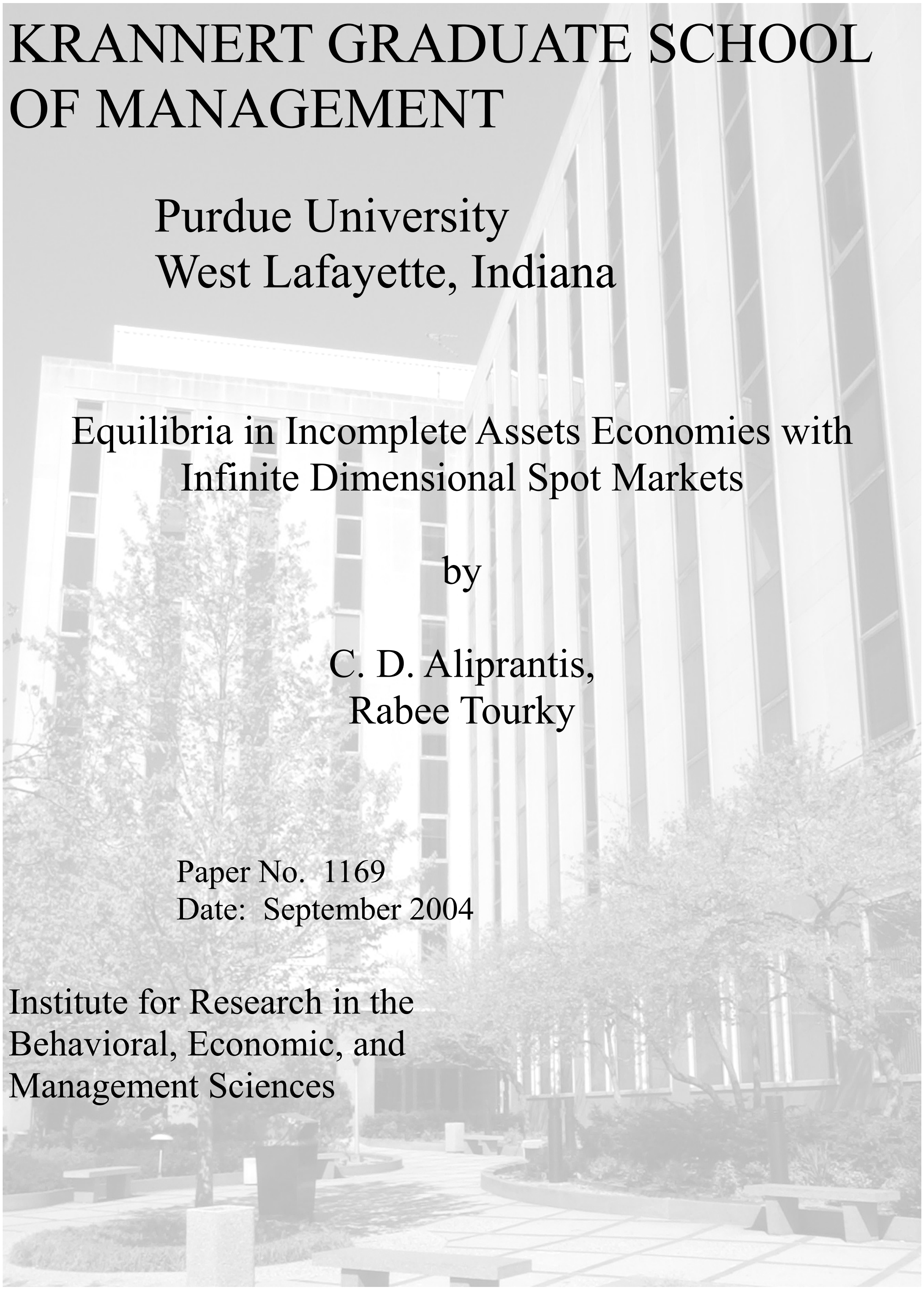

KRANNERT GRADUATE SCHOOL OF MANAGEMENT

Purdue University
West Lafayette, Indiana

Equilibria in Incomplete Assets Economies with Infinite Dimensional Spot Markets

by

C. D. Aliprantis,
Rabee Tourky

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EQUILIBRIA IN INCOMPLETE ASSETS ECONOMIES WITH INFINITE DIMENSIONAL SPOT MARKETS

CHARALAMBOS D. ALIPRANTIS AND RABEE TOURKY

ABSTRACT. The paper studies the two period incomplete markets model where assets are claims on state contingent commodity bundles and there are no bounds on portfolio trading. The important results on the existence of equilibrium in this model assume that there is a finite number of commodities traded in each spot market and that preferences are given by smooth utility functions. With these assumptions an equilibrium exists outside an "exceptional" set of assets structures and initial endowments. The present paper extends these results by allowing for general infinite dimensional commodity spaces in each spot market. These include all the important commodity spaces studied in the literature on the existence of Walrasian equilibrium—in each spot market the consumption sets are the positive cone of an arbitrary locally solid Riesz space or of an ordered topological vector space with order unit or of a locally solid Riesz space with quasi-interior point. The paper establishes that even with our very general commodity spaces there exists an equilibrium for a "very" dense set of assets structures. Our approach is in the main convex analytic and the results do not require that preferences be smooth or complete or transitive. The typical situation in infinite dimensional commodity spaces does not readily allow for the kind of differential analysis and smoothness assumptions used in the finite dimensional setting. In the general settings that we study it seems that one is restricted to convex analytic techniques and assumptions. Therefore, the concepts and techniques studied in this paper also have important finite dimensional applications.

1. INTRODUCTION

In this paper we investigate the existence of equilibrium in the two period incomplete markets economy studied by Duffie and Shafer (1985) and Magill and Shafer (1990) allowing for an infinite number of commodities in each spot market. In this economy there are finitely many assets that are claims on state contingent commodity bundles and finitely many states of the world. There are no bounds on portfolio trading and the value of a portfolio depends on the prevailing prices in spot markets. Our purpose is to establish the existence of equilibrium in sufficiently general infinite dimensional settings with weak assumptions on agents that are comparable to those typically made in the study of the existence of Walrasian equilibrium.

It is well understood that when more than one commodity is traded in each spot market and there are no bounds on portfolio trading an equilibrium may fail to exist even when preferences satisfy very strong assumptions. The classical counter example is that of Hart (1975), which highlights the dramatic discontinuity of the budget sets in the incomplete markets model. However, in their paper Duffie and Shafer (1985) show that outside a negligible set of assets structures and initial

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endowments there exists an equilibrium. In particular, counter examples such as Hart (1975) are not robust to perturbations of the securities and endowments. Duffie and Shafer assume that there are finitely many commodities in each spot market and that preferences are given by smooth, strictly convex, monotone utility functions. For an overview of the literature on the existence of equilibrium in incomplete markets see for instance Duffie (1996), Geanakoplos (1990), and Magill and Shafer (1991).

In this paper we consider economies with incomplete markets and infinite dimensional commodity spaces in each spot market. Such a situation arises in a host of economic applications. It arises in hybrid models of incomplete markets and Arrow–Debreu economies. For instance in models with commodity differentiation the commodity space in each spot market is naturally infinite dimensional (cf., Mas-Colell (1975)). It also arises in cases where there are infinitely many states of the world. Though we restrict our attention to finitely many states of the world our model covers the situation in which there are infinitely many states of the world but dividends are paid in an interim period according to their expected value over a finite number of events; see the Appendix A.

Infinite dimensional commodity spaces have been well studied in the context of the Arrow–Debreu economy. The most important results come from the literature that follows the work of Mas-Colell (1986), where one assumes that preferences satisfy some cone condition typically termed properness and that the commodity space is a Riesz space. A principal motivation for these works is to establish fundamental results such as the existence of equilibrium in a general enough setting that includes all the important applications in economics. For an overview of this literature see for example Aliprantis, Brown, and Burkinshaw (1990), Aliprantis, Cornet, and Tourky (2002), Aliprantis, Tourky, and Yannelis (2000), and Mas-Colell and Zame (1991).

We prove in this paper four main theorems on the existence of equilibrium, which cover the most important commodity spaces in economics. The first is in the finite dimensional setting and extends the results on the existence of equilibrium in economies with incomplete markets to the case of non-smooth and unordered preferences. The second is a theorem for ordered topological vector spaces with order units. This extends the classical result of Bewley (1972) on the existence of Walrasian equilibrium to the incomplete markets framework. Our third result looks at locally solid Riesz commodity spaces with strictly positive total endowments. We establish this result with a variant of the very weak pointwise properness assumption on preferences and extend the works of Araujo and Monteiro (1989) and Yannelis and Zame (1986) to the incomplete markets framework. Finally, we look at the very general locally solid Riesz spaces and extend the ideas of Mas-Colell (1986). This result is established using a variant of the uniform properness assumption of Yannelis and Zame (1986).

