maximizes $\boldsymbol{p}^*\cdot\boldsymbol{z}$ on Z_{ik}, then $\boldsymbol{y}_k \stackrel{def}{=} (1/s_{ik})\boldsymbol{z}_{ik}$ maximizes profits on Y_k.*
3. *Thus, for all $(i,k) \in M \times L$:, $\Pi_{ik} = \Pi_k$ and for any $\boldsymbol{p} \in \Pi_k$:*

$$\widehat{\pi}_{ik}(\boldsymbol{p}) = s_{ik}\pi_k(\boldsymbol{p}), \tag{5}$$

4. *and, for all $i \in M$, and for any $\boldsymbol{z}_i \in Z_i$ and any $\boldsymbol{p} \in \Pi$:*

$$\boldsymbol{p}\cdot\boldsymbol{z}_i \leq \sum_{k\in L} s_{ik}\pi_k(\boldsymbol{p}).$$

2.5. Proposition. *If $\mathcal{E} = (\langle X_i, P_i\rangle, \langle Y_k\rangle, \langle \boldsymbol{r}_i\rangle, [s_{ik}])$ is a private ownership economy, then, given the definitions of this section:*

$$Y \equiv \sum_{k\in L} Y_k \subseteq \sum_{i\in M} Z_i. \tag{6}$$

Moreover, if Y_k is convex and contains $\mathbf{0}$, for each $k \in L$, then for each $S \in \mathcal{S}$ we have:

$$\sum_{i\in S} Z_i \subseteq Y. \tag{7}$$

Proof. I will leave the proof of the first part of this proposition as an 'exercise for the interested reader.' As to the 'moreover part' of the conclusion, suppose that $\boldsymbol{z}_i \in Z_i$, for each $i \in S$. Then, by the definitions of Z_i and Z_{ik}, for each $i \in S$ there exist $\boldsymbol{y}_k^i$, for $k = 1, \ldots, \ell$, such that:

$$\boldsymbol{z}_i = \sum_{k\in L} s_{ik}\boldsymbol{y}_k^i. \tag{8}$$

However, since each Y_k is convex and contains $\mathbf{0}$, and $\sum_{i\in M} s_{ik} = 1$:

$$\boldsymbol{y}_k \stackrel{\text{def}}{=} \sum_{i\in S} s_{ik}\boldsymbol{y}_k^i + \Big(\sum_{i\in M\setminus S} s_{ik}\Big)\mathbf{0} = \sum_{i\in S} s_{ik}\boldsymbol{y}_k^i,$$

is an element of Y_k, for each $k \in L$. Moreover:

$$\sum_{i\in S}\boldsymbol{z}_i = \sum_{i\in S}\sum_{k\in L} s_{ik}\boldsymbol{y}_k^i = \sum_{k\in L}\sum_{i\in S} s_{ik}\boldsymbol{y}_k^i = \sum_{k\in L}\boldsymbol{y}_k.$$

□

3 The Core of a Production Economy.

In this section we will state a number of fairly standard and familiar definitions. *We will assume throughout the remainder of this article that the following condition holds:*

$$X_i \cap [\boldsymbol{r}_i + Z_i] \neq \emptyset \quad \text{for } i = 1, \ldots, m; \tag{9}$$

in other words, for each $i \in M$, we suppose that there exist $\bar{\boldsymbol{x}}_i \in X_i$ and $\bar{\boldsymbol{z}}_i \in Z_i$ such that:

$$\bar{\boldsymbol{x}}_i = \boldsymbol{r}_i + \bar{\boldsymbol{z}}_i. \tag{10}$$

The assumption expressed as equation (9) is fairly restrictive; in a modern industrialized society, individuals specialize in the expectation of being able to purchase (or trade for) necessities which they themselves do not produce. On the other hand, in much of the literature, it is supposed that $X_i = \mathbb{R}^n_+$ for each i, and that $\boldsymbol{r}_i \in \mathbb{R}^n_+$ as well; consequently, the condition in (9) generalizes the assumption commonly-used. Moreover, it greatly simplifies our analysis, by making the following definition apply to any $S \in \mathcal{S}$.[1]

3.1. Definition. Let S be a non-empty subset of M (so that $S \in \mathcal{S}$). We will say that $\langle(\boldsymbol{x}_i, \boldsymbol{z}_i)\rangle_{i \in S}$ is **attainable for S**, or **feasible for S**, iff:

$$\boldsymbol{x}_i \in X_i \text{ and } \boldsymbol{z}_i \in Z_i \text{ for all } i \in S, \tag{11}$$

and:

$$\sum_{i \in S} \boldsymbol{x}_i = \sum_{i \in S} \boldsymbol{r}_i + \sum_{i \in S} \boldsymbol{z}_i. \tag{12}$$

3.2. Definition. Let $\langle \boldsymbol{x}_i^* \rangle_{i \in M}$ be a consumption allocation for $\mathcal{E}$, and let $S \in \mathcal{S}$ be a coalition. We shall say that $\langle \boldsymbol{x}_i^* \rangle_{i \in M}$ **can be improved upon by the coalition S** (or is **blocked by S**) iff there exists an allocation, $\langle(\boldsymbol{x}_i, \boldsymbol{z}_i)\rangle_{i \in S}$, which is feasible for S, and satisfies:

$$(\forall i \in S) \colon \boldsymbol{x}_i P_i \boldsymbol{x}_i^*. \tag{13}$$

3.3. Definition. The **core of an economy** $\mathcal{E} = (\langle X_i, P_i \rangle, \langle Y_k \rangle, \langle \boldsymbol{r}_i \rangle, [s_{ik}])$, is defined as the set of all attainable consumption allocations for $\mathcal{E}$ which cannot by improved upon (or blocked) by any coalition, $S \in \mathcal{S}$. We shall denote the set of all core allocations for $\mathcal{E}$ by '$\boldsymbol{C}(\mathcal{E})$.'

