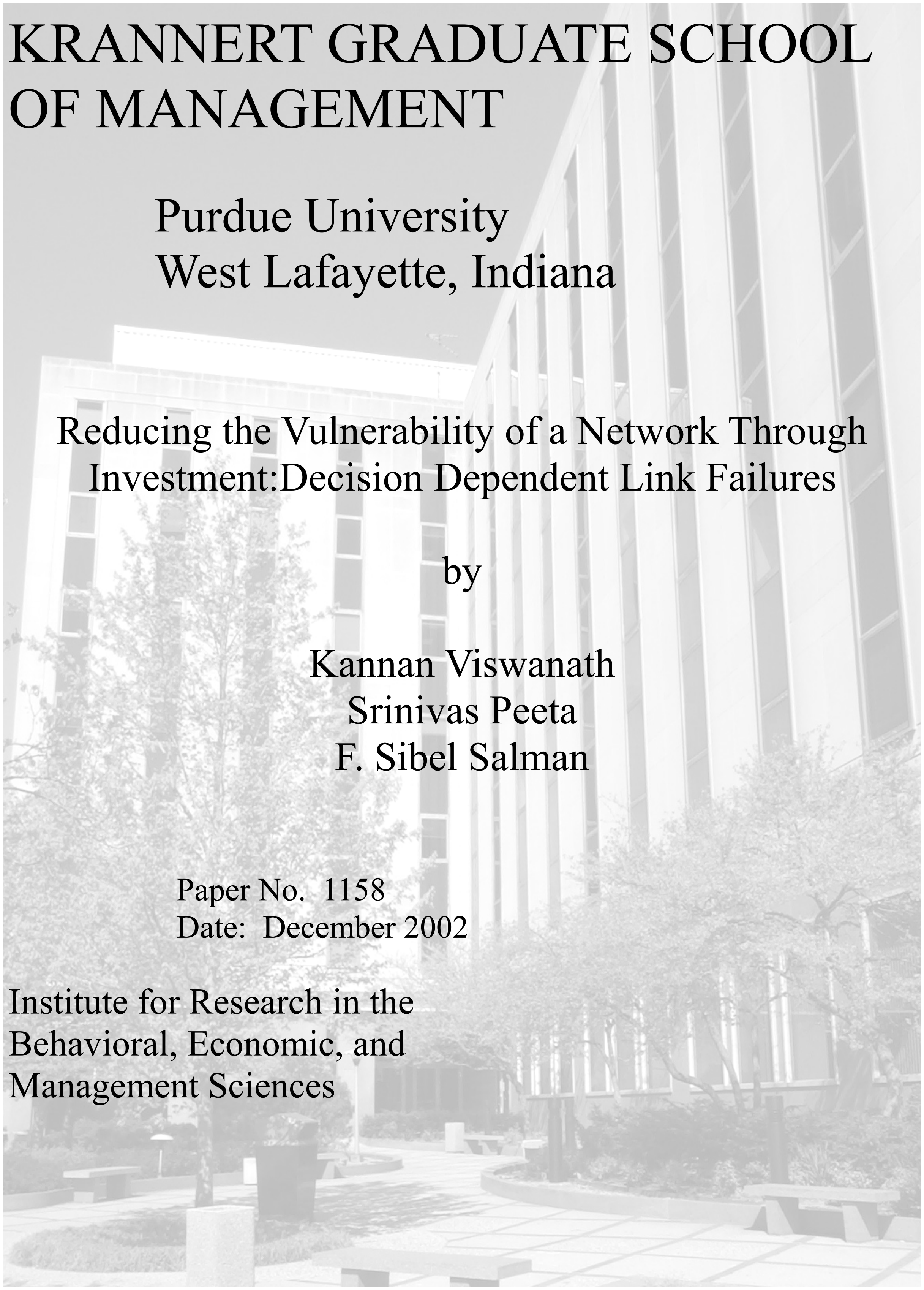

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by

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Reducing the Vulnerability of a Network through Investment: Decision Dependent Link Failures

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Abstract

We consider a problem that arises in disaster-related strategic planning: investing in the links of a stochastic network to improve its expected post-disaster performance. We are given a network whose links are subject to random failures, where the failure probability of a link is reduced by investing in the link. The operational links define a network realization, in which a specified pair of origin-destination (O-D) nodes need to be connected with a shortest path. There is a fixed penalty cost for any network realization that does not have connectivity between the O-D nodes. Our objective is to allocate a given budget to the links of the network such that the expected shortest path between the O-D nodes is minimized. We formulate the problem as a two-stage stochastic integer program with recourse, and propose a solution procedure by optimizing the expected shortest path over the connected network realizations, subject to a bound on the probability of disconnected realizations. In the proposed iterative procedure, given a feasible investment decision, we characterize the benefit from investing in a particular link by measuring the improvement in the expected shortest path.

Keywords: Decision-dependent link failure, stochastic integer programming

1. Introduction

Consider the following strategic planning problem of enhancing the survivability and performability of an infrastructure network in a populated geographical region under risk factors caused by a disaster, such as an earthquake. In the pre-disaster planning stage, a planner has to identify the set of links whose vulnerable components need to be upgraded in order for the network to remain connected between significant origin-destination (O-D) nodes in the aftermath of a potential disaster. This is critical from the perspective of emergency response operations. Moreover, the path that enables connectivity in the survived network, for any O-D pair, should be fast enough for efficient response. The basic problem facing the planner is how to allocate resources to upgrade the links considering the randomness in the network state so as to optimize the post-disaster network performance. For example, in earthquake management, the vulnerability of a highway transportation network is reduced by retrofitting the bridges. However, due to significant monetary implications, it is critical to develop cost-effective methods to select which links to improve. The associated cost-benefit trade-off is a common issue for the design of most systems that are governed by randomness and risk.