The typical situation in infinite dimensional commodity spaces is that the positive cone of the commodity space has an empty interior. Therefore, the analysis does not readily allow for the kind of smoothness assumptions on preferences that are made in the finite dimensional model. Moreover, in the general commodity spaces that we consider one seems to be restricted to convex analytic techniques. We do not avoid this restriction and do not mimic smoothness type assumptions on preferences—noting, however, that the properness of preferences is related to the notion of subdifferentiability. Rather, we assume that preferences are given by convex valued mappings that need not be transitive or complete. In this regard the paper also contributes to the literature of the existence of equilibrium in incomplete markets with finite dimensional spot markets.

In our main theorems we show that for a "very" dense set of assets structures there exists an equilibrium. That is, for any assets structure there exist directions such that every small perturbation of the assets structure in those directions gives a structure for which an equilibrium exists. This, of course, may not mean that "generically" for assets structures there exists an equilibrium. However, it is very closely related to the idea of largeness in the intuitive geometric sense. It implies much more than simple density in the space of assets structures. We establish that the set of assets structures for which an equilibrium fails to exist is perforated by gaps of fixed finite codimension.

The idea of looking at the existence of equilibrium in a dense set of assets structures is borrowed from the work of Magill and Quinzii (1996) who study infinite horizon incomplete markets with finite dimensional commodity spaces. The common feature with that paper is that in both settings it is not apparent how the typical finite dimensional arguments can be used to prove genericity. Our result is also interesting in light of the recent paper of Busch and Govindan (2002) that gives a counter example on the existence of equilibrium that is robust to small perturbations of endowments—but, of course, not robust to perturbations of the assets structure.

Our proofs add to, collect, and adapt ideas developed in mathematical economics over the last three decades. The proofs have four steps. First, we follow Duffie and Shafer (1985) and Magill and Shafer (1990) and reduce the problem to the existence of a pseudo-equilibrium. Then we define the notion of an abstract game with subspaces and prove the existence of equilibrium for such a game using the subspace fixed point theorem of Husseini, Larsy, and Magill (1990). At this point our analysis faces serious technical difficulties that do not arise in the finite dimensional incomplete markets literature or in the Arrow–Debreu infinite dimensional literature. A major problem is that the strict budget correspondence fails to have an open graph and the weak budget correspondences fail to be lower hemicontinuous.[1] We solve this problem by defining a market game whose better response correspondences are constructed using the order structure of the commodity space. Importantly, a "Nash" equilibrium for this market game both exists and implies the existence of a pseudo-equilibrium for the economy. So we broadly follow the strategy of Gale and Mas-Colell (1975) by reducing the problem of pseudo-equilibrium existence to that of the existence of an equilibrium for an abstract game. Finally, we complete the proofs by extending ideas pioneered by Aliprantis, Brown, and Burkinshaw (1987), Araujo and Monteiro (1989), Bewley (1972), Mas-Colell (1986), and Yannelis and Zame (1986).

In the last two decades, perhaps the most spectacular highlights of the literature on mathematical economics comprise the works on incomplete markets with finite dimensional commodity spaces and the works on complete markets with infinite dimensional commodity spaces. That is, the works that follow Duffie and Shafer (1985) and Mas-Colell (1986), respectively. The present paper contributes to both of these literatures by extending the theory of incomplete markets to the general setting of Aliprantis and Brown's (1982) Riesz commodity-price duality and extending important vector lattice theoretic techniques beyond the Arrow–Debreu model of general equilibrium.

There are other areas of investigation that are closely related to the present work. The first is on existence of equilibrium with a continuum of states of the world and a finite number of commodities in each spot market, see for instance Hellwig (1996), Monteiro (1996), Mas-Colell and Monteiro (1996), and Mas-Colell and Zame (1996). The important results in this setting assume that preferences are given by state-dependent von Neumann–Morgenstern utility functions and place some restrictions on short sales. A surprise in the present paper is that there is no need for arbitrary restrictions on portfolio trading even if commodity spaces are infinite dimensional. The second related area of investigation is the work on incomplete markets with only purely financial securities. In the finite dimensional setting such models admit theorems on the existence

[1]The discontinuities arise even with the budget sets defined in the pseudo-equilibrium setting.

of equilibrium that are basically as general as the standard results on the Arrow–Debreu model, see for instance Werner (1985, 1989). Furthermore, with purely financial assets the technical difficulties in the infinite dimensional setting are closely related to those that arise in the Arrow–Debreu model, see for instance Aliprantis, Brown, Polyrakis, and Werner (1998). Finally the work is related in an obvious way to the study of equilibrium existence in infinite horizon economies and finite dimensional commodity spaces, see for instance Levine (1989), Magill and Quinzii (1994), Hernández and Santos (1996), Levine and Zame (1996).

The paper is organized as follows. The model is described in the next section. We follow with a list of our major theorems. We then study a bilinear valuation operator arizing in the theory of incomplete markets. This is followed with an analysis of pseudo-equilibria and a study of our market game. We conclude the paper with a mathematical and applications Appendix.

2. The economic model

Our economy is a securities market with two time periods; 0 and 1 or "today" and "tomorrow." There is a finite number of states of the world $\{1, \ldots, S\}$ that can be realized in period 1. For convenience we incorporate period 0 into the set of states of the world. We denote by s elements of the set $\mathcal{S} = \{0, 1, \ldots, S\}$ and call each of them a ***state***. We emphasize, however, that the number of period 1 uncertain states of the world is precisely S, i.e., the states $1, \ldots, S$.

In each state s there is a spot market where consumers trade commodity bundles represented by the vectors in a Hausdorff locally convex space (E_s, τ_s). The space E_s is also an ordered vector space with positive cone E_s^+. The commodity space is the Cartesian product $E = \prod_{s=0}^{S} E_s$ endowed with its product ordering and product topology. The typical vector $x \in E$ will be written as $x = (x_0, x_1, \ldots, x_S)$, where $x_s \in E_s$ for each $s \in \mathcal{S}$. We shall denote by E_+ the positive cone of E (i.e., $E_+ = \prod_{s=0}^{S} E_s^+$) and by τ its product topology (i.e., $\tau = \prod_{s=0}^{S} \tau_s$). Unless otherwise indicated, all topological concepts concerning the spaces E and E_s will be understood in terms of the topologies τ and τ_s, respectively. We adapt some notation from game theory. If $z \in E_s$ and $x \in E$, then the expression $y = (z, x_{-s})$ will denote the vector $y \in E$ whose s^{th} coordinate y_s is precisely z and for $s' \neq s$ we have $y_{s'} = x_{s'}$. The tree structure of the two period securities market is shown in Figure 1.

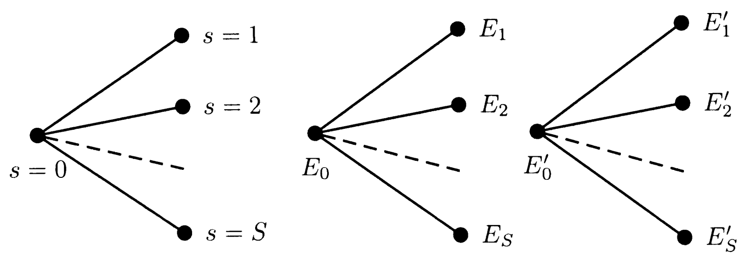

FIGURE 1. A two period model of uncertainty: states, spot commodity spaces, spot price spaces

A ***spot price system***, or simply a ***spot price***, is any nonzero vector $p = (p_0, p_1, \ldots, p_S) \in E'_+$, where $E' = \prod_{s=0}^{S} E'_s$ is the topological dual of E.[2] As usual, a vector $x \in E_+$ is called ***strictly positive*** (in symbols $x \gg 0$ or $0 \ll x$), if $0 < p \in E'$ implies $p \cdot x = \sum_{s=0}^{S} p_s \cdot x_s > 0$. The notation $x \gg y$ (or $y \ll x$) means $x - y \gg 0$. Likewise a spot price $p \in E'$ is said to be ***strictly positive*** (denoted $p \gg 0$), if for all $x \in E$ satisfying $x > 0$ we have $p \cdot x > 0$.[3]

Our securities market will have m consumers indexed by i. Each consumer i has E_+ as her consumption set and her ***strict preferences*** are represented by a correspondence $P_i \colon E_+ \twoheadrightarrow E_+$. Also, each consumer i owns an **initial endowment** $\omega^i = (\omega_0^i, \omega_1^i, \ldots, \omega_S^i) \in E_+$. Denoting the ***total endowment*** of the economy by

$$\omega = \sum_{i=1}^{m} \omega^i = \Big(\sum_{i=1}^{m} \omega_0^i, \sum_{i=1}^{m} \omega_1^i, \ldots, \sum_{i=1}^{m} \omega_S^i \Big) = (\omega_0, \omega_1, \ldots, , \omega_S) \in E_+ \,,$$

we shall say that a vector $x = (x^1, x^2, \ldots, x^m) \in E_+^m$, where $x^i = (x_0^i, x_1^i, \ldots, x_S^i) \in E_+$ for each consumer i, is an ***allocation*** if

$$\sum_{i=1}^{m} x^i = \Big(\sum_{i=1}^{m} x_0^i, \sum_{i=1}^{m} x_1^i, \ldots, \sum_{i=1}^{m} x_S^i \Big) = \omega \,,$$

or, equivalently, if $\sum_{i=1}^{m} x_s^i = \sum_{i=1}^{m} \omega_s^i$ for each $s = 0, 1, \ldots, S$.