3.4. Definition. An $(m+\ell+1)n$-tuple, $(\langle \boldsymbol{x}_i^* \rangle, \langle \boldsymbol{y}_k^* \rangle, \boldsymbol{p}^*)$ is a **Walrasian** (or **competitive**) **equilibrium for the private ownership economy $\mathcal{E}$** iff:

1. $\boldsymbol{p}^* \neq \boldsymbol{0}$,
2. $(\langle \boldsymbol{x}_i^* \rangle, \langle \boldsymbol{y}_k^* \rangle) \in A(\mathcal{E})$,
3. for each k ($k = 1, \ldots, \ell$), we have: $\boldsymbol{p}^* \cdot \boldsymbol{y}_k^* = \pi_k(\boldsymbol{p}^*)$, and
4. for each i ($i = 1, \ldots, m$), we have:
   a. $\boldsymbol{p}^* \cdot \boldsymbol{x}_i^* \leq w_i(\boldsymbol{p}^*) \stackrel{\text{def}}{=} \boldsymbol{p}^* \cdot \boldsymbol{r}_i + \sum_{k \in L} s_{ik} \pi_{ik}(\boldsymbol{p}^*)$, and:
   b. $(\forall \boldsymbol{x}_i \in X_i) \colon \boldsymbol{x}_i P_i \boldsymbol{x}_i^* \Rightarrow \boldsymbol{p}^* \cdot \boldsymbol{x}_i > w_i(\boldsymbol{p}^*)$.

[1] It should be noted that if the set of *viable* coalitions, $\mathcal{S}^*$, is a proper subset of $\mathcal{S}$, then Theorem 4.1, below, remains essentially intact.

3.5. Definitions. Given an economy, $\mathcal{E} = (\langle X_i, P_i\rangle, \langle Y_k\rangle, \langle \boldsymbol{r}_i\rangle, [s_{ik}])$, we define the set of all **Walrasian allocations for $\mathcal{E}$, $\mathcal{W}(\mathcal{E})$**, by:

$$\mathcal{W}(\mathcal{E}) = \big\{ \big(\langle \boldsymbol{x}_i^*\rangle, \langle \boldsymbol{y}_k^*\rangle\big) \in A(\mathcal{E}) \mid \big(\exists \boldsymbol{p}^* \in \mathbb{R}^n\big): \\ (\langle \boldsymbol{x}_i^*\rangle, \langle \boldsymbol{y}_k^*\rangle, \boldsymbol{p}^*) \text{ is a Walrasian equilibrium for } \mathcal{E}\big\}$$

We then define the set of **Walrasian consumption allocations for $\mathcal{E}$, $\boldsymbol{W}(\mathcal{E})$**, by:

$$\boldsymbol{W}(\mathcal{E}) = \big\{\langle \boldsymbol{x}_i^*\rangle_{i\in M} \in \boldsymbol{X}^*(\mathcal{E}) \mid (\exists \langle \boldsymbol{y}_k^*\rangle_{k\in L})\colon \big(\langle \boldsymbol{x}_i^*\rangle, \langle \boldsymbol{y}_k^*\rangle\big) \in \mathcal{W}(\mathcal{E})\big\}.$$

4 The Core in Replicated Economies.

Given an economy, $\mathcal{E} = (\langle X_i, P_i\rangle, \langle Y_k\rangle, \langle \boldsymbol{r}_i\rangle, [s_{ik}])$, we consider the sequence of related economies, $\mathcal{E}_q$, defined in the following way.

$\mathcal{E}_1 = \mathcal{E}$,

......

$\mathcal{E}_q = \langle (X_{hi}, P_{hi}, \boldsymbol{r}_{hi}, Z_{hi})\rangle_{(h,i)\in,Q\times M}$, where :

$Q = \{1\ldots,q\}, X_{hi} = X_i, P_{hi} = P_i,\ \boldsymbol{r}_{hi} = \boldsymbol{r}_i$ and $Z_{hi} = Z_i$ for $h = 1,\ldots,q; i = 1,\ldots,m$.

Thus, in $\mathcal{E}_q$, the agents (consumers) have a double index; agent (h,i) is the h^{th} agent of the i^{th} type; and each agent of the i^{th} type has the same economic characteristics as does the i^{th} agent in the original economy. We will refer to $\mathcal{E}_q$ as the **q-fold replication of $\mathcal{E}$**. In dealing with $\mathcal{E}_q$, we will use the notation '$\langle(\boldsymbol{x}_{hi}, \boldsymbol{z}_{hi})\rangle_{(h,i)\in Q\times M}$' to denote allocations for $\mathcal{E}_q$

We will show that, in a sense to be explained shortly, as $q \to \infty$, $\boldsymbol{C}(\mathcal{E}_q)$ 'shrinks to $\boldsymbol{W}(\mathcal{E})$.' Our basic approach follows Debreu and Scarf [1963] in considering the sets $\boldsymbol{C}_q$, defined as the set of all feasible allocations, $\langle \boldsymbol{x}_i\rangle_{i\in M} \in \boldsymbol{X}^*(\mathcal{E})$ such that the allocation $\langle \boldsymbol{x}_{hi}\rangle_{(h,i)\in Q\times M}$ given by:

$$\boldsymbol{x}_{hi} = \boldsymbol{x}_i \quad \text{for } h = 1,\ldots,q; i = 1,\ldots,m; \tag{14}$$

is in $\boldsymbol{C}(\mathcal{E}_q)$. In other words, $\boldsymbol{C}_q$ is the projection on $\boldsymbol{X}^*(\mathcal{E})$ of the allocations $\langle \boldsymbol{x}_{hi}\rangle_{(h,i)\in Q\times M}$ from $\boldsymbol{C}(\mathcal{E}_q)$ which have the property that:

$$\boldsymbol{x}_{hi} = \boldsymbol{x}_{h'i} \quad \text{for } h, h' = 1,\ldots,q; i = 1,\ldots,m. \tag{15}$$

Notice, incidentally, that if $\langle \boldsymbol{x}_i\rangle_{i\in M}$ is a feasible consumption allocation for $\mathcal{E}$, and we define $\langle \boldsymbol{x}_{hi}\rangle_{(h,i)\in Q\times M}$ as in equation (14), then $\langle \boldsymbol{x}_{hi}\rangle_{(h,i)\in Q\times M}$ is a feasible consumption allocation for $\mathcal{E}_q$.