In this paper we focus on the case where the performance criterion is the connectivity of a single O-D node pair through a shortest path. We refer to this problem as the single commodity problem. The problem formulation given here can be easily extended to the more general case where the connectivity of a set of O-D nodes (via shortest paths) is the criterion. It is also possible to consider other criteria at the same time, such as maximizing the population that is accessible in the surviving network. One can also consider several likely disaster scenarios and specify link failure probabilities under each disaster scenario. We plan to extend the solution procedure proposed here to these more general problems in subsequent studies.

We model the single commodity problem as a two-stage discrete stochastic program. In the first stage, we determine whether to invest and upgrade the structures on each link to reduce vulnerability. There is a budget limitation for the investments, and the objective is to minimize the expected shortest-path length between the O-D nodes over all network realizations. In the second stage, the link survival probabilities are determined by the given investment plan, and a surviving network (induced by the operational links) is realized based on this decision-dependent probability distribution. In a given network realization, if the O-D pair is connected, then the value function for the corresponding second stage problem equals the shortest path length between the O-D nodes; otherwise, it equals a large penalty cost.

We propose a solution procedure based on optimizing the expected shortest path over the connected network realizations, subject to a bound on the probability of disconnected realizations. Our solution procedure is iterative. Given a feasible investment decision, by means of a path-based approach, we characterize the benefit from investing in a particular link by measuring the improvement in the expected shortest path.

2. Literature Review

The design and analysis of networks that are subject to random component failures have been studied in the network reliability literature (Ball, Colbourn, and Provan, 1995). Most work in this area concentrates on measuring the reliability and performance of the network, rather than designing the network to reduce vulnerability and to improve network performance. Since explicit expressions for network reliability are complex to compute, even in the case of independent link failures, typical design models use surrogates in place of explicit reliability expressions, such as the number of node- and/or edge-disjoint paths or maximum number of links in a path, (Colbourn, 1999). In particular, for telecommunication networks, the design of survivable networks studied in Grotschel et al. (1995a) and Soni et al. (1999) aim to design networks with built-in redundancy without modeling random failures and performance under uncertainty. Alevras et al. (1997) require the network to have additional capacity to re-route demand under single node or edge failures. Grotschel et al. (1995b) provide a detailed survey of survivable network design addressing these issues.

Problems of recourse in a stochastic network typically arise in various transportation planning problems. Wollmer [1980] presented a two-stage stochastic linear program with random supply of commodities to determine the optimal investment that minimizes the expected total cost in a multi-commodity flow network. In his work, the initial supply of commodities are increased by investing in them, and the total cost is defined as the sum of the investment used to increase the resources at the nodes and the minimum flow cost corresponding to the realizations of the random supply variables. Wollmer [1991] considered the problem of capacity expansion by investing in the arcs of a network whose initial capacities are random variables. The increase in arc capacity is a linear function of the investment. The objective was to maximize the linear combination of the expected maximum flow and negative of the total investment cost. The problem was formulated as a two stage linear program under uncertainty with recourse. A constraint generation based solution algorithm that takes advantage of problem structure was presented. More general version of the problem in which the increase in capacity due to investment is also treated as a random variable is discussed. The restriction they impose is that the capacity increase of an arc is a concave stochastic function of the investment. Wallace [1987] formulated the problem of investing a given budget in a network with arcs subject to random failure to increase expected maximum flow as a two stage stochastic program with network recourse. The investing is for increasing the capacity of existing arcs or building new arcs. The properties of the recourse problem were characterized. Upper and lower bounds are provided for the recourse problem using Jensen's inequality and the results of Aneja and Nair [1980], Carey and Hendrickson [1984] as evaluating the exact solution is computationally intractable. Wallace [1987] developed a piecewise linear upper bound on the recourse function for a minimum cost network flow problem in which the supply, demand and arc capacities are stochastic. Computational results illustrated that these bounds are a bit weaker than the standard Madansky bound but much faster to evaluate.

The feature of the model presented in this work that is different from those aforementioned is that the probability distribution of the second stage random variables is affected by the first-stage decision variables. To the best of our knowledge, all existing studies in the stochastic programming literature with recourse consider randomness to be exogenous to first-stage decision variables.

In the following sections we formulate our problem, discuss some properties and propose a solution procedure.

3. Problem Formulation

We are given an undirected network $G = (N, E)$ with node set N and edge set E, where $[i, j] \in E$ denotes an undirected link between nodes i and j. Let O and D be the origin and destination nodes in G.

Each link of the network will exist in either the operational or the non-operational state after the occurrence of a disaster event. A non-operational state corresponds to the failure of the link and reduces the capacity of the link from one to zero. An operational link is said to *survive* the disaster event. The probability that link $[i, j]$ survives is p_{ij}. However, if the link is upgraded for a positive cost c_{ij}, its probability of survival increases to q_{ij}. There is a limited budget B allocated for the upgrading activities.