We are now ready to introduce the notion of our portfolio space.

Definition 2.1. *A* ***portfolio space*** *is any finite dimensional vector space M of dimension $\mathfrak{J} \le S$.*

Of course, during our study M will be a fixed portfolio space. The number $\mathfrak{J}$ (the dimension of M) is interpreted as being the number of (non-redundant) available securities. Here are the usual pertinent notions associated with the portfolio space.

(1) A ***portfolio*** is simply an arbitrary vector of M.
(2) A ***portfolio trade*** is any vector $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m) \in M^m$ satisfying $\sum_{i=1}^{m} \theta_i = 0$.
(3) An ***assets price*** is a linear functional $q \in M'$, the dual of the portfolio space M.
(4) An ***assets structure*** is a linear operator $T \colon M \to \prod_{s=1}^{S} E_s$.
(5) The ***marketed space induced by*** T is the range $T(M)$ of the operator T.

The collection of all assets structures will be denoted by $\mathcal{G}$. All of our equilibrium results shall be stated in terms of "generic" subsets of $\mathcal{G}$.

Lemma 2.2. *For any fixed basis $\{\eta_1, \eta_2, \ldots, \eta_{\mathfrak{J}}\}$ of the portfolio space M, every assets structure T can be identified with a vector $(T\eta_1, T\eta_2, \ldots, T\eta_{\mathfrak{J}})$ of $\big(\prod_{s=1}^{S} E_s\big)^{\mathfrak{J}}$. In particular, $\mathcal{G}$ can be identified with the vector space $\big(\prod_{s=1}^{S} E_s\big)^{\mathfrak{J}}$.*

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[2] As usual, we let $E'_+ = \prod_{s=0}^{S} (E'_s)^+$, where $(E'_s)^+ = \{p \in E'_s \colon\ p(x) \ge 0 \text{ for all } x \in E_s^+\}$ is the ***dual cone*** of the cone E_s^+. Following the standard notation in economics, we shall write $p \cdot x$ instead of $p(x)$.

[3] Keep in mind that in a partially ordered set $(X, \ge)$ the notation $x > y$ means $x \ge y$ and $x \ne y$. Also, if a and b are elements in a partially ordered set $(X, \ge)$ satisfying $a \le b$, then we shall use the following standard notation to denote the order intervals associated with a and b: $[a, b] = \{x \in X \colon\ a \le x \le b\}$, $(a, b] = \{x \in X \colon\ a < x \le b\}$, and $[a, b) = \{x \in X \colon\ a \le x < b\}$. The "open" order interval (a, b) will have a special meaning for us here. It will be defined as $(a, b) = \{x \in E \colon\ a \ll x \ll b\}$.

With such an identification, the number $\mathfrak{J}$ represents the number of the non-redundant securities $T\eta_1, T\eta_2, \ldots, T\eta_{\mathfrak{J}}$. For the rest of our discussion in this section, T will denote a fixed assets structure. As we shall see in the subsequent sections, the important notion for the analysis is the marketed space $T(M)$. We shall assume that M has a fixed basis $\{\eta_1, \eta_2, \ldots, \eta_{\mathfrak{J}}\}$ so that T can be identified with the $\mathfrak{J}$-dimensional vector $T = (T_1, T_2 \ldots, T_{\mathfrak{J}})$, where $T_i = T\eta_i \in \prod_{s=1}^{S} E_s$ for $i = 1, 2, \ldots, \mathfrak{J}$.

Following the standard operator theory notation (see for instance Abramovich and Aliprantis (2002a,b)), we shall also denote the value $T(\theta)$ by $T\theta$. Given an arbitrary portfolio θ the vector $T\theta = (T\theta(1), T\theta(2), \ldots, T\theta(S))$ is a period 1 contingent claim where $T\theta(s)$ is the commodity bundle assigned for the portfolio θ by the assets structure T at state s. If moreover, the spot price in state s is $p(s)$, then $p(s) \cdot [T\theta(s)]$ is the payoff of the portfolio θ in state s.

Having introduced enough notation and terminology above, we can define the general framework of our economy as follows.

Definition 2.3. *An* ***assets economy with spots markets*** (*or simply an* ***economy***) *is a tuple* $\mathcal{E} = \big(\mathcal{S}, (\langle E_s, E_s'\rangle)_{s\in\mathcal{S}}, (\omega^i)_{i=1}^m, (P_i)_{i=1}^m, M, T\big)$, *where:*

(a) $\mathcal{S} = \{0, 1, \ldots, S\}$ *is the states of the world.*
(b) $\langle E_s, E_s'\rangle$ *represents the spot markets commodity–price duality system in each state* s.
(c) m *is the number of consumers, where each consumer* i *is endowed with an initial endowment* $\omega^i \in E_+ = \prod_{s=0}^{S} E_s^+$ *and a preference correspondence* $P_i\colon E_+ \to E_+$.
(d) M *is the* J*-dimensional portfolio space and* $T\colon M \to \prod_{s=1}^{S} E_s$ *is the assets structure linear operator.*

We shall assume throughout this work the following:

AI: *The ordered Hausdorff locally convex space* E *has* τ*-closed and* τ*-bounded order intervals and there exists a Hausdorff linear topology* μ *on* E *for which each order interval in* E *is* μ*-compact.*[a]

AII: *For each consumer* i *and each* s *we have* $\omega_s^i > 0$.

AIII: *The preference correspondence* $P_i\colon E_+ \twoheadrightarrow E_+$ *of each consumer* i *is:*
(a) ***irreflexive***, *i.e.,* $x \notin P_i(x)$ *for each* $x \in E_+$.
(b) ***strictly monotone***, *i.e.,* $x + y \in P_i(x)$ *for all* $x, y \in E_+$ *with* $y > 0$.
(c) ***comprehensive***, *i.e,* $P_i(x) + E_+ \subseteq P_i(x)$ *for each* $x \in E_+$.
(d) ***convex- and open-valued*** in E_+ for some Hausdorff linear topology on E (which might be different than τ).

[a]Notice that this condition is automatically satisfied if each E_s has weakly compact intervals.

The next thing in line is to introduce the budget sets for our consumers. For each consumer i, assets price $q \in M'$, spot price $p \in E'$, and portfolio $\theta \in M$, we define the following budget sets:

$$\beta_i(p,q,\theta) = \left\{ x \in E_+ : \begin{array}{rrcl} & p_0 \cdot x_0 & \leq & p_0 \cdot \omega_0^i - q \cdot \theta \\ \forall s \geq 1, & p_s \cdot x_s & \leq & p_s \cdot \omega_s^i + p_s \cdot [T\theta(s)] \end{array} \right\},$$

$$\beta_i(p,q) = \bigcup_{\theta \in M} \beta_i(p,q,\theta).$$

The bundles in $\beta_i(p,q,\theta)$ are called the consumer i's ***affordable*** bundles when she purchases the portfolio θ and the prevailing spot price is p and assets price is q. That is, the budget set $\beta_i(p,q,\theta)$ is the set of all commodity bundles in E_+ that are affordable to consumer i if she buys the portfolio θ in period zero when the prevailing spot price is p and the assets price is q. The budget set $\beta_i(p,q)$ consists of all consumption bundles $x \in E_+$ for which their exists a portfolio θ so that x is affordable to consumer i if she buys the portfolio θ.

The equilibrium concept for our economy is defined as follows.

Definition 2.4. *An* ***equilibrium*** *is a 4-tuple $(p,q,x,\boldsymbol{\theta})$ such that:*

(1) *p is a spot price.*
(2) *q is an assets price.*
(3) *$\boldsymbol{\theta} = (\theta_1,\dots,\theta_m) \in M^m$ is a portfolio trade.*
(4) *$x = (x^1,x^2,\dots,x^m) \in E_+^m$ is an allocation such that for each consumer i the bundle x^i is affordable if she buys the portfolio θ_i and no preferred bundle to x^i is affordable for all portfolio purchases. In other words, for each consumer i we have:*
   (a) *$x^i \in \beta_i(p,q,\theta_i)$, and*
   (b) *$P_i(x^i) \cap \beta_i(p,q) = \emptyset$.*

As expected budget equalities hold at equilibria.