The following result generalizes a 'well-known' version of the 'First Fundamental Theorem of Welfare Economics.' Moreover, it establishes the fact that if $\boldsymbol{W}(\mathcal{E}) \neq \emptyset$, then $\boldsymbol{C}_q \neq \emptyset$, for $q = 1,2,\ldots$. I have stated it here without proof, since it can be proved by fairly standard arguments.

4.1. Theorem. *For any economy,* $\mathcal{E} = (\langle X_i, P_i\rangle, \langle Y_k\rangle, \langle \boldsymbol{r}_i\rangle, [s_{ik}])$, *we have:*

1. $\boldsymbol{W}(\mathcal{E}) \subseteq \boldsymbol{C}_q$, *and*
2. $\boldsymbol{C}_{q+1} \subseteq \boldsymbol{C}_q$,

for $q = 1, 2, \ldots$.

Notice that it follows from 4.1 that for all q:

$$\boldsymbol{C}_q = \bigcap_{s=1}^{q} \boldsymbol{C}_s;$$

and thus it is natural to write:

$$\bigcap_{q=1}^{\infty} \boldsymbol{C}_q = \lim_{q\to\infty} \boldsymbol{C}_q. \tag{16}$$

Debreu and Scarf [1963] showed that given any exchange economy, $\mathcal{E} = \langle (P_i, \boldsymbol{r}_i)\rangle_{i\in M}$, satisfying certain assumptions, we will have:

$$\bigcap_{q=1}^{\infty} C_q \subseteq W(\mathcal{E});$$

which, when combined with Theorem 4.1 and equation (16) means that under the Debreu-Scarf conditions, we have:

$$\lim_{q\to\infty} \boldsymbol{C}_q = \boldsymbol{W}(\mathcal{E}). \tag{17}$$

We will prove a generalization of their result; one which applies to a private ownership economy with production. However, we will begin by introducing the idea of a 'quasi-competitive equilibrium,' and proving that an Edgeworth allocation is a quasi-competitive equilibrium allocation; where we follow Aliprantis *et. al.* [1987a, 1987b] in defining the set of **Edgeworth Allocations for** $\boldsymbol{\mathcal{E}}$, $\boldsymbol{X}^E(\mathcal{E})$, by:

$$\boldsymbol{X}^E(\mathcal{E}) = \bigcap_{q=1}^{\infty} C_q. \tag{18}$$

4.2. Definition. We shall say that $(\langle \boldsymbol{x}_i^*\rangle, \langle \boldsymbol{y}_k^*\rangle, \boldsymbol{p}^*)$ is a **quasi-competitive equilibrium for the economy** $\mathcal{E} = (\langle X_i, P_i\rangle, \langle Y_k\rangle, \langle \boldsymbol{r}_i\rangle, [s_{ik}])$, iff $(\langle \boldsymbol{x}_i^*\rangle, \langle \boldsymbol{y}_k^*\rangle, \boldsymbol{p}^*)$ satisfies conditions 1–3 of Definition 3.4, and:

4′. for each $i \in M$, we have:
   a. $\boldsymbol{p}^* \cdot \boldsymbol{x}_i^* \leq w_i(\boldsymbol{p}^*) \equiv \boldsymbol{p}^* \cdot \boldsymbol{r}_i + \sum_{k\in L} s_{ik}\pi(\boldsymbol{p}^*)$, and:
   b. either:

$$w_i(\boldsymbol{p}^*) = \min \boldsymbol{p}^* \cdot X_i,$$

   or:

$$(\forall \boldsymbol{x}_i \in X_i)\colon \boldsymbol{x}_i P_i \boldsymbol{x}_i^* \Rightarrow \boldsymbol{p}^* \cdot \boldsymbol{x}_i > w_i(\boldsymbol{p}^*).$$

We will denote the set of all consumption allocations, $\langle \boldsymbol{x}_i^*\rangle \in \boldsymbol{X}^*(\mathcal{E})$, for which there exists a production allocation $\langle \boldsymbol{y}_k^*\rangle_{k\in L}$ and a price vector $\boldsymbol{p}^*$ such that $(\langle \boldsymbol{x}_i^*\rangle, \langle \boldsymbol{y}_k^*\rangle, \boldsymbol{p}^*)$ is a quasi-competitive equilibrium for $\mathcal{E}$ by '$\boldsymbol{W}^\dagger(\mathcal{E})$.'

In our initial result, we will establish conditions sufficient to ensure that $\boldsymbol{X}^E(\mathcal{E}) \subseteq \boldsymbol{W}^\dagger(\mathcal{E})$. In our proof, which owes a great deal to McKenzie [1988] and Nikaido [1968, Theorem 17.4, p. 291], we will need to make use of the following mathematical result; the proof of which is omitted, since it is fairly 'well-known.' In the statement of the result, however, I have introduced one further bit of notation; we will denote the unit matrix in $\mathbb{R}^m$ by 'Δ_m,' that is:

$$\Delta_m = \Big\{\boldsymbol{a} \in \mathbb{R}^m_+ \mid \sum\nolimits_{i=1}^m a_i = 1\Big\}.$$

4.3. Proposition. *If $C_i \subseteq \mathbb{R}^n$ is convex and non-empty, for $i = 1, \ldots, m$, then the convex hull of $C \stackrel{def}{=} \bigcup_{i=1}^m C_i$, $co(C)$, is given by:*

$$co(C) = \big\{\boldsymbol{x} \in \mathbb{R}^n \mid (\exists \boldsymbol{a} \in \Delta_m \;\&\; \boldsymbol{x}_i \in C_i,\ \text{for } i = 1, \ldots, m) \colon \boldsymbol{x} = \sum\nolimits_{i=1}^m a_i \boldsymbol{x}_i\big\}. \tag{19}$$

While the above result may seem obvious, it should be noted that the conclusion no longer holds if the C_i's are not all convex; that is, the convex hull of C is not generally given by the formula in equation (19) if the sets C_i are not all convex.