We define deterministic binary decision variables to indicate whether each link is upgraded. We refer to the vector $y = (y_{ij})$, $y \subset \{0, 1\}^{|E|}$ as the "investment vector", where y_{ij} is defined as follows.

$$y_{ij} = \begin{cases} 1, \text{ if link } [i, j] \text{ is upgraded} \\ 0, \text{ otherwise} \end{cases}$$

We represent the post-disaster state of link $[i,j]$ by a random variable ξ_{ij} defined as follows.

$$\xi_{ij} = \begin{cases} 1, \text{ if link } [i,j] \text{ is operational after the disaster event} \\ 0, \text{ if link } [i,j] \text{ is non-operational after the disaster event} \end{cases}$$

The vector of the random variables ξ_{ij} over all links $[i, j]$ in E, denoted by $\xi = (\xi_{ij})$, $\xi \subset \{0, 1\}^{|E|}$, defined on the probability space (Ξ, A, P), that induces a subnetwork of G, which we refer to as the "surviving network". Let $E(\xi) = \{[i, j] \in E : \xi_{ij} = 1\}$ denote the surviving edges, and $G(\xi) = (N, E(\xi))$ denote the surviving network. Let $\tilde{\xi}_{ij}$ be a realization of ξ_{ij} and $\tilde{\xi}$ be a realization of ξ. The support of ξ, denoted by $\Xi = \{\tilde{\xi}^1, \tilde{\xi}^2, \ldots, \tilde{\xi}^{|\Xi|}\}$ is a finite and discrete set.

Let t_{ij} be the unit transportation cost on edge $[i, j]$. One unit of flow must be transported in $G(\xi)$ from the origin O to the destination D with minimum cost. If there exists no path between the O-D nodes in $G(\xi)$, then a penalty cost M is incurred. We can assume that M is sufficiently large, e.g. larger than the sum of t_{ij}'s over all $[i, j]$ in E. To differentiate the direction of flow, we consider two directed links (i, j) and (j, i) corresponding to each original undirected link $[i, j]$. Let $A(\xi)$ denote the set of (uncapacitated) arcs corresponding to $E(\xi)$. We define flow variables $x_{ij}(\xi_{ij})$ for each arc $(i, j) \in A(\xi)$ and define the flow vector $x(\xi) = (x_{ij}(\xi_{ij})), \forall\ (i, j) \in A(\xi)$.

The problem of investing in links so as to minimize the expected shortest path between the O-D nodes can be formulated as a two-stage stochastic program with recourse as below.

Problem P

$$F = \min_{y} \ \mathrm{f}(y) = \min_{y} \mathrm{E}_{\xi}[\mathrm{f}(y, \xi)] \qquad (1)$$

subject to

$$\sum_{[i,j]\,\varepsilon\, E} c_{ij} y_{ij} \leq B \qquad (2)$$

$$\sum_{j \varepsilon N} x_{ij}(\xi_{ij}) - \sum_{j \varepsilon N} x_{ji}(\xi_{ij}) = \begin{cases} 1, \text{if } i = O \\ -1, \text{if } i = D \\ 0, \text{otherwise} \end{cases} \qquad \forall\ i \in N, \forall\ \xi \in \Xi \qquad (3)$$

$$x_{ij}(\xi_{ij}) \leq \xi_{ij} \qquad \forall\ [i, j] \in E, \forall\ \xi \in \Xi \qquad (4)$$

$$x_{ji}(\xi_{ij}) \leq \xi_{ij} \qquad \forall\ [i, j] \in E, \forall\ \xi \in \Xi \qquad (5)$$

$$x_{ij}(\xi_{ij}), x_{ji}(\xi_{ji}) \geq 0 \qquad \forall\ [i, j] \in E, \forall\ \xi \in \Xi \qquad (6)$$

$$y_{ij} = 0 \text{ or } 1 \qquad \forall\ [i, j] \in E \qquad (7)$$

$$\xi_{ij} = 0 \text{ or } 1 \qquad \forall\ [i, j] \in E, \forall\ \xi \in \Xi \qquad (8)$$

where,

$$\mathrm{f}(y, \xi) = \begin{cases} \mathrm{M} \text{ if } \mathrm{X}(\xi) = \phi \\ \underset{\forall\ x(\xi) \in X(\xi)}{\mathrm{Min}} \sum_{[i,j] \in E} (t_{ij} x_{ij}(\xi_{ij}) + t_{ij} x_{ji}(\xi_{ji})) \end{cases} \qquad (9)$$

and $X(\xi) = \{\ x(\xi) \mid$ subject to constraints (3), (4), (5) and (6) $\}$ is defined as the set of all feasible solutions corresponding to the O-D paths in $G(\xi)$.

The first stage decisions are 0-1 binary decision variables and the second stage variables are continuous variables in [0, 1]. The function $\mathrm{f}(y, \xi)$ denotes the recourse function due to a given first stage decision y and a network realization ξ. Its value is equal to the least transportation cost in the surviving network $G(\xi)$, if such a path exists; or a penalty cost M, if O and D are disconnected in $G(\xi)$. This helps to ensure complete recourse for the problem. For a given y_{ij} the conditional probability of ξ_{ij} is given as follows.

$$P(\xi_{ij} = \tilde{\xi}_{ij} \mid y_{ij}) = \{ \tilde{\xi}_{ij} [(1-y_{ij})p_{ij} + y_{ij}q_{ij}] + (1- \tilde{\xi}_{ij})[(1-y_{ij})(1-p_{ij}) + y_{ij}(1-q_{ij})] \} \tag{10}$$

As the link failures are independent of each other, the conditional probability of ξ for a given first stage y vector is simply the product of the individual probabilities for the links in the network.

$$P(\xi = \tilde{\xi} \mid \mathrm{y}) = \prod_{\forall [i,j] \varepsilon E} P(\xi_{ij} = \tilde{\xi}_{ij} \mid y_{ij})$$

$$= \prod_{\forall [i,j] \varepsilon E} \{ \tilde{\xi}_{ij} [(1-y_{ij})p_{ij} + y_{ij}q_{ij}] + (1- \tilde{\xi}_{ij})[(1-y_{ij}) (1-p_{ij}) + y_{ij}(1-q_{ij})] \} \tag{11}$$

This equation clearly illustrates the nature of dependence of the probability of a network realization on the first stage decision variables.