Lemma 2.5. *If $(p,q,x,\boldsymbol{\theta})$ is an equilibrium, then:*

(1) *$p \gg 0$, i.e., the spot price p is strictly positive, and*
(2) *for each consumer i we have the budget equalities*
   (a) *$p_0 \cdot x_0^i = p_0 \cdot \omega_0^i - q \cdot \theta_i$, and*
   (b) *$p_s \cdot x_s^i = p_0 \cdot \omega_s^i + p_s \cdot [T\theta_i(s)]$ for $s = 1,2,\dots,S$.*

Proof. Assume that $(p,q,x,\boldsymbol{\theta})$ is an equilibrium.

(1) If $p = (p_0,p_1,\dots,p_S)$ is not strictly positive, then there exists some state s^* such that p_{s^*} is not strictly positive. That is, there exists some vector $0 < v \in E_{s^*}$ such that $p_{s^*} \cdot v = 0$. Now if we let $u = (v, 0_{-s^*}) \in E_+$, then note that $u > 0$ and $x^i + u \in \beta_i(p,q,\theta_i) \cap P_i(x^i)$ for each i, which is impossible.

(2) From the definition of equilibrium, we get

$$\sum_{i=1}^m p_0 \cdot x_0^i + \sum_{s=1}^S \sum_{i=1}^m p_s \cdot x_s^i \leq \sum_{i=1}^m (p_0 \cdot \omega_0^i - q \cdot \theta_i) + \sum_{s=1}^S \sum_{i=1}^m \big(p_s \cdot \omega_s^i + p_s \cdot [T\theta_i(s)]\big).$$

Now note that

$$\sum_{i=1}^m p_0 \cdot x_0^i + \sum_{s=1}^S \sum_{i=1}^m p_s \cdot x_s^i = \sum_{s=0}^S \sum_{i=1}^m p_s \cdot x_s^i = \sum_{i=1}^m \sum_{s=0}^S p_s \cdot x_s^i = \sum_{i=1}^m p \cdot x^i = p \cdot \omega,$$

and taking into account that $\sum_{i=1}^m \theta_i = 0$, we see that

$$\begin{aligned}
&\sum_{i=1}^m (p_0\cdot\omega_0^i - q\cdot\theta_i) + \sum_{s=1}^S\sum_{i=1}^m \big(p_s\cdot\omega_s^i + p_s\cdot[T\theta_i(s)]\big)\\
&\qquad = \sum_{i=1}^m p_0\cdot\omega_0^i - q\cdot\Big[\sum_{i=1}^m\theta_i\Big] + \sum_{s=1}^S\sum_{i=1}^m p_s\cdot\omega_s^i + \sum_{s=1}^S\sum_{i=1}^m p_s\cdot[T\theta_i(s)]\\
&\qquad = \sum_{s=0}^S\sum_{i=1}^m p_s\cdot\omega_s^i + \sum_{s=1}^S p_s\cdot\Big[\sum_{i=1}^m T\theta_i(s)\Big]\\
&\qquad = p\cdot\omega + \sum_{s=1}^S p_s\cdot\Big[T\Big(\sum_{i=1}^m\theta_i\Big)(s)\Big]\\
&\qquad = p\cdot\omega\,.
\end{aligned}$$

So, we have shown that

$$\sum_{i=1}^m p_0\cdot x_0^i + \sum_{s=1}^S\sum_{i=1}^m p_s\cdot x_s^i = \sum_{i=1}^m (p_0\cdot\omega_0^i - q\cdot\theta_i) + \sum_{s=1}^S\sum_{i=1}^m \big(p_s\cdot\omega_s^i + p_s\cdot[T\theta_i(s)]\big) = p\cdot\omega\,,$$

and from this we infer that budget equalities must hold true. ■

3. The major results

Our results will be stated in terms of a notion of "largeness." We shall introduce this fundamental concept for our study first.

Definition 3.1. *Let X be a nonempty subset of a vector space Y. A vector $x \in X$ is called an* ***internal point*** *of X with respect to Y if for every $y \in Y$ there exists some $\alpha_0 > 0$ (depending on y) such that $x + \alpha y \in X$ holds for all $-\alpha_0 < \alpha < \alpha_0$.*[4] *The nonempty set X is called:*

(i) ***algebraically open in*** *Y, if every vector of X is an internal point of X with respect to Y, and*
(ii) ***algebraically nowhere dense in*** *Y, if X has no internal points with respect to Y and $Y \setminus X$ is algebraically open in Y.*

It should be clear that if X is algebraically nowhere in Y, then $X \cap Y \neq \emptyset$. Recall that a vector subspace Z of a vector space Y is said to have ***codimension*** n, if there exists an n-dimensional subspace L of Y such that $L \oplus Z = Y$, i.e., Y is the direct sum of L and Z, or equivalently if the quotient vector space Y/Z has dimension n.

Definition 3.2. *Let n be a natural number. A subset X of some vector space Y is said to be* ***n-strongly dense*** *if there exists a family $\{Y^\lambda\}_{\lambda\in\Lambda}$ of vector subspaces of Y with the following properties:*

(1) *Each Y^λ has codimension less than or equal to n and $Y = \bigcup_{\lambda\in\Lambda} Y^\lambda$.*

[4]This is equivalent to saying that for each $y \in Y$ there exists some $0 < \alpha_0 < 1$ such that $(1-\alpha)x + \alpha y \in X$ for all $0 \le \alpha \le \alpha_0$. This is a very convenient alternate definition that will be employed extensively in our work.

(2) *For each* λ *the set* $Y^\lambda \setminus X$ *is algebraically nowhere dense in* Y_λ. *That is, each* $Y^\lambda \setminus X$ *has no internal points with respect to* Y^λ *and every vector of* $X \cap Y^\lambda$ *is an internal point of* $X \cap Y^\lambda$ *with respect to* Y^λ.

(3) *If* $y \in Y^\lambda \setminus X$ *holds for some* λ, *then there exists some* $x \in Y^\lambda$ *such that for all* $0 < \alpha < 1$ *we have* $\alpha y + (1-\alpha)x \in X$.

We shall say that the set X *is* **strongly dense** *in* Y *if it is* n*-strongly dense for some* n.[5]

We list a simple property of strongly dense sets.

Lemma 3.3. *If* X *is strongly dense in* Y, *then* X *is dense in* Y *for any topology on* Y *for which line segments are continuous paths.*[6] *In particular,* X *is dense in* Y *for any linear topology on* Y.

Proof. Let X be strongly dense in Y and let $\mathcal{T}$ be a topology on Y for which line segments are continuous paths. Pick $y \in Y$. For some λ we have $y \in Y^\lambda$. If $y \in X$, then we are done. So, we can assume that $y \notin X$, i.e., $y \in Y_\lambda \setminus X$. Since $Y^\lambda \setminus X$ has no internal points with respect to Y_λ, there exist some $x \in Y^\lambda$ and some $0 < \alpha_0 < 1$ such that $z_\alpha = (1-\alpha)y + \alpha x \notin Y^\lambda \setminus X$ for all $0 < \alpha < \alpha_0$. So $z_\alpha \in X$ for all $0 < \alpha < \alpha_0$ and from $z_\alpha \xrightarrow[\alpha \downarrow 0]{\mathcal{T}} y$ we get $y \in \overline{X}$, i.e., $\overline{X} = Y$. ■

With this notion of strong density, we are ready to state our first major result for the finite dimensional setting.

Theorem 3.4. *If* E *is finite dimensional and for each* i *we have* $\omega^i \gg 0$ *and* P_i *is lower hemicontinuous, then for a* $\mathfrak{J}(S-\mathfrak{J})$*-strongly dense set of assets structures in* $\mathcal{G}$ *there exists an equilibrium.*

This theorem extends the work of Duffie and Shafer (1985) to the case of unordered preferences and the results of Gale and Mas-Colell (1975) and Shafer and Sonnenschein (1975) to the model of general equilibrium with incomplete markets.

Let us now move to the infinite dimensional case. First we need to make two continuity assumptions on preferences. Recall that the typical assumption used in the study of the infinite dimensional Arrow–Debreu model is that the sets of the form $P_i^{-1}(y) = \{x \in E_+ \colon\ y \in P_i(x)\}$ (the **lower sections** of P_i) are μ-open in E_+. Such an assumption is needed to deal with the "discontinuity" of the Walrasian budget set when we endow the price space with its weak* topology and the commodity space with the topology μ; see for example Araujo (1985). In the incomplete markets setting the budget sets are more "dramatically discontinuous." Therefore, it seems that the usual continuity assumption needs strengthening. We shall achieve this goal with the help of two more conditions.

A1: If F is any finite dimensional subspace of E, then for each i the set

$$\big\{(x,y) \in E_+ \times F \colon\ y \in P_i(x)\big\}$$

is open in $(E_+, \mu) \times F$.

[5]An example of a strongly dense set is the space of nonsingular $m \times m$ matrices. We do not know the exact relationship between strongly dense and nowhere dense sets. This issue remains open even in $\mathbb{R}^2$. However, Yeneng Sun (oral communication) has indicated to us that he can establish some results in this direction.