4.4. Theorem. *If $\mathcal{E} = (\langle X_i, P_i\rangle, \langle Y_k\rangle, \langle \boldsymbol{r}_i\rangle, [s_{ik}])$ is an economy such that:*

1. Y_k is convex, for $k = 1, \ldots, \ell$;

and, for each $i \in M$:

2. P_i is locally non-saturating and weakly convex, and:

3. $X_i \cap [\boldsymbol{r}_i + Z_i] \neq \emptyset$,

then:

$$\boldsymbol{X}^E(\mathcal{E}) \equiv \bigcap\nolimits_{q=1}^\infty \boldsymbol{C}_q \subseteq \boldsymbol{W}^\dagger(\mathcal{E}).$$

Proof. Suppose $\langle \boldsymbol{x}_i^* \rangle_{i \in M} \in \boldsymbol{C}_q$ for all q, define $\mathbb{P}_i = P_i \boldsymbol{x}_i^* - \boldsymbol{r}_i - Z_i$, for each $i \in M$,[2] and:

$$\mathbb{P} = co\Big(\bigcup\nolimits_{i=1}^m \mathbb{P}_i\Big);$$

that is, $\mathbb{P}$ is the convex hull of the union of the $\mathbb{P}_i$'s. The tricky part of the proof is to establish the fact that $\mathbf{0} \notin \mathbb{P}$.

Suppose, by way of establishing a contradiction, that $\mathbf{0} \in \mathbb{P}$. Then, since each $\mathbb{P}_i$ is a convex set (and non-empty, by the assumption that each P_i is locally non-saturating), it follows from Proposition 4.3 that there exist $\boldsymbol{a} \in \Delta_m$, $\boldsymbol{x}_i \in X_i$, and $\boldsymbol{z}_i \in Z_i$ for $i = 1, \ldots, m$, such that:

$$\sum\nolimits_{i=1}^m a_i(\boldsymbol{x}_i - \boldsymbol{r}_i - \boldsymbol{z}_i) = \mathbf{0}, \tag{20}$$

[2]That is:

$$\mathbb{P}_i = \big\{\boldsymbol{v} \in \mathbb{R}^n \mid (\exists \boldsymbol{x}_i \in X_i \;\&\; \boldsymbol{z}_i \in Z_i) \colon \boldsymbol{x}_i P_i \boldsymbol{x}_i^* \;\&\; \boldsymbol{v} = \boldsymbol{x}_i - \boldsymbol{r}_i - \boldsymbol{z}_i\big\}.$$

and:

$$\boldsymbol{x}_i P_i \boldsymbol{x}_i^* \quad \text{for } i = 1, \ldots, m. \tag{21}$$

We will show that these two conditions allow us to construct a coalition in $\mathcal{E}_{q^*}$, for some (finite) integer, q^*, which can improve upon $\langle \boldsymbol{x}_i^* \rangle_{i \in M}$; contradicting the assumption that $\langle \boldsymbol{x}_i^* \rangle_{i \in M} \in \boldsymbol{C}_q$, for all q.

Accordingly, we begin by noting that (20) implies:

$$\sum\nolimits_{i=1}^{m} a_i(\boldsymbol{x}_i - \boldsymbol{r}_i) = \sum\nolimits_{i=1}^{m} a_i \boldsymbol{z}_i. \tag{22}$$

We then define $I = \{i \in M \mid a_i > 0\}$, and, for each $i \in I$ and each positive integer, q, we let b_i^q be the smallest integer greater than or equal to qa_i. Now, by assumption 3, for each $i \in I$ there exist $\widehat{\boldsymbol{x}}_i \in X_i$ and $\widehat{\boldsymbol{z}}_i \in Z_i$ such that:

$$\widehat{\boldsymbol{x}}_i = \boldsymbol{r}_i + \widehat{\boldsymbol{z}}_i. \tag{23}$$

We make use of the $\widehat{\boldsymbol{x}}_i$ to define, for each $i \in I$ and each positive integer, q:

$$\boldsymbol{x}_i^q = \left(\frac{qa_i}{b_i^q}\right)\boldsymbol{x}_i + \left[1 - \left(\frac{qa_i}{b_i^q}\right)\right]\widehat{\boldsymbol{x}}_i; \tag{24}$$

and note that, since each P_i is lower semi-continuous, and since:

$$\frac{qa_i}{b_i^q} \to 1 \ \text{ as } q \to \infty,$$

it follows from (21) that for each $i \in I$, there exists a positive integer, q_i such that for all $q \geq q_i$,

$$\boldsymbol{x}_i^q P_i \boldsymbol{x}_i^*. \tag{25}$$

But now let:

$$q^* = \max_{i \in I} q_i,$$

let $b^* = \max\{b_1^{q^*}, \ldots, b_m^{q^*}\}$ and consider the coalition, S, in $\mathcal{E}_{b^*}$ consisting of $b_i^{q^*}$ consumers of each type $i \in I$, and the allocation $\langle \bar{\boldsymbol{x}}_{hi} \rangle_{(h,i) \in S}$ defined by:

$$\bar{\boldsymbol{x}}_{hi} = \boldsymbol{x}_i^{q^*} \quad \text{for } h = 1, \ldots, b_i^{q^*}, \text{ and each } i \in I. \tag{26}$$