The first stage investment decisions are taken in the presence of uncertainty about future realizations of ξ. The future effects of the investment plan are measured by the value function $f(y)$, which computes the expected value of taking decision y. We formulate the first stage problem as follows.

P $\quad \min \ \mathrm{f}(\mathrm{y}) = \mathrm{E}_\xi [\mathrm{f}(y, \xi)] = \sum_{\xi \varepsilon \Xi} P(\xi = \tilde{\xi} \mid y) \, \mathrm{f}(y, \tilde{\xi})$

subject to

$$\sum_{[i,j] \varepsilon E} c_{ij} y_{ij} \le B$$

$$y_{ij} = 0 \text{ or } 1, \qquad \forall [i, j] \in E$$

The first stage problem is a pure binary program with a knapsack-type constraint and a nonlinear objective function. In the second stage, for a given realization $\tilde{\xi}$, the function $\mathrm{f}(y, \tilde{\xi})$ can be evaluated easily by finding a shortest path from O to D in $G(\tilde{\xi})$.

If we define the set of feasible investments as

$$Y = \{ y \mid \sum_{[i,j] \varepsilon E} c_{ij}y_{ij} \le B, y \in \{0,1\}^{|E|} \}, \tag{12}$$

then the problem **P** can be conveniently expressed as:

$$F = \min_{y} \{ E_\xi[\mathrm{f}(y, \xi)] \mid y \in Y, \xi \in \{0,1\}^{|E|} \} \tag{13}$$

Technically, it is possible to solve the first stage problem by enumerating all feasible y vectors, since y takes values from a finite discrete set (of size at most $2^{|E|}$), and by evaluating the value function for each feasible y. However, this requires exponential computational time because of the exponential number of the feasible investments and the network realizations $|\Xi|$.

4. Preliminaries

In this section we justify the need for defining a penalty cost for network realizations in which the O-D pair is disconnected (infeasible realization) by showing that ignoring infeasibility in modeling leads to an optimistic appraisal of the true expected performance. We also show that optimizing the investment decision by incorporating the penalty cost for infeasible network realizations will yield an investment plan that reduces the probability of infeasible realizations. We use the following notation. Let S be the set of network realizations that have O-D connectivity (so that $X(\xi) \neq \varnothing$), and $S^c = \Xi / S$, be the set of network realizations that do not have O-D connectivity. Let $f_1(y, \xi) = f(y, \xi)$ when $\xi \in S$ and let $f_1(y) = E_\xi[f_1(y, \xi)]$.

Proposition 1 The optimal solution to the problem which does not consider infeasible network realizations in evaluating the expectation (the problem that minimizes $f_1(y)$) strictly overestimates the actual expected performance (measured by f(.)).

Proof

Let $y^* = \arg\min_y \{ E_\xi[f(y, \xi)] \mid y \in Y, \xi \in \{0,1\}^{|E|} \}$ (14)

and $y_1^* = \arg\min_y \{ E_\xi[f_1(y, \xi)] \mid y \in Y, \xi \in \{0,1\}^{|E|} \}$ (15)

Now $f(y_1^*) = E_\xi[f(y_1^*, \xi) \mid \xi \in S]\, P(S \mid y_1^*) + E_\xi[f(y_1^*, \xi) \mid \xi \in S^c]\, P(S^c \mid y_1^*)$ (16)

$= E_\xi[f(y_1^*, \xi) \mid \xi \in S]\, P(S \mid y_1^*) + M\, P(S^c \mid y_1^*)$ (17)

Since we assume M is sufficiently large, that is larger than the sum of t_{ij}'s over all $[i, j]$ in E, the following holds: $M > \max_{y \in Y} \{ f_1(y, \xi) \} > E_\xi[f_1(y, \xi)]$ for any $y \in Y$. Hence,

$f(y_1^*) > E_\xi[f(y_1^*, \xi) \mid \xi \in S]\, P(S \mid y_1^*) + E_\xi[f(y_1^*, \xi) \mid \xi \in S]\, P(S^c \mid y_1^*)$ (18)

$> E_\xi[f(y_1^*, \xi) \mid \xi \in S] = f_1(y_1^*)$ (20)

The inequality $f(y_1^*) > f_1(y_1^*)$ proves the proposition.

Proposition 2 The following always holds. $P(S^c \mid y_1^*) \geq P(S^c \mid y^*)$.

Proof

We know that $f(y^*) \leq f(y_1^*)$ from (14).

$\Rightarrow E_\xi[f(y^*, \xi) \mid \xi \in S]\, P(S \mid y^*) + E_\xi[f(y^*, \xi) \mid \xi \in S^c]\, P(S^c \mid y^*)$

$\quad \leq E_\xi[f(y_1^*, \xi) \mid \xi \in S]\, P(S \mid y_1^*) + E_\xi[f(y_1^*, \xi) \mid \xi \in S^c]\, P(S^c \mid y_1^*)$

$\Rightarrow E_\xi[f(y^*, \xi) \mid \xi \in S]\, P(S \mid y^*) + M\, P(S^c \mid y^*)$

$\quad \leq E_\xi[f(y_1^*, \xi) \mid \xi \in S]\, P(S \mid y_1^*) + M\, P(S^c \mid y_1^*)$

$\Rightarrow M.P(S^c \mid y^*) - M\, P(S^c \mid y_1^*)$

$\quad \leq E_\xi[f(y_1^*, \xi) \mid \xi \in S]\, P(S \mid y_1^*) - E_\xi[f(y^*, \xi) \mid \xi \in S]\, P(S \mid y^*)$

$\Rightarrow M\, [\, P(S^c \mid y^*) - P(S^c \mid y_1^*)\,]$

$\quad \leq E_\xi[f(y_1^*, \xi) \mid \xi \in S]\, P(S \mid y_1^*) - E_\xi[f(y^*, \xi) \mid \xi \in S]\, P(S \mid y^*) = f_1(y_1^*) - f_1(y^*)$

From (15) $f_1(y_1^*) \leq f_1(y^*)$, and M is finite and positive. Hence, $P(S^c \mid y^*) \leq P(S^c \mid y_1^*)$.