[6]That is, for each $x, y \in Y$ the mapping $\alpha \mapsto \alpha x + (1-\alpha)y$, $0 \le \alpha \le 1$, is continuous.

Now define for each i and each state s the lower section correspondence $P_{is}^{-1}\colon E_+^s \twoheadrightarrow E_+$ by

$$P_{is}^{-1}(z) = \{x \in E_+ \colon (z, x_{-s}) \in P_i(x)\},$$

and consider the following condition.

A2: *For each i, each s and each $z \in E_+^s$, the set $P_{is}^{-1}(z)$ is μ-open in E_+.*

Some remarks concerning conditions **A1** and **A2** are in order.

- *If $u\colon E_+ \to \mathbb{R}$ is a quasi-concave and continuous utility function, then the strict preference correspondence $P\colon E_+ \twoheadrightarrow E_+$ with $P(x) = \{y \in E_+\colon\ u(y) > u(x)\}$ satisfies assumption* **A1** *for $\mu = \sigma(E, E')$.*

To see this, let F be any vector susbspace of E and suppose that some $(x, y) \in E_+ \times F$ satisfies $y \in P(x)$, i.e., $u(y) > u(x)$. Let a net $(x_\lambda, y_\lambda) \in E_+ \times F$ satisfy $(x_\lambda, y_\lambda) \xrightarrow{\sigma(E,E')\times\tau} (x, y)$. We need to show that eventually for λ large enough $u(y_\lambda) > u(x_\lambda)$.

To this end, pick $z \in E_+$ such that $u(y) > u(z) > u(x)$. Notice that he convex closed set $U = \{h \in E_+\colon\ u(h) \geq u(z)\}$ is also $\sigma(E, E')$-closed. Hence, the set $E_+ \setminus U$ is $\sigma(E, E')$-open in E_+ and since $x \in E_+ \setminus U$, it follows that $x_\lambda \in E_+ \setminus U$ for all λ eventually large. This implies $u(x_\lambda) < u(z)$ for all λ eventually large. On the other hand, from $y_\lambda \xrightarrow{\tau} y$, we see that $u(y_\lambda) > u(z)$ holds for all λ eventually large. Consequently, for all λ eventually large we have $u(y_\lambda) > u(x_\lambda)$.

- *A preference correspondence $P\colon E_+ \twoheadrightarrow E_+$ satisfies* **A1** *and* **A2** *for $\mu = \sigma(E, E')$ if one of the following holds true:*
  (a) *For each s there exists a continuous quasi-concave utility function $u_s\colon E_+^s \to \mathbb{R}$ such that $P(x) = \{y \in E_+\colon\ \sum_{s=0}^S u_s(y_s) > \sum_{s=0}^S u_s(x_s)\}$ holds for each $x \in E_+$.*
  (b) *P is the strict preference correspondence induced by a continuous utility function $u\colon E_+ \to \mathbb{R}$ and for each $z \in E_s$ the set $\{x \in E_+\colon\ u(x) \geq u(z, x_{-s})\}$ is τ-closed and convex.*
  (c) *P has a weakly open graph in $E_+ \times E_+$.*
  (d) *E is finite dimensional and P has an open graph in $E_+ \times E_+$.*

We shall indicate how one can prove the above statements

(a) Define $u\colon E_+ \to \mathbb{R}$ by $u(x) = \sum_{s=1}^S u_s(x_s)$ and note that u is quasi-concave and continuous. Therefore, by the previously proven fact, **A1** holds true.
Fix $z \in E_s^+$ and consider the τ-closed convex set $U(z) = \{x \in E_+\colon\ u(x) \geq u(z, x_{-s})\}$. It follows that $U(z)$ is $\sigma(E, E')$-closed and its complement in E_+ (which is precisely the set $P_s^{-1}(z)$) must be $\sigma(E, E')$-open in E_+. Thus, **A2** is satisfied.
(b) It follows as in (b).
(c) If (c) is true, the **A1** is clearly satisfied. Now let $\{x_\lambda\}$ be a $\sigma(E, E')$-convergent net to $x \in P_s^{-1}(z)$. For λ large enough it must be the case that $(z, (x_\lambda)_{-s}) \in P(x_\lambda)$, since $\{(z, (x_\lambda)_{-s})\}$ weakly converges to (z, x_{-s}) and $(z, x_{-s}) \in P(x)$, it follows that $x_\lambda \in P_s^{-1}(z)$ for λ large enough. This implies that $P_s^{-1}(z)$ is $\sigma(E, E')$-open in E_+, i.e., **A2** is true.
(d) Repeat the arguments in part (c).

We now state the first of our theorems for infinite dimensional commodity spaces.

Theorem 3.5. *If ω is an interior point of E_+ and conditions **A1** and **A2** are satisfied, then for a $\mathfrak{J}(S-\mathfrak{J})$-strongly dense set of assets structures in $\mathcal{G}$ there exists an equilibrium.*

Theorem 3.5 extends results in the works of Bewley (1972), Florenzano (1983), Khan (1984), Toussaint (1984) to the incomplete markets framework.

If E_+ does not have an interior point, then we need to assume some type of properness on preferences and to strengthen the assumptions on the order structure of E. The following condition on preferences is related to the very weak pointwise properness assumption studied in the literature.

> **A3**: *For each i, each s, and each $x \in E_+$, there exists an open convex cone*[a] *$C_s^{i,x} \subseteq E_s$ such that:*
> (a) $\omega_s \in C_s^{i,x}$, *and*
> (b) $z \in (x_s + C_s^{i,x}) \cap E_+^s$ *implies* $(z, x_{-s}) \in P_i(x)$.
>
> *We shall call $C_s^{i,x}$ the **cone of pointwise properness** of consumer i in state s at x.*
>
> [a] A nonempty subset C of a vector space is called an ***open convex cone*** if C is an open convex set and $x \in C$ implies $\alpha x \in C$ for all $\alpha > 0$.

Let $\mathcal{G}_\omega$ be the set of all assets structures in $\mathcal{G}$ such that for every $\theta \in M$ there exists $\alpha \geq 0$ satisfying $-\alpha\omega \leq T\theta \leq \alpha\omega$. In other words,

$$\mathcal{G}_\omega = \Big\{T \in \mathcal{G}\colon\ T(M) \subseteq E_\omega = \bigcup_{n=1}^{\infty} n[-\omega, \omega]\Big\}.$$

If we recall that the set $E_\omega = \bigcup_{n=1}^{\infty} n[-\omega, \omega]$ is called the **principal ideal** generated by ω in E, then $\mathcal{G}_\omega$ consists simply of all assets structures having ranges lying in the ideal E_ω.

Lemma 3.6. *Each $\mathcal{G}_\omega$ is a vector subspace of the space of assets structures $\mathcal{G}$.*

The next version of our major result is stated in the Riesz space framework.

Theorem 3.7. *If each E_s is a Hausdorff locally convex-solid Riesz space, $\omega \gg 0$ and conditions* **A1, A2, A3** *are satisfied, then for a $\mathfrak{J}(S-\mathfrak{J})$-strongly dense set of assets structures in $\mathcal{G}_\omega$ there exists an equilibrium.*

Theorem 3.7 extends the works of Araujo and Monteiro (1989) and Yannelis and Zame (1986) to the incomplete markets framework.

If E_+ has an interior point, then $v \gg 0$ is equivalent to saying that v is an interior point of E_+.[7] If, on the other hand, E is a locally convex-solid Riesz space, then $v \gg 0$ holds true if and only if v is a quasi-interior point, i.e., if the principal ideal E_v generated by v is τ-dense in E; see Aliprantis and Burkinshaw (1985, p. 259). Most (but not all) commodity spaces in economics have a quasi-interior point. These include the important classes of L_p-spaces and $C(\Omega)$-spaces. The basic limitation of Theorem 3.7 is that some large commodity spaces don't have quasi-interior points. For such cases we need a uniform properness type of assumption like the following one.

[7] Indeed, if $v \gg 0$ and v does not belong to the nonempty interior of the convex set E_+, then by the separation theorem there exists a nonzero $x' \in E'$ satisfying $x'(v) \leq x'(x)$ for all $x \in E_+$. This implies $0 < x' \in E'$ and $x'(v) = 0$, a contradiction.

> **A4**: *For each i and each s, there exists an open convex cone $C_s^i \subseteq E_s$ such that:*
> (a) $\omega_s \in C_s^i$, *and*
> (b) *for every* $x \in E_+$ *if* $z \in (x_s + C_s^i) \cap E_+^s$, *then* $(z, x_{-s}) \in P_i(x)$.
>
> *We call C_s^i the* **cone of uniform properness** *of consumer i in state s.*

We are now ready to state the fourth version of our major theorem.

Theorem 3.8. *If each E_s is a Hausdorff locally convex-solid Riesz space and conditions* **A1, A2, A3** *are satisfied, then for a $\mathfrak{J}(S-\mathfrak{J})$-strongly dense set of assets structures in $\mathcal{G}$ there exists an equilibrium.*

Theorem 3.8 extends the works of Aliprantis, Brown, and Burkinshaw (1987), Mas-Colell (1986), Yannelis and Zame (1986), and Jones (1987) on the existence of Walrasian equilibrium to our incomplete markets framework.