We have $\bar{\boldsymbol{x}}_{hi} P_{hi} \boldsymbol{x}_i^*$ for each h and each $i \in I$; while by using (24), (23) and (22) in turn, we have:

$$\begin{aligned}
\sum\nolimits_{i \in I} \sum\nolimits_{h=1}^{b_i^{q^*}} \bar{\boldsymbol{x}}_{hi} &= \sum\nolimits_{i \in I} b_i^{q^*} \boldsymbol{x}_i^{q^*} = \sum\nolimits_{i \in I} \left[(q^* a_i)\boldsymbol{x}_i + (b_i^{q^*} - q^* a_i)\widehat{\boldsymbol{x}}_i\right] \\
&= q^*\Big(\sum\nolimits_{i \in I} a_i(\boldsymbol{x}_i - \boldsymbol{r}_i)\Big) - q^* \sum\nolimits_{i \in I} a_i \widehat{\boldsymbol{z}}_i + \sum\nolimits_{i \in I} b_i^{q^*}(\boldsymbol{r}_i + \widehat{\boldsymbol{z}}_i) \\
&= q^*\Big(\sum\nolimits_{i \in I} a_i \boldsymbol{z}_i\Big) - q^* \sum\nolimits_{i \in I} a_i \widehat{\boldsymbol{z}}_i + \sum\nolimits_{i \in I} b_i^{q^*}(\boldsymbol{r}_i + \widehat{\boldsymbol{z}}_i) \\
&= \sum\nolimits_{i \in I} b_i^{q^*} \boldsymbol{r}_i + \sum\nolimits_{i \in I} b_i^{q^*} \left[\left(\frac{q^* a_i}{b_i^{q^*}}\right)\boldsymbol{z}_i + \widehat{\boldsymbol{z}}_i - \left(\frac{q^* a_i}{b_i^{q^*}}\right)\widehat{\boldsymbol{z}}_i\right].
\end{aligned}$$

Thus, since each Z_i is convex, it follows that the coalition S can improve upon $\langle \boldsymbol{x}_i^* \rangle_{i \in M}$; contradicting the assumption that $\langle \boldsymbol{x}_i^* \rangle_{i \in M} \in C_q$ for all positive integers, q. Therefore $\mathbf{0} \notin \mathbb{P}$.

Since we have now established the fact that $\mathbf{0} \notin \mathbb{P}$, it follows from the 'Separating Hyperplane Theorem' that there exists a non-zero $\boldsymbol{p}^* \in \mathbb{R}^n$ satisfying:

$$(\forall \boldsymbol{v} \in \mathbb{P})\colon \boldsymbol{p}^* \cdot \boldsymbol{v} \geq 0. \tag{27}$$

From the definition of $\mathbb{P}$, it then follows immediately that for each $i \in M$, we have:

$$(\forall \boldsymbol{x}_i \in X_i \ \& \ \boldsymbol{z}_i \in Z_i)\colon \boldsymbol{x}_i P_i \boldsymbol{x}_i^* \Rightarrow \boldsymbol{p}^* \cdot \boldsymbol{x}_i \geq \boldsymbol{p}^* \cdot (\boldsymbol{r}_i + \boldsymbol{z}_i). \tag{28}$$

Moreover, since P_i is locally non-saturating, it then follows easily that, for each i and each $\boldsymbol{z}_i \in Z_i$:

$$\boldsymbol{p}^* \cdot \boldsymbol{x}_i^* \geq \boldsymbol{p}^* \cdot \boldsymbol{r}_i + \boldsymbol{p}^* \cdot \boldsymbol{z}_i. \tag{29}$$

Now, since $\langle \boldsymbol{x}_i^* \rangle_{i \in M} \in \boldsymbol{C}_q$, for each q, it follows from the definitions that there exists $\langle \boldsymbol{y}_k^* \rangle_{k \in L}$ such that:

$$\sum_{i \in I} \boldsymbol{x}_i^* = \sum_{i \in I} \boldsymbol{r}_i + \sum_{k \in L} \boldsymbol{y}_k^*. \tag{30}$$

Defining:

$$\boldsymbol{z}_i^* = \sum_{k \in L} s_{ik} \boldsymbol{y}_k^*, \tag{31}$$

we then have from (29) that:

$$\boldsymbol{p}^* \cdot (\boldsymbol{x}_i^* - \boldsymbol{r}_i - \boldsymbol{z}_i^*) \geq 0 \quad \text{for } i = 1, \ldots, m. \tag{32}$$

However, since $\sum_{i \in I} \boldsymbol{z}_i^* = \sum_{k \in L} \boldsymbol{y}_k^*$, it follows from (30) that:

$$\sum_{i \in I} \boldsymbol{p}^* \cdot (\boldsymbol{x}_i^* - \boldsymbol{r}_i - \boldsymbol{z}_i^*) = \boldsymbol{p}^* \cdot \Big(\sum_{i \in I} \boldsymbol{x}_i^* - \sum_{i \in I} \boldsymbol{r}_i - \sum_{k \in L} \boldsymbol{y}_k^* \Big) = 0;$$

and from (32) and our definitions that:

$$\boldsymbol{p}^* \cdot \boldsymbol{x}_i^* = \boldsymbol{p}^* \cdot \boldsymbol{r}_i + \boldsymbol{p}^* \cdot \boldsymbol{z}_i^* = \boldsymbol{p}^* \cdot \boldsymbol{r}_i + \sum_{k \in L} s_{ik} \boldsymbol{p}^* \cdot \boldsymbol{y}_k^* \quad \text{for } i = 1, \ldots, m. \tag{33}$$

Now, let $j \in L$ be arbitary, let $\boldsymbol{y}_j \in Y_j$, and define, for each i:

$$\boldsymbol{z}_i = \sum_{k \neq j} s_{ik} \boldsymbol{y}_k^* + s_{ij} \boldsymbol{y}_j. \tag{34}$$

Then we see that $\boldsymbol{z}_i \in Z_i$ for each i, so that by (29), we have:

$$\boldsymbol{p}^* \cdot \boldsymbol{x}_i^* \geq \boldsymbol{p}^* \cdot \boldsymbol{r}_i + \boldsymbol{p}^* \cdot \boldsymbol{z}_i \quad \text{for } i = 1, \ldots, m. \tag{35}$$

Adding the inequalities in (35), and making use of our definitions of the $\boldsymbol{z}_i$, we have:

$$\begin{aligned}\sum_{i\in I}\boldsymbol{p}^*\cdot\boldsymbol{x}_i^* = \boldsymbol{p}^*\cdot\sum_{i\in I}\boldsymbol{x}_i^* &\geq \boldsymbol{p}^*\cdot\sum_{i\in I}\boldsymbol{r}_i + \boldsymbol{p}^*\cdot\sum_{i\in I}\boldsymbol{z}_i \\ &= \boldsymbol{p}^*\cdot\sum_{i\in I}\boldsymbol{r}_i + \boldsymbol{p}^*\cdot\Big[\sum_{i\in M}\Big(\sum_{k\neq j}s_{ik}\boldsymbol{y}_k^* + s_{ij}\boldsymbol{y}_j\Big)\Big] \\ &= \boldsymbol{p}^*\cdot\sum_{i\in I}\boldsymbol{r}_i + \sum_{k\neq j}\boldsymbol{p}^*\cdot\boldsymbol{y}_k^* + \boldsymbol{p}^*\cdot\boldsymbol{y}_j. \quad (36)\end{aligned}$$