5. Solution by Restricting the Probability of Infeasibility

Propositions 1 and 2 justify minimizing f(y) in problem **P** (using the penalty cost M as defined in (9)). However, for solution purposes we focus on the problem that optimizes $f_1(y)$, subject to a constraint that bounds the probability of infeasible network realizations. In this section we describe this problem and obtain bounds for the expected performance of the optimal investment decision for problem **P**.

Lemma 1 If $P(S^c \mid y) \leq \alpha$, for some $0 \leq \alpha \leq 1$ and $y \in Y$, then

i) $f(y) \leq f_1(y) + M\alpha,$

ii) $f(y) \geq f_1(y)\,(1-\alpha).$

Proof

$$f(y) = E_\xi[f(y, \xi) \mid \xi \in S]\, P(S \mid y) + E_\xi[f(y, \xi) \mid \xi \in S^c]\, P(S^c \mid y) \qquad \forall\, y \in Y \qquad (24)$$

$$= E_\xi[f(y, \xi) \mid \xi \in S]\, P(S \mid y) + M\, P(S^c \mid y) \qquad \forall\, y \in Y \qquad (25)$$

$$\leq E_\xi[f(y, \xi) \mid \xi \in S]\, P(S \mid y) + M\alpha \qquad \forall\, y \in Y \qquad (26)$$

$$\leq E_\xi[f(y, \xi) \mid \xi \in S] + M\alpha \qquad \forall\, y \in Y \qquad (27)$$

$$= f_1(y) + M\alpha \qquad \forall\, y \in Y \qquad (28)$$

Since $P(S \mid y) = 1 - P(S^c \mid y) \geq 1-\alpha$, from (25), we get

$$f(y) \geq f_1(y)\,(1-\alpha) + M\, P(S^c \mid y) \qquad \forall\, y \in Y \qquad (29)$$

$$\geq f_1(y)\,(1-\alpha) \qquad \forall\, y \in Y \qquad (30)$$

Hence proved.

Let us define $Y(\alpha) = \{y \mid \sum_{[i,j] \varepsilon E} c_{ij}y_{ij} \le B, P(S^c \mid y) \le \alpha,\ y \in \{0,1\}^{|E|}\}$.

Lemma 2 For any $\alpha \in [0, 1]$, $f(y^*) \le \min_{y \in Y(\alpha)} \{ f_1(y) \} + M\alpha$.

Proof

From lemma 1i), $f(y) \le f_1(y) + M\alpha,\ \forall\ y \in Y(\alpha)$. Since $Y(\alpha) \subseteq Y$, $f(y^*) \le f_1(y) + M\alpha,\ \forall\ y \in Y(\alpha)$. $\therefore$ For a given $\alpha \in [0, 1]$, the best value for the upper bound on the optimal objective function value, $f(y^*)$ for **P,** is given by the lowest value of the rhs.

$$\Rightarrow f(y^*) = \min_{y \in Y} f(y) \le \min_{y \in Y(\alpha)} \{ f_1(y) \} + M\alpha \quad \forall\ \alpha \in [0, 1] \tag{31}$$

Hence proved.

Lemma 3 If $\alpha_1 \le \alpha_2$ then $Y(\alpha_1) \subseteq Y(\alpha_2)$.

The proof follows easily, hence it is omitted.

We now propose an approach to obtain bounds on the optimal expected performance. This can be developed from lemma 2 as follows. Observe from equation (31) that by searching over the α's we can obtain a lower value for the upper bound. We define

$$g(\alpha) = \min_{y \in Y(\alpha)} \{ f_1(y) \} + M\alpha\ \forall\ \alpha \in [0, 1] \text{ and let } y(\alpha) = \min_{y \in Y(\alpha)} \{ f_1(y) \}.\ ) \tag{32}$$

From lemma 2 we have

$$f(y^*) \le g(\alpha) \qquad \forall\ \alpha \in [0, 1] \tag{33}$$

$$\Rightarrow \qquad f(y^*) \le \min_{\forall \alpha \in (0,1)} g(\alpha) \tag{34}$$

Let $\alpha^* = \underset{\forall \alpha \in (0,1)}{\operatorname{argmin}}\ g(\alpha)$, therefore

$$f(y^*) \le g(\alpha^*) = \min_{y \in Y(\alpha^*)} \{ f_1(y) \} + M\alpha^* \tag{35}$$

$g(\alpha^*)$ would therefore be the best known upper bound on the optimal expected performance for problem **P**. The best set of investment decisions known would be $y(\alpha^*)$. The search over the α's can be made efficient using the following proposition.

Proposition 3 Let **0** and **1** denote the vector of 0's and 1's with dimension $|E|$ then the following condition is true, $P(S^c \mid y = \mathbf{1}) \le P(S^c \mid y) \le P(S^c \mid y = \mathbf{0})\ \forall\ y \in Y$.

Thus $\alpha^* \in (\alpha_{\min}, \alpha_{\max})$ where $\alpha_{\min} = P(S^c \mid y = \mathbf{1})$ and $\alpha_{\max} = P(S^c \mid y = \mathbf{0})$.