Our task now is to establish these theorems.

4. The valuation bilinear operator

We shall study the bilinear operator known as the ***valuation operator*** using the box notation of Duffie and Shafer (1985). It is the bilinear operator $\Box\colon E' \times E \to \mathbb{R}^{S+1}$ defined

$$p \,\Box\, x = \begin{bmatrix} p_0 \cdot x_0 \\ p_1 \cdot x_1 \\ \vdots \\ p_S \cdot x_S \end{bmatrix}$$

for all $p = (p_0, p_1, \ldots, p_S) \in E'$ and $x = (x_0, x_1, \ldots, x_S) \in E$.

Clearly, $(p, x) \mapsto p \,\Box\, x$ is indeed a bilinear operator, i.e., it is linear in each variable separately. Also, this bilinear operator is separately continuous (i.e., continuous in each variable) but is not in general jointly continuous when E is infinite dimensional. We adhere to the following notation regarding subsets of E. If X is a subset of E, then the image of X in $\mathbb{R}^{S+1}$ under the valuation operator will be denoted $p \,\Box\, X$, i.e.,

$$p \,\Box\, X = \{p \,\Box\, x \colon x \in X\}.$$

Also, if $p = (p_1, \ldots, p_S)$ and $x = (x_1, \ldots, x_S)$, then $p \,\Box\, x$ is the vector in $\mathbb{R}^S$ given by

$$p \,\Box\, x = \begin{bmatrix} p_1 \cdot x_1 \\ p_2 \cdot x_2 \\ \vdots \\ p_S \cdot x_S \end{bmatrix}.$$

The objective of this section is to present the basic order theoretic properties of the valuation operator. We start with its positivity properties.

Lemma 4.1. *The valuation operator enjoys that following positivity properties.*

(1) *If* $p \geq 0$ *and* $x \geq 0$, *then* $p \,\square\, x \geq 0$.
(2) *For each fixed* $p \in E'_+$ *the operator* $x \mapsto p \,\square\, x$ *is positive, i.e.,* $p \,\square\, x \geq 0$ *for each* $x \in E_+$.
(3) *If* $p \gg 0$, *then the operator* $x \mapsto p \,\square\, x$ *is strictly positive, i.e.,* $x > 0$ *implies* $p \,\square\, x > 0$.
(4) *For each fixed* $x \in E_+$ *the operator* $p \mapsto p \,\square\, x$ *is positive.*
(5) *If* $x \gg 0$, *then the operator* $p \mapsto p \,\square\, x$ *is strictly positive.*

When $p \gg 0$ the operator $x \mapsto p \,\square\, x$ has full range.

Lemma 4.2. *If* $p \gg 0$, *then* $x \mapsto p \,\square\, x$ *carries* E_+ *onto* $\mathbb{R}^{S+1}_+$ *(and so it is a surjective operator).*

Proof. Let $p = (p_0, p_1 \ldots, p_s)$ be strictly positive. This is equivalent to saying that each p_s is strictly positive. Let $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_S) \in \mathbb{R}^{S+1}$. Put $x_s = \frac{\alpha_s}{p_s \cdot \omega_s^1} \omega_s^1 \in E_s^+$ for each s and notice that if $x = (x_0, x_1, \ldots, x_S)$, then $p \,\square\, x = \alpha$. ■

Lemma 4.3. *If* $p \gg 0$, *then for each* J*-dimensional subspace* L *of* $\mathbb{R}^{S+1}$ *the subspace*

$$H = \{x \in E\colon\ p \,\square\, x \in L\}.$$

has codimension $S + 1 - J$.

Proof. It suffices to show that there exists a $(S - J + 1)$-dimensional vector subspace F of E such that $E = H \oplus F$. To this end, pick a basis $\{r_1, \ldots, r_J\}$ of L and then select vectors $r_{J+1}, \ldots, r_{S+1}$ such that $\{r_1, \ldots, r_J, r_{J+1}, \ldots, r_{S+1}\}$ is a basis for $\mathbb{R}^{S+1}$. Since the valuation operator $x \mapsto p \,\square\, x$ is surjective, for each j there exists some $x_j \in E$ such that $p \,\square\, x_j = r_j$. Clearly, the vectors $x_{J+1}, \ldots, x_{S+1}$ are linearly independent and so the vector subspace $F = \text{Span}\{x_{J+1}, \ldots, x_{S+1}\}$ of E has dimension $S - J + 1$. We shall finish the proof by showing that $E = H \oplus F$.

To this end, note first that if $W = \text{Span}\{r_{J+1}, \ldots, r_{S+1}\}$ in R^{S+1}, then $\mathbb{R}^{S+1} = L \oplus W$. Now let $x \in E$. Pick $y \in L$ and $w \in W$ such that $p \,\square\, x = y + w$. Choose some $f \in F$ such that $p \,\square\, f = w$ and note that the vector $v = x - f$ satisfies $p \,\square\, v = p \,\square\, x - p \,\square\, f = p \,\square\, x - w = y \in L$, i.e, $v \in H$. So $x = v + f \in H + F$. This shows that $E = H + F$.

To see that $H \cap F = \{0\}$, let $z \in H \cap F$. Then $p \,\square\, z \in L \cap W = \{0\}$, i.e., $p \,\square\, z = 0$. Since $\square\colon F \to W$ is a linear isomoprhism, we see that $z = 0$. Thus, $H \cap F = \{0\}$ and hence $E = H \oplus F$. Consequently, H has codimension $S - J + 1$. ■

Lemma 4.4. *Assume that* $\{x^1, x^2, \ldots, x^\ell\}$ *and* $\{y^1, y^2, \ldots, y^\ell\}$ *are collections of vectors in* E. *If for some* $p \in E'$ *the collection of vectors* $\{p \,\square\, x^1, p \,\square\, x^2, \ldots, p \,\square\, x^\ell\}$ *is linearly independent, then there exists* $0 < \alpha_0 < 1$ *such that for each* $\alpha_0 \leq \alpha \leq 1$ *the collection*

$$\{p \,\square\, (\alpha x^1 + (1 - \alpha) y^1), p \,\square\, (\alpha x^2 + (1 - \alpha) y^2), \ldots, p \,\square\, (\alpha x^\ell + (1 - \alpha) y^\ell)\}$$

is linearly independent.

Proof. Assume by way of contradiction that this is false. This means that there exists a sequence $\{\alpha_n\}$ with $0 < \alpha_n \uparrow 1$ and vectors $\lambda_n \in \mathbb{R}^m$ satisfying $\|\lambda_n\| = 1$ for each n, such that

$$\sum_{i=1}^{\ell} \lambda_n^i \big[p \,\square\, (\alpha_n x^i + (1 - \alpha_n) y^i)\big] = 0\,. \qquad (\star)$$

We can assume that $\lambda_n \to \lambda$ with $\|\lambda\| = 1$. Letting $n \to \infty$ in $(\star)$ yields $\sum_{i=1}^{\ell} \lambda^i [p \,\square\, x^i] = 0$, which is a contradiction. ■

Recall that if $R\colon X \to Y$ is an arbitrary linear operator between two vector spaces and the vectors $Rx^1, Rx^2, \ldots, Rx^\ell$ are linearly independent, then $x^1, x^2, \ldots, x^\ell$ are linearly independent. From this observation and the preceding result we get the following.

Corollary 4.5. *Assume that* $X = \{x^1, x^2, \ldots, x^\ell\}$ *and* $Y = \{y^1, y^2, \ldots, y^\ell\}$ *are collections of vectors in* E. *If* X *is a linearly independent set, then there exists some* $0 < \alpha_0 < 1$ *such that for each* $\alpha_0 \le \alpha \le 1$ *the collection*

$$\{\alpha x^1 + (1-\alpha)y^1, \alpha x^2 + (1-\alpha)y^2, \ldots, \alpha x^\ell + (1-\alpha)y^\ell\}$$

is linearly independent.

The rest of the section is devoted to several important convergence properties of the valuation operator. These properties will be used in the proofs of the major theorems. To do this, we need the notion of the Grassmanian manifold $\mathbb{G}_J$, which is the manifold of all J-dimensional vector subspaces of $\mathbb{R}^{S+1}$; for details see the Appendix B.

Lemma 4.6. *Regarding the valuation operator and the Grassmanian we have the following.*

(a) *If* E *is finite dimensional, then the correspondence* $(p, L) \mapsto\!\!\!\to \{x \in E\colon\ p\,\square\, x \in L\}$, *from* $E' \times \mathbb{G}_J$ *to* E, *has a closed graph.*

(b) *Let* $(p^\lambda, u^\lambda, L^\lambda) \xrightarrow[\lambda]{} (p, u, L)$ *hold in* $(E', \sigma(E', E)) \times (E, \tau^*) \times \mathbb{G}_J$, *where* $p \gg 0$ *and* τ^* *is an arbitrary linear topology on* E. *If* $p\,\square\,(x-u) \in L$ *for some* $x \in E$, *then there exists a net* $\{x^\lambda\}$ *of* E *satisfying* $x^\lambda \xrightarrow{\tau^*} x$ *and* $p^\lambda\,\square\,(x^\lambda - u^\lambda) \in L^\lambda$ *eventually for each* λ.