From (30) and (36) we then see that $\boldsymbol{p}^*\cdot\boldsymbol{y}_j^* \geq \boldsymbol{p}^*\cdot\boldsymbol{y}_j$, and thus $\boldsymbol{y}_k^*$ maximizes $\boldsymbol{p}^*\cdot\boldsymbol{y}_k$ on Y_k, for $k = 1,\ldots,\ell$.

Finally, let $i \in M$ be arbitrary. Then from (33) and the conclusion of the previous paragraph, we have:

$$\boldsymbol{p}^*\cdot\boldsymbol{x}_i^* = w_i(\boldsymbol{p}^*) \overset{\text{def}}{=} \boldsymbol{p}^*\cdot\boldsymbol{r}_i + \sum\nolimits_{k=1}^{\ell} s_{ik}\pi(\boldsymbol{p}^*).$$

Furthermore, it follows from (28) that:

$$(\forall \boldsymbol{x}_i \in X_i)\colon \boldsymbol{x}_i P_i \boldsymbol{x}_i^* \Rightarrow \boldsymbol{p}^*\cdot\boldsymbol{x}_i \geq w_i(\boldsymbol{p}^*);$$

and thus from Proposition 2.3, we see that either:

$$w_i(\boldsymbol{p}^*) = \min \boldsymbol{p}^*\cdot X_i,$$

or:

$$(\forall \boldsymbol{x}_i \in X_i)\colon \boldsymbol{x}_i P_i \boldsymbol{x}_i^* \Rightarrow \boldsymbol{p}^*\cdot\boldsymbol{x}_i > w_i(\boldsymbol{p}^*).$$

Therefore, $(\langle\boldsymbol{x}_i^*\rangle, \langle\boldsymbol{y}_k^*\rangle, \boldsymbol{p}^*)$ is a quasi-competitive equilibrium for $\mathcal{E}$. □

We can strengthen the conclusion of the theorem just proved to conclude that if $\langle\boldsymbol{x}_i^*\rangle$ is an Edgeworth allocation and satisfies an irreducibility condition, then $\langle\boldsymbol{x}_i^*\rangle \in \boldsymbol{W}(\mathcal{E})$. The condition in question is defined as follows.

4.5. Definition. We shall say that the economy, $\mathcal{E} = (\langle X_i, P_i\rangle, \langle Y_k\rangle, \langle\boldsymbol{r}_i\rangle, [s_{ik}])$ is **irreducible at the consumption allocation $\langle\boldsymbol{x}_i^*\rangle \in \boldsymbol{X}^*(\mathcal{E})$** iff, given any partition of the consumers, $\{S_1, S_2\}$,[3] there exists $\langle(\boldsymbol{x}_i, \boldsymbol{z}_i)\rangle_{i\in M}$ such that:

$$\boldsymbol{x}_i \in X_i \;\&\; \boldsymbol{z}_i \in Z_i \quad \text{for } i = 1,\ldots,m, \tag{37}$$

$$\sum_{i\in S_1}(\boldsymbol{x}_i - \boldsymbol{r}_i - \boldsymbol{z}_i) = \sum_{i\in S_2}(\boldsymbol{r}_i + \boldsymbol{z}_i - \boldsymbol{x}_i), \tag{38}$$

and:

$$(\forall i \in S_1)\colon \boldsymbol{x}_i P_i \boldsymbol{x}_i^*. \tag{39}$$

We will denote the set of consumption allocations at which $\mathcal{E}$ is irreducible by '$\boldsymbol{X}^I(\mathcal{E})$.'

[3] By a partition of the consumers, $\{S_1, S_2\}$, we mean $S_j \subseteq M \;\&\; S_j \neq \emptyset$, for $j = 1, 2$, $S_1 \cap S_2 = \emptyset$, and $S_1 \cup S_2 = M$.

In effect, the economy $\mathcal{E}$ is irreducible at $\langle \boldsymbol{x}_i^* \rangle \in \boldsymbol{X}^*(\mathcal{E})$ iff, given any partition of the consumers into two groups, S_1 and S_2, there is a feasible trade between the two groups which would make each of the consumers in S_1 better off than they are at $\langle \boldsymbol{x}_i^* \rangle$. Of course, this same trade may make each of the consumers in S_2 worse off than they are at $\langle \boldsymbol{x}_i^* \rangle$!

4.6. Theorem. *If* $\mathcal{E} = (\langle X_i, P_i \rangle, \langle Y_k \rangle, \langle \boldsymbol{r}_i \rangle, [s_{ik}])$ *is an economy such that:*

1. Y_k *is convex, for* $k = 1, \ldots, \ell$*;*

2. $int(X) \cap [\boldsymbol{r} + Y] \neq \emptyset$*,*

and, for each $i \in M$*:*

3. P_i *is locally non-saturating, weakly convex and lower semi-continuous, and:*

4. $X_i \cap [\boldsymbol{r}_i + Z_i] \neq \emptyset$*,*

then:

$$\boldsymbol{X}^I(\mathcal{E}) \cap \left[\bigcap_{q=1}^{\infty} \boldsymbol{C}_q\right] \subseteq \boldsymbol{W}(\mathcal{E}).$$

Proof. Suppose:

$$\langle \boldsymbol{x}_i^* \rangle_{i \in M} \in \boldsymbol{X}^I(\mathcal{E}) \cap \left[\bigcap_{q=1}^{\infty} \boldsymbol{C}_q\right].$$