The problem that needs to be solved can be expressed as:

P1
$$\min_{\forall \alpha \in (\alpha_{\min}, \alpha_{\max})} \{ \min_{y \in Y(\alpha)} \{ f_1(y) \} + M\alpha \} \tag{36}$$

The first step towards solving this problem would be to solve the inner minimization problem, which is the expected performance over feasible network realizations with a constraint on the probability of infeasibility due to any investment decision in addition to the budget limitation. For sake of understanding, the problem formulation is shown below.

SP(*α*)

$$F(\alpha) = \min_y f_1(y) = \min_y E_\xi[f_1(y, \xi)] = \min_y E_\xi[f(y^*, \xi) \mid \xi \in S] \tag{37}$$

subject to

$$\sum_{[i,j] \in E} c_{ij} y_{ij} \leq B \tag{38}$$

$$P(S^c \mid y) \leq \alpha \tag{39}$$

$$y_{ij} = 0 \text{ or } 1 \qquad \forall\ [i,j] \in E \tag{40}$$

$$\xi_{ij} = 0 \text{ or } 1 \qquad \forall\ (i, j) \in A, \forall\ \xi \in \Xi \tag{41}$$

here $$f_1(y, \xi) = \min_{\forall x(\xi) \in X(\xi)} \sum_{[i,j] \in E} (t_{ij} x_{ij}(\xi) + t_{ji} x_{ji}(\xi)) \tag{42}$$

The above 2nd stage recourse problem is a linear program for a feasible first stage investment vector and a realization of the vector of random variables ξ, as the constraint set is totally unimodular and the objective function is linear in $x(\xi)$. The following theorem concerns the property of the objective function value of **SP(*α*)**.

Theorem 1 $F(\alpha)$ is a piecewise constant function of α.

Proof Let $Y = \{ y^{(1)}, y^{(2)}, y^{(3)} \ldots y^{(p)} \}$ be the set of feasible investment decisions such that, $P(S^c \mid y^{(1)}) \geq P(S^c \mid y^{(2)}) \geq \ldots \geq P(S^c \mid y^{(p)})$. Obviously $y^{(1)} = 0$. Represent $P(S^c \mid y^{(m)})$ as α_m.

$$\Rightarrow Y(\alpha_m) = \{ y^{(m)}, y^{(m+1)}, \ldots, y^{(p)} \} \tag{43}$$

Let y_n^* be the optimal solution to **SP(*α_m*)** with optimal objective function value of $F(\alpha_m)$. Consider $\alpha \in (\alpha_m, \alpha_{m+1})$ then $Y(\alpha) = Y(\alpha_m)$ and therefore y_n^* be the optimal solution to **SP(*α*)** with an objective function value of $F(\alpha)$ (from lemma 3 and (43)).

$\therefore$ $F(\alpha_m)$ is optimal objective function value in (α_m, α_{m+1}). In fact we can do better than this, in fact y_n* is optimal $\forall\ \alpha \in (\alpha_m, \alpha_r)$ where r = $\underset{m}{\text{argmin}} \mid y^{(m)} = y_n$*. (again from lemma 3 and (43)). Note that $r \geq m+1$.

Corollary 1 $F(\alpha) + M\alpha$ is a piecewise linear function of α.

The proof follows from Theorem 1 and is omitted.

This is a useful result as it is sufficient to search for α's that causes discontinuity.

6. A Path-based Approach to Solve SP(α)

We now proceed to discuss the solution procedure for **SP(α)** using a path based approach. We require the following extended notation. Let π represents a path from O to D $G(N, E)$ and π_k be the k[th] shortest path from O to D in G and $T(\pi_k)$ be the distance corresponding to path π_k. Also let K be the set of paths from O to D in G. We define an indicator variable $\mathrm{I}(k, \tilde{\xi}_{ij})$ as,

$$= \begin{cases} 1, \text{if the shortest path in the realized network } \tilde{\xi} \text{ is the } k^{\text{th}} \text{ shortest path in } G(N, E) \\ 0, \text{otherwise} \end{cases}$$

from (37), $F(\alpha) = \min\limits_{y} \mathrm{E}_\xi[\mathrm{f}_1(y, \xi)]$

$$= \min_{y \in Y} \sum_{\tilde{\xi} \varepsilon S} \mathrm{P}(\xi = \tilde{\xi} \mid y)\, \mathrm{f}_1(y, \xi)$$

$$= \min_{y \in Y} \sum_{\tilde{\xi} \varepsilon S} \mathrm{P}(\xi = \tilde{\xi} \mid y)\{\ \mathrm{Min} \sum_{[i,\, j] \varepsilon E} (t_{ij} x_{ij}(\xi) + t_{ji} x_{ji}(\xi))\ \} \tag{44}$$

$$= \min_{y \in Y} \sum_{\tilde{\xi} \varepsilon S} \mathrm{P}(\xi = \tilde{\xi} \mid y)\{ \sum_{k=1}^{|K|} \mathrm{I}(k, \tilde{\xi}_{ij}) \mathrm{T}(\pi_k)\ \} = \sum_{k=1}^{|K|} \sum_{\tilde{\xi} \varepsilon S} \mathrm{P}(\xi = \tilde{\xi} \mid y)\, \mathrm{I}(k, \tilde{\xi}_{ij}) \mathrm{T}(\pi_k)$$

$$= \min_{y \in Y} \sum_{k=1}^{|K|} \{ \sum_{\tilde{\xi} \varepsilon S} \mathrm{I}(k, \tilde{\xi}_{ij}) \mathrm{P}(\xi = \tilde{\xi} \mid y)\ \} \mathrm{T}(\pi_k)$$

$$= \min_{y \in Y} \sum_{k=1}^{|K|} \mathrm{P}(\pi = \pi_k \mid y) \mathrm{T}(\pi_k) = \sum_{k=1}^{|K|} \mathrm{P}(\pi_k \mid y) T(\pi_k) \tag{45}$$

$$\Rightarrow \mathrm{E}_\xi[\mathrm{f}_1(y, \xi)] = \sum_{k=1}^{|K|} \mathrm{P}(\pi_k \mid y) \mathrm{T}(\pi_k) \tag{46}$$