Moreover, the net $\{x^\lambda\}$ *can be chosen to lie in the linear span of the vectors*

$$\{x, u, (\omega_0, 0_{-0}), (\omega_1, 0_{-1}), \ldots, (\omega_S, 0_{-S})\} \cup \{u^\lambda\},$$

where

$$\omega = \sum_{i=1}^m \omega^i = \Big(\sum_{i=1}^m \omega_0^i, \sum_{i=1}^m \omega_1^i, \ldots, \sum_{i=1}^m \omega_S^i\Big) = (\omega_0, \omega_1, \ldots, \omega_S) \in E_+,$$

is the total endowment of our economy.

Proof. (a) Notice that all spaces are metrizable. Assume that $(p^n, L^n, x^n) \to (p, L, x)$ holds true in $E' \times \mathbb{G}_J \times E$ and that $p^n\,\square\, x^n \in L^n$ for all n. Since the valuation operation on any finite dimensional vector space is jointly continuous, it follows that $p^n\,\square\, x^n \to p\,\square\, x$. Now using that the correspondence $F \mapsto\!\!\!\to F$, from $\mathbb{G}_J$ to $\mathbb{R}^{S+1}$ has a closed graph (see Lemma B.5 in Appendix B), we infer that $p\,\square\, x \in L$.

(b) Since $L^\lambda \to L$ in $\mathbb{G}_J$ it follows from Lemma B.5 (2) of the Appendix B that there exists a net $\{r^\lambda\}$ satisfying $r^\lambda \in L^\lambda$ for each λ and $r^\lambda \to p\,\square\,(x-u) = r = (r_0, r_1, \ldots, r_S) \in L \subseteq \mathbb{R}^{S+1}$. Since $p \gg 0$ it must be the case that $p\,\square\,\omega \gg 0$. Therefore, by truncating the net, we can assume that $p^\lambda\,\square\,\omega \gg 0$ for all λ. Now define the vectors $y^\lambda \in E$ by

$$y^\lambda = \Big(\frac{r_0^\lambda - p_0^\lambda \cdot (x_0 - u_0)}{p_0^\lambda \cdot \omega_0}\omega_0, \frac{r_1^\lambda - p_1^\lambda \cdot (x_1 - u_1)}{p_1^\lambda \cdot \omega_1}\omega_1, \ldots, \frac{r_S^\lambda - p_S^\lambda \cdot (x_S - u_S)}{p_S^\lambda \cdot \omega_S}\omega_S\Big).$$

Clearly, $p^\lambda\,\square\, y^\lambda = r^\lambda - p^\lambda\,\square\,(x-u)$. From $p^\lambda\,\square\,\omega \to p\,\square\,\omega \gg 0$, it follows that $y^\lambda \to 0$ in E with respect to any linear topology on E.

Letting $x^\lambda = y^\lambda + x - u + u^\lambda \in E$, we see that $x^\lambda \xrightarrow{\tau^*} x$ and

$$p^\lambda\,\square\,(x^\lambda - u^\lambda) = p^\lambda\,\square\,(y^\lambda + x - u) = p^\lambda\,\square\, y^\lambda + p^\lambda\,\square\,(x-u) = r^\lambda \in L^\lambda.$$

Hence, the net $\{x^\lambda\}$ has the desired properties.

Finally, the net $\{y^\lambda\}$ lies in the finite dimensional space spanned by $(\omega_s, 0_{-s})_{s=0}^S$ and therefore $\{x^\lambda\}$ lies in the linear span of

$$\{x, u, (\omega_0, 0_{-0}), (\omega_1, 0_{-1}), \ldots, (\omega_S, 0_{-S})\} \cup \{u^\lambda\},$$

and the proof is finished. ■

The lack of joint continuity of the valuation map in infinite dimensional spaces poses a problem. That is, if $(x^\lambda, p^\lambda) \to (x, p)$ holds in $E \times (E', \sigma(E', E))$, then it does not necessarily follow that $p^\lambda \Box x^\lambda \to p \Box x$. Fortunately, the situation is better for some well behaved nets.

Lemma 4.7. *Assume that a net $\{(x^\lambda, p^\lambda)\} \subseteq E \times E'$ satisfies $p_\lambda \xrightarrow{\sigma(E',E)} p$ and $p^\lambda \Box x^\lambda \to r$ in $\mathbb{R}^{S+1}$, and let $e \in E$. If for every $0 < \alpha < 1$ and every state s there exists λ_0 (depending on α and s) such that $\lambda \geq \lambda_0$ implies $p_s^\lambda \cdot (x_s + \alpha e_s) > p_s^\lambda \cdot x_s^\lambda$, then $p \Box x \geq r$.*

Proof. Fix a state s and $0 < \alpha < 1$. Pick λ_0 such that $p_s^\lambda \cdot (x_s + \alpha e_s) > p_s^\lambda \cdot x_s^\lambda$, for all $\lambda \geq \lambda_0$. Since $p_s^\lambda \cdot (x_s + \alpha e_s) \xrightarrow[\lambda]{} p_s \cdot (x_s + \alpha e_s)$ and by assumption $p_s^\lambda \cdot x_s^\lambda \xrightarrow[\lambda]{} r_s$, we see that $p_s \cdot (x_s + \alpha e_s) \geq r_s$ for all s and all $0 < \alpha < 1$. Letting $\alpha \downarrow 0$, we get $p_s \cdot x_s \geq r_s$ for all s. That is, $p \Box x \geq r$. ■

We continue with more convergence properties of the valuation operator.

Lemma 4.8. *Assume that $((x^{i,\lambda})_{i=1}^m, (u^{i,\lambda})_{i=1}^m, p^\lambda, L^\lambda) \xrightarrow[\lambda]{} ((x^i)_{i=1}^m, (u^i)_{i=1}^m, p, L)$ holds in the space $(E_+, \mu)^m \times (E_+)^m \times (E'_+, \sigma(E', E)) \times \mathbb{G}_J$, and let $e \in E$. Suppose also that:*

1. *For each λ, s, and i, we have $p_s^\lambda \cdot u_s^{i,\lambda} > 0$.*
2. *For each i the convergence $u^{i,\lambda} \xrightarrow[\lambda]{} u^i$ takes place in a finite dimensional vector space.*
3. *There exists some $u^* \in E_+$ satisfying $\sum_{i=1}^m u^{i,\lambda} \leq u^*$ for each λ.*
4. *For each λ and each i we have $p^\lambda \Box (x^{i,\lambda} - u^{i,\lambda}) \in L^\lambda$.*
5. *For each λ we have $\sum_{i=1}^m x^{i,\lambda} = \sum_{i=1}^m u^{i,\lambda}$.*
6. *For each i each $0 < \alpha < 1$ and all s, there exists some index λ_0 (depending on α and s) such that $\lambda \geq \lambda_0$ implies $p_s^\lambda \cdot (x_s^i + \alpha e_s) > p_s^\lambda \cdot x_s^{i,\lambda}$.*

Then $p^\lambda \Box (x^{i,\lambda} - u^{i,\lambda}) \to p \Box (x^i - u^i)$ and $p \Box (x^i - u^i) \in L$ for each i.