Then it follows from Theorem 4.4 that, given $\langle \boldsymbol{y}_k^* \rangle_{k \in L}$ such that $\big(\langle \boldsymbol{x}_i^* \rangle, \langle \boldsymbol{y}_k^* \rangle\big) \in A(\mathcal{E})$, there exists $\boldsymbol{p}^* \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$ such that $(\langle \boldsymbol{x}_i^* \rangle, \langle \boldsymbol{y}_k^* \rangle, \boldsymbol{p}^*)$ is a quasi-competitive equilibrium for $\mathcal{E}$. Furthermore, from assumption 2 we see that there exists $\widehat{\boldsymbol{x}} \in X^*$ and $\theta \in \mathbb{R}_{++}$ such that:

$$\boldsymbol{x}^\dagger \stackrel{\text{def}}{=} \widehat{\boldsymbol{x}} - \theta \boldsymbol{p}^* \in X^*. \tag{40}$$

Thus we see that:

$$\boldsymbol{p}^* \cdot \boldsymbol{x}^\dagger = \boldsymbol{p}^* \cdot [\widehat{\boldsymbol{x}} - \theta \boldsymbol{p}^*] = \boldsymbol{p}^* \cdot \widehat{\boldsymbol{x}} - \theta \boldsymbol{p}^* \cdot \boldsymbol{p}^* < \boldsymbol{p}^* \cdot \widehat{\boldsymbol{x}} \leq \boldsymbol{p}^* \cdot \boldsymbol{x}^*,$$

where the last inequality follows easily from the definition of a quasi-competitive equilibrium. Therefore:

$$(\exists i \in M)\colon w_i(\boldsymbol{p}^*) = \boldsymbol{p}^* \cdot \boldsymbol{x}_i^* > \min \boldsymbol{p}^* \cdot X_i. \tag{41}$$

Now, define $S_i \subseteq M$ $(i = 1, 2)$ by:

$$S_1 = \{i \in M \mid w_i(\boldsymbol{p}^*) > \min \boldsymbol{p}^* \cdot X_i\},$$

and:

$$S_2 = \{i \in M \mid w_i(\boldsymbol{p}^*) = \min \boldsymbol{p}^* \cdot X_i\},$$

respectively. By (41), $S_1 \neq \emptyset$. Suppose by way of obtaining a contradiction, that $S_2 \neq \emptyset$ as well. Then, since $\langle \boldsymbol{x}_i^* \rangle_{i \in M} \in \mathfrak{X}^I(\mathcal{E})$, there exists $\langle (\boldsymbol{x}_i, \boldsymbol{z}_i) \rangle_{i \in M}$ such that:

$$\sum_{i \in S_1} (\boldsymbol{x}_i - \boldsymbol{r}_i - \boldsymbol{z}_i) = \sum_{i \in S_2} (\boldsymbol{r}_i + \boldsymbol{z}_i - \boldsymbol{x}_i), \tag{42}$$

and:

$$(\forall i \in S_1)\colon \boldsymbol{x}_i P_i \boldsymbol{x}_i^*. \tag{43}$$

However, by definition of S_1 and the fact that $(\langle \boldsymbol{x}_i^* \rangle, \langle \boldsymbol{y}_k^* \rangle, \boldsymbol{p}^*)$ is a quasi-competitive equilibrium, we have, for each $i \in S_1$:

$$\boldsymbol{p}^* \cdot \boldsymbol{x}_i > w_i(\boldsymbol{p}^*) == \boldsymbol{p}^* \cdot \boldsymbol{r}_i + \sum_{k=1}^{\ell} s_{ik} \boldsymbol{p}^* \cdot \boldsymbol{y}_k^*. \tag{44}$$

Moreover, it follows from profit maximization and Proposition 2.4.4 that for each $i \in M$:

$$\sum_{k=1}^{\ell} s_{ik} \boldsymbol{p}^* \cdot \boldsymbol{y}_k^* \geq \boldsymbol{p}^* \cdot \boldsymbol{z}_i, \tag{45}$$

where $\boldsymbol{z}_i \in Z_i$ is from (42). Thus, from (44) and (45) we see that, for each $i \in S_1$:

$$\boldsymbol{p}^* \cdot (\boldsymbol{x}_i - \boldsymbol{r}_i - \boldsymbol{z}_i) > 0;$$

and it then follows from (42) that:

$$\sum_{i \in S_2} \boldsymbol{p}^* \cdot \boldsymbol{x}_i < \sum_{i \in S_2} \boldsymbol{p}^* \cdot \boldsymbol{r}_i + \sum_{i \in S_2} \boldsymbol{p}^* \cdot \boldsymbol{z}_i \leq \sum_{i \in S_2} \boldsymbol{p}^* \cdot \boldsymbol{r}_i + \sum_{i \in S_2} \sum_{k \in L} s_{ik} \boldsymbol{p}^* \cdot \boldsymbol{y}_k^* = \sum_{i \in S_2} w_i(\boldsymbol{p}^*).$$

But this means that for at least one $i \in S_2$, we must have:

$$\boldsymbol{p}^* \cdot \boldsymbol{x}_i^* = w_i(\boldsymbol{p}^*) > \min \boldsymbol{p}^* \cdot X_i;$$

contradicting our definition of S_2. Therefore, $S_2 = \emptyset$, and it follows that $(\langle \boldsymbol{x}_i^* \rangle, \langle \boldsymbol{y}_k^* \rangle, \boldsymbol{p}^*)$ is a Walrasian (competitive) equilibrium for $\mathcal{E}$. □

Our final result could almost be called a corollary of 4.4 and 4.1. In it we will make use of the following definition.