Equation (45) simply states that the objective function value, the expected value of the shortest distance over the set of feasible network realizations is equal to summation over all *O-D* paths in *G*, the product of the likelihood of the kth shortest path in *G* being the shortest path in *G'* and the distance of the kth shortest path in *G*. This makes intuitive sense. In effect this transformation saves us from solving the inner optimization problem of identifying the shortest path in the realized network. For practical considerations the evaluation of $|K|$ probabilities and distance corresponding to the routes in equation (46) may not be required as longer routes may not be used in the operational context vis-à-vis emergency response.

Even though in a strict sense gradient for the expected value with respect to the investment decisions does not exist we consider a gradient like direction based on finite-difference approximation taken from the discrete points of the feasible region. We now treat the *y* variables to be continuous in order to evaluate the effect of investing in a link on the objective function approximately.

$$\Rightarrow \frac{\partial E_{\xi}[f_1(y,\xi)]}{\partial y_{ij}} = \sum_{k=1}^{|K|} T(\pi_k)\frac{\partial P(\pi_k \mid y)}{\partial y_{ij}} \tag{47}$$

$$\Rightarrow b_{ij}(y) = \frac{\partial E_{\xi}[f_1(y,\xi)]}{\partial y_{ij}} = \sum_{k=1}^{|K|} T(\pi_k)\frac{\partial P(\pi_k \mid y)}{\partial y_{ij}} \tag{48}$$

We use $b_{ij}(y)$ to quantify the as the marginal effect of investment in link $[i, j]$ on the objective function. From (48) this requires evaluation of the derivative of a likelihood function with respect to an investment variable. This is done as below.

$$P(\pi_k \mid y) = \sum_{\xi \varepsilon \Xi} I(k, \tilde{\xi})\, P(\xi = \tilde{\xi} \mid y) \tag{49}$$

$$\frac{\partial P(\pi_k \mid y)}{\partial y_{ij}} = \sum_{\xi \varepsilon \Xi} I(k, \tilde{\xi}_{ij})\frac{\partial P(\xi = \tilde{\xi} \mid y)}{\partial y_{ij}} \tag{50}$$

Note: $\frac{\partial P(\xi = \tilde{\xi} \mid y)}{\partial y_{ij}} =$

$$\frac{\partial\{\prod_{\forall [m,n] \in E} P(\xi_{mn} = \tilde{\xi}_{mn} \mid y_{mn})\}}{\partial y_{ij}} = \prod_{\forall [m,n] \in E \mid [m,n] \neq [i,j]} P(\xi_{mn} = \tilde{\xi}_{mn} \mid y_{mn})\frac{\partial P(\xi_{ij} = \tilde{\xi}_{ij} \mid y_{ij})}{\partial y_{ij}} \tag{51}$$

Now $\frac{\partial P(\xi_{ij}=\tilde{\xi}_{ij} \mid y_{ij})}{\partial y_{ij}}$

$$= \frac{\partial}{\partial y_{ij}} \{ \tilde{\xi}_{ij} [(1-y_{ij})p_{ij} + y_{ij}q_{ij}] + (1-\tilde{\xi}_{ij})[(1-y_{ij})(1-p_{ij}) + y_{ij}(1-q_{ij})] \}$$

$$= \tilde{\xi}_{ij} [- p_{ij} + q_{ij}] + (1-\tilde{\xi}_{ij})[-(1-p_{ij}) + (1-q_{ij})] = \tilde{\xi}_{ij} [\Delta p_{ij}] + (1-\tilde{\xi}_{ij}) [-\Delta p_{ij}] \quad (52)$$

$$\therefore \frac{\partial P(\xi=\tilde{\xi} \mid y)}{\partial y_{ij}} = \prod_{\forall [m,n]\in E \mid [m,n]\neq[i,j]} P(\xi_{mn}=\tilde{\xi}_{mn} \mid y_{mn}) \quad \{ \tilde{\xi}_{ij} [\Delta p_{ij}] + (1-\tilde{\xi}_{ij})[-\Delta p_{ij}] \}$$

$$= P(\xi= \tilde{\xi} \mid y) \frac{1}{P(\xi_{ij} = \tilde{\xi}_{ij} \mid y_{ij})} \{ \tilde{\xi}_{ij} [\Delta p_{ij}] + (1-\tilde{\xi}_{ij})[-\Delta p_{ij}] \} \quad (53)$$

(here $\Delta p_{ij} = p_{ij} - q_{ij}$)

The results below can be obtained from substitution in equation (53).

$$\frac{\partial P(\xi=\tilde{\xi} \mid y)}{\partial y_{ij}}\Big|_{y_{ij}=0,\tilde{\xi}_{ij}=0} = P(\xi= \tilde{\xi} \mid y) \frac{-\Delta p_{ij}}{1-p_{ij}} \quad (54)$$

$$\frac{\partial P(\xi=\tilde{\xi} \mid y)}{\partial y_{ij}}\Big|_{y_{ij}=0,\tilde{\xi}_{ij}=1} = P(\xi= \tilde{\xi} \mid y) \frac{\Delta p_{ij}}{p_{ij}} \quad (55)$$

$$\frac{\partial P(\xi=\tilde{\xi} \mid y)}{\partial y_{ij}}\Big|_{y_{ij}=1,\tilde{\xi}_{ij}=0} = P(\xi= \tilde{\xi} \mid y) \frac{-\Delta p_{ij}}{1-q_{ij}} \quad (56)$$

$$\frac{\partial P(\xi=\tilde{\xi} \mid y)}{\partial y_{ij}}\Big|_{y_{ij}=1,\tilde{\xi}_{ij}=1} = P(\xi= \tilde{\xi} \mid y) \frac{\Delta p_{ij}}{q_{ij}} \quad (57)$$

Some insights can be obtained from above and make intuitive sense. Comparing (54) and (56) the following can be concluded. The marginal effect of an additional investment in a failed link reduces the probability of the network realization, $\tilde{\xi}$ to a greater extent when the level of investment is higher (i.e. $\frac{\Delta p_{ij}}{1-p_{ij}} < \frac{\Delta p_{ij}}{1-q_{ij}}$). From (55) and (57), the marginal effect of an additional investment in a survived link increases the probability of the network realization, $\tilde{\xi}$ to a greater when the level of investment is lower (i.e. $\frac{\Delta p_{ij}}{p_{ij}} > \frac{\Delta p_{ij}}{q_{ij}}$). The result from (53) can be incorporated into (50).

$$\frac{\partial P(\pi_k | y)}{\partial y_{ij}} = \sum_{\xi \varepsilon \Xi} I(k, \tilde{\xi}_{ij}) P(\xi = \tilde{\xi} | y) \frac{1}{P(\xi_{ij} = \tilde{\xi}_{ij} | y_{ij})} \{ \tilde{\xi}_{ij} [\Delta p_{ij}] + (1 - \tilde{\xi}_{ij})[-\Delta p_{ij}] \} \quad (58)$$

Now use (58) in (48) and we get,

$$b_{ij}(y) = \sum_{k=1}^{|K|} T(\pi_k) \sum_{\xi \varepsilon \Xi} I(k, \tilde{\xi}_{ij}) P(\xi = \tilde{\xi} | y) \frac{1}{P(\xi_{ij} = \tilde{\xi}_{ij} | y_{ij})} \{ \tilde{\xi}_{ij} [\Delta p_{ij}] + (1 - \tilde{\xi}_{ij})[-\Delta p_{ij}] \} \quad (59)$$

$$= \sum_{\xi \varepsilon \Xi} \sum_{k=1}^{|K|} P(\xi = \tilde{\xi} | y) T(\pi_k) I(k, \tilde{\xi}_{ij}) \frac{\tilde{\xi}_{ij} [\Delta p_{ij}] + (1 - \tilde{\xi}_{ij})[-\Delta p_{ij}]}{P(\xi_{ij} = \tilde{\xi}_{ij} | y_{ij})} \quad (60)$$

$$\Rightarrow b_{ij}(y) = \sum_{\xi \varepsilon \Xi} P(\xi = \tilde{\xi} | y) \sum_{k=1}^{|K|} T(\pi_k) I(k, \tilde{\xi}_{ij}) \frac{\tilde{\xi}_{ij} [\Delta p_{ij}] + (1 - \tilde{\xi}_{ij})[-\Delta p_{ij}]}{P(\xi_{ij} = \tilde{\xi}_{ij} | y_{ij})} \quad (61)$$

$$\Rightarrow b_{ij}(y) = E_{\xi} [\sum_{k=1}^{|K|} T(\pi_k) I(k, \tilde{\xi}_{ij}) \frac{\tilde{\xi}_{ij} [\Delta p_{ij}] + (1 - \tilde{\xi}_{ij})[-\Delta p_{ij}]}{P(\xi_{ij} = \tilde{\xi}_{ij} | y_{ij})} | \xi \in S] \quad (62)$$

A closer look at (62) shows that $b_{ij}(y)$ is the change in expected shortest distance between O and D taken over the set of feasible realizations, S due to an unit investment in link $[i, j]$. The total change in the expected shortest distance is simply the sum of the marginal effects across all links in the network.

$$\therefore \; d\, E_{\xi}[f_1(y, \xi)] = \sum_{[i,j] \varepsilon E} \frac{\partial E_{\xi}[f_1(y, \xi)]}{\partial y_{ij}} dy_{ij} \quad (63)$$

$$\Rightarrow E_{\xi}[f_1(y, \xi)] = \sum_{[i,j] \varepsilon E} b_{ij}(y) y_{ij} \quad (64)$$

SP(α) can be re-expressed as follows.

$$F(\alpha) = \min_{y} f_1(y) = \sum_{[i,j] \varepsilon E} b_{ij}(y) y_{ij} \quad (65)$$

subject to

$$\sum_{[i,j] \varepsilon E} c_{ij} y_{ij} \leq B \quad (66)$$

$$P(S^c | y) \leq \alpha \quad (67)$$

$$y_{ij} = 0 \text{ or } 1 \qquad \forall\, [i, j] \in E \quad (68)$$

$$\xi_{ij} = 0 \text{ or } 1 \qquad \forall\, (i, j) \in A, \forall\, \xi \in \Xi \quad (69)$$

7. Conclusions and Future Work

We are still working on developing a solution procedure to **SP(α)**. Potential solution approach would be based on a sample average approximation method. For a given y,

$b_{ij}(y)$ can be approximated statistically using a Monte Carlo simulation, moreover this is perhaps the only possible practical approach especially when the number of outcomes of the random vector ξ is large. This approximate problem can be considered to be a knapsack problem with the addition of the probabilistic constraint. The important issue to address would be to find the optimal solution $y^*(\alpha)$ and also $b_{ij}(y^*(\alpha))$ simultaneously using an iterative scheme. The convergence of such a method also merits investigation.

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