4.7. Definitions. We will say that the j^{th} commodity is a **numéraire good for $\boldsymbol{P_i}$** iff for all $\boldsymbol{x} \in X_i$ and all $\theta \in \mathbb{R}_{++}$,[4] we have:

$$\boldsymbol{x} + \theta \boldsymbol{e}_j \in X_i \text{ and } (\boldsymbol{x} + \theta \boldsymbol{e}_j) P_i \boldsymbol{x}, \tag{46}$$

where $\boldsymbol{e}_j$ is the j^{th} unit coordinate vector.[5] We shall say that the j^{th} commodity is a **numéraire good for the economy**, $\mathcal{E} = (\langle X_i, P_i \rangle, \langle Y_k \rangle, \langle \boldsymbol{r}_i \rangle, [s_{ik}])$ iff it is a numéraire good for each $i \in M$, and for each $i \in M$ there exists $\theta_i > 0$ such that:

$$X_i \cap [(\boldsymbol{r}_i - \theta_i \boldsymbol{e}_j) + Z_i] \neq \emptyset. \tag{47}$$

[4]Where $\mathbb{R}_{++} = \{x \in \mathbb{R} \mid x > 0\}$.

[5]The vector having all coordinates equal to zero except for the j^{th} coordinate, which is equal to one.

4.8. Theorem. *If* $\mathcal{E} = (\langle X_i, P_i\rangle, \langle Y_k\rangle, \langle \boldsymbol{r}_i\rangle, [s_{ik}])$ *is an economy such that:*

1. Y_k *is convex, for* $k = 1, \ldots, \ell$*;*

2. $int(X) \cap [\boldsymbol{r} + Y] \neq \emptyset$*,*

3. for some $j' \in \{1, \ldots, n\}$*, the commodity* j' *is a numéraire good for* $\mathcal{E}$*,*

and, for each $i \in M$*:*

4. P_i *is weakly convex and lower semi-continuous,*

then:

$$\boldsymbol{X}^E(\mathcal{E}) \equiv \left[\bigcap_{q=1}^{\infty} \boldsymbol{C}_q\right] = \boldsymbol{W}(\mathcal{E}).$$

Proof. Since it is an immediate implication of Theorem 4.1 that $W(\mathcal{E}) \subseteq \boldsymbol{X}^E(\mathcal{E})$, we need only prove that $\boldsymbol{X}^E(\mathcal{E}) \subseteq W(\mathcal{E})$. Accordingly, let $j' \in \{1, \ldots, n\}$ be the numéraire good for $\mathcal{E}$, and note that it then follows from (47) and (48) of Definition 4.7 that, for each $i \in M$:

$$X_i \cap [\boldsymbol{r}_i + Z_i] \neq \emptyset.$$

Consequently, since each P_i is locally non-saturating by virtue of the fact that commodity j' is a numéraire good for $\mathcal{E}$, it follows from Theorem 4.4 that if $\langle \boldsymbol{x}_i^*\rangle_{i\in M} \in \boldsymbol{X}^E(\mathcal{E})$, then there exists $\boldsymbol{p}^* \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$ and $\langle \boldsymbol{y}_k^*\rangle_{k\in L}$ such that $(\langle \boldsymbol{x}_i^*\rangle, \langle \boldsymbol{y}_k^*\rangle, \boldsymbol{p}^*)$ is a quasi-competitive equilibrium for $\mathcal{E}$. It also follows from assumption 2 that there exists $\boldsymbol{x} \in X \stackrel{\text{def}}{=} \sum_{i\in M} X_i$, $\theta \in \mathbb{R}_{++}$, and $\boldsymbol{y} \in \sum_{k\in L} Y_k$ such that:

$$\boldsymbol{x} - \theta\boldsymbol{p}^* \in X \quad \text{and} \quad \boldsymbol{x} = \boldsymbol{r} + \boldsymbol{y}.$$

Thus, as in the proof of 4.6 we see that there must exist at least one $h \in M$ such that:

$$w_h(\boldsymbol{p}^*) = \boldsymbol{p}^* \cdot \boldsymbol{r}_h + \sum_{k\in L} s_{hk}\pi_k(\boldsymbol{p}^*) > \min \boldsymbol{p}^* \cdot X_i;$$

so that, by Proposition 2.3:

$$(\forall \boldsymbol{x}_h \in X_h)\colon \boldsymbol{x}_h P_h \boldsymbol{x}_h^* \Rightarrow \boldsymbol{p}^* \cdot \boldsymbol{x}_h > w_h(\boldsymbol{p}^*).$$

However, since commodity j' is a numéraire good for P_h, we recall that for any $\Delta x_{j'} > 0$, we have:

$$(\boldsymbol{x}_h^* + \Delta x_{j'}\boldsymbol{e}_{j'})P_h\boldsymbol{x}_h^*,$$

where $\boldsymbol{e}_{j'}$ is the $(j')^{th}$ unit coordinate vector. It then follows that we must have $p_{j'}^* > 0$.

Now let $i \in M$ be arbitrary. Then, by definition of a numéraire good for $\mathcal{E}$, there exists $\bar{\boldsymbol{x}}_i \in X_i$, $\bar{\boldsymbol{z}}_i \in Z_i$, and $\theta_{j'} > 0$ such that:

$$\bar{\boldsymbol{x}}_i = \boldsymbol{r}_i - \theta_{j'}\boldsymbol{e}_{j'} + \bar{\boldsymbol{z}}_i.$$

and, since $p_j^* > 0$, it then follows that:

$$\boldsymbol{p}^* \cdot \bar{\boldsymbol{x}}_i < \boldsymbol{p}^* \cdot \boldsymbol{r}_i + \boldsymbol{p}^* \cdot \bar{\boldsymbol{z}}_i. \tag{48}$$

Moreover, it follows from Proposition 2.4 and the definition of a quasi-competitive equilibrium that:

$$\boldsymbol{p}^* \cdot \bar{\boldsymbol{z}}_{\boldsymbol{i}} \leq \sum_{k \in L} s_{ik} \pi_k(\boldsymbol{p}^*). \tag{49}$$

From (48), (49), and Proposition 2.3 it now follows that:

$$(\forall \boldsymbol{x}_i \in X_i) \colon \boldsymbol{x}_i P_i \boldsymbol{x}_i^* \Rightarrow \boldsymbol{p}^* \cdot \boldsymbol{x}_i > w_i(\boldsymbol{p}^*),$$

and we see that $(\langle \boldsymbol{x}_i^* \rangle, \langle \boldsymbol{y}_k^* \rangle, \boldsymbol{p}^*)$ is a Walrasian equilibrium for $\mathcal{E}$. □

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