Proof. From (5) and the fact that μ is a linear topology, we see that $\sum_{i=1}^m x^i = \sum_{i=1}^m u^i = u \in E$. From (5) and (3) we get $\sum_{i=1}^m x^{i,\lambda} \leq u^*$. Using that $p^\lambda \geq 0$, it follows that $p^\lambda \Box x^{i,\lambda} \leq p^\lambda \Box u^*$ for all λ and all i, i.e., $p_s^\lambda \cdot x_s^{i,\lambda} \leq p_s^\lambda \cdot u_s^*$. Also, from (1) (3) (5) we see that $0 < p_s^\lambda \cdot u_s^*$. Hence,

$$0 \leq \frac{p_s^\lambda \cdot x_s^{i,\lambda}}{p_s^\lambda \cdot u_s^*} \leq 1 \tag{†}$$

for all λ, i, and s. Now for each λ and i consider the vector $z^{i,\lambda} \in E_+$ defined by

$$z^{i,\lambda} = \Big(\frac{p_0^\lambda \cdot x_0^{i,\lambda}}{p_0^\lambda \cdot u_0^*} u_0^*, \frac{p_1^\lambda \cdot x_1^{i,\lambda}}{p_1^\lambda \cdot u_1^*} u_1^*, \ldots, \frac{p_S^\lambda \cdot x_S^{i,\lambda}}{p_S^\lambda \cdot u_S^*} u_S^*\Big).$$

Clearly, $p^\lambda \Box z^{i,\lambda} = p^\lambda \Box x^{i,\lambda}$. Moreover, the vectors $z^{i,\lambda}$ lie in the finite dimensional vector subspace space V of E spanned by the set of vectors

$$\{(u_0^*, 0_{-0}), (u_1^*, 0_{-1}), \ldots, (u_S^*, 0_{-S})\}.$$

From (†) and the definition of the $z^{i,\lambda}$, we see that $0 \leq z^{i,\lambda} \leq u^*$. Now, from $z^{i,\lambda} \in [0, u^*] \cap V$ and the fact that order intervals of E are τ-bounded and τ-closed, it follows that $\{z^{i,\lambda}\}$ lies in the closed and bounded subset $[0, u^*] \cap V$ of the finite dimensional vector space V. Consequently, for each i the net $\{z^{i,\lambda}\}$ has a cluster vector $z^i \in [0, u^*] \cap V$. Clearly, the net $\{p^\lambda \Box z^{i,\lambda}\}$ has $r^i = p \Box z^i$ as a cluster point in $\mathbb{R}^{S+1}$. Since $p^\lambda \Box z^{i,\lambda} = p^\lambda \Box x^{i,\lambda}$, we see that r^i is also a cluster point of the net $\{p^\lambda \Box x^{i,\lambda}\}$. So, using (6), it follows from Lemma 4.7 that for all i we have

$$p \Box x_i \geq r^i \,. \tag{$\star$}$$

Since $p^\lambda \Box z^{i,\lambda} = p^\lambda \Box x^{i,\lambda}$, we see that

$$\sum_{i=1}^{m} p^\lambda \Box z^{i,\lambda} = \sum_{i=1}^{m} p^\lambda \Box x^{i,\lambda} = \sum_{i=1}^{m} p^\lambda \Box u^{i,\lambda}$$

holds for all λ. Now according to (2) the convergence $u^{i,\lambda} \xrightarrow[\lambda]{} u^i$ takes place in a finite dimensional vector space. This implies $\sum_{i=1}^{m} p^\lambda \Box u^{i,\lambda} \xrightarrow[\lambda]{} \sum_{i=1}^{m} p^\lambda \Box u^i = p \Box u$. Therefore,

$$\sum_{i=1}^{m} r^i = p \Box u = \sum_{i=1}^{m} p \Box x_i \,. \tag{$\star\star$}$$

From ($\star$) and ($\star\star$), it follows that $p \Box x_i = r^i$ for each i. In particular, r^i is a unique cluster point of the net $\{p^\lambda \Box x^{i,\lambda}\}$ and $p^\lambda \Box x^{i,\lambda} \xrightarrow[\lambda]{} p \Box x^i$ for each i. Finally, since $p^\lambda \Box (x^{i,\lambda} - u^{i,\lambda}) \in L^\lambda$, it must be the case by Lemma 4.6 (and the fact that $\{z^{i,\lambda}\}$ and $\{u^{i,\lambda}\}$ both converge in finite dimensional vector spaces and $p^\lambda \Box z^{i,\lambda} = p^\lambda \Box x^{i,\lambda}$) that $p \Box (x^i - u^i) \in L$. ■

We conclude this section with an important supporting property of the valuation operator.

Lemma 4.9. *Assume that E_+ has internal points and fix $L \in \mathbb{G}_J$. Suppose that there exist some i, some $v > 0$, and some $p \gg 0$ such that:*

(a) $p \Box v \in L$.

(b) *For each internal vector $y \in E_+$ satisfying $y \in P_i(x)$ and $p \Box (y - \omega^i) \in L$ we have $p \cdot y \geq p \cdot \omega^i$.*

If $y \in P_i(x)$ with $p \Box (y - \omega^i) \in L$, then $p \cdot y > p \cdot \omega^i$.

Proof. Let $y \in E_+$ satisfy $y \in P_i(x)$ and $p \Box (y - \omega^i) \in L$. For each $0 < \lambda < 1$ define the vector $z^\lambda = \lambda y + (1-\lambda)\omega^i \in E_+$. Clearly, $p \Box (z^\lambda - \omega^i) \in L$ for each λ. Furthermore, $p \Box z^\lambda \gg 0$ since $p \Box \omega^i \gg 0$. For each $0 < \lambda < 1$ let $h^\lambda = \lambda z^\lambda + (1-\lambda)e$, where e is an internal point of E_+. Notice that each h^λ is also an internal point of E_+. Also, for each λ define the vector

$$y^\lambda = \Big(\frac{p_0 \cdot z_0^\lambda}{p_0 \cdot h_0^\lambda}\, h_0^\lambda, \frac{p_1 \cdot z_1^\lambda}{p_1 \cdot h_1^\lambda}\, h_1^\lambda, \ldots, \frac{p_S \cdot z_S^\lambda}{p_S \cdot h_S^\lambda}\, h_S^\lambda \Big) \,.$$

Since $p \Box z^\lambda \gg 0$ and $p \Box h^\lambda \gg 0$, it follows that the vectors y^λ are all internal points of E_+. They also satisfy $y^\lambda \xrightarrow[\lambda \uparrow 1]{} y$ with respect to any linear topology on E. Since $y \in P_i(x)$ there is $0 < \lambda^* < 1$ such that $\lambda^* \leq \lambda < 1$ implies $y^\lambda \in P_i(x)$. Furthermore, from $p \Box (y^\lambda - \omega^i) = p \Box (z^\lambda - \omega^i) \in L$ and our hypothesis (b) we get $p \cdot y^\lambda \geq p \cdot \omega^i$ for $\lambda^* \leq \lambda < 1$. This implies $p \cdot y \geq p \cdot \omega^i$. Thus, we have shown up to this point that:

$$y \in P_i(x) \text{ and } p \Box (y - \omega^i) \in L \implies p \cdot y \geq p \cdot \omega^i \,. \tag{$\diamond$}$$

Assume by way of contradiction that for some $y \in P_i(x)$ with $p \Box (y - \omega^i) \in L$ we have $p \cdot y = p \cdot \omega^i$. Next, consider the internal point u of E_+ defined by

$$u = \Big(\frac{p_0 \cdot \omega_0^i}{p_0 \cdot y_0^{\lambda^*}}\, y_0^{\lambda^*}, \frac{p_1 \cdot \omega_1^i}{p_1 \cdot y_1^{\lambda^*}}\, y_1^{\lambda^*}, \ldots, \frac{p_S \cdot \omega_S^i}{p_S \cdot y_S^{\lambda^*}}\, y_S^{\lambda^*} \Big) \,.$$

Notice that $p\Box(u-\omega^i)=0$ and that u is an internal point of E_+, since y^{λ^*} is an internal point of E_+. In particular, $p\Box(u-\omega^i)\in L$ and $p\cdot u=p\cdot\omega^i$. Take $v>0$ with $p\Box v\in L$ and for each $0<\alpha<1$ let $g^\alpha=\alpha u-(1-\alpha)(v+\omega^i)$. Since u is an internal point of E_+, there exists some $0<\alpha^*<1$ such that $g^{\alpha^*}\in E_+$. Furthermore,

$$p\Box(g^{\alpha^*}-\omega^i)=p\Box(\alpha^*u-\alpha^*\omega^i-(1-\alpha^*)(v+\omega^i-\omega^i))=-p\Box((1-\alpha^*)v)\in L\,,$$

and $p\cdot(g^{\alpha^*}-\omega^i)=p\cdot(1-\alpha^*)(-v)<0$. The latter implies $p\cdot g^{\alpha^*}<p\cdot\omega^i$.

Finally, for $0<\delta<1$ close enough to zero it must be the case $\delta g^{\alpha^*}+(1-\delta)y\in P_i(x)$. But then we have $p\Box(\delta g^{\alpha^*}+(1-\delta)y-\omega^i)\in L$ and $p\cdot(\delta g^{\alpha^*}+(1-\delta)y)<p\cdot\omega^i$, contrary to ($\diamond$). ■

5. Pseudo-equilibrium

We start by noticing that the marketed space $T(M)$ need not contain any nonzero vectors in the Edgeworth box $[0,\omega]$. This geometrical deficiency creates several serious technical problems. Therefore, we need to extend the marketed space by adding to it some nonzero vectors from the Edgeworth box. We do this in the next definition.

Definition 5.1. *An* ***extended marketed space*** *is any vector subspace $\mathcal{M}$ of E of dimension $J\le S$ such that there exists a vector $v\in\mathcal{M}$ satisfying $0<v\le\omega$; i.e., $\mathcal{M}\cap(0,\omega]\ne\emptyset$.*[8]

For the rest of the paper we shall fix an extended marketed space $\mathcal{M}$. We find it convenient to modify the notion of non-arbitrage equilibrium studied in Husseini, Larsy, and Magill (1990). For each $p\in E'_+$, let

$$B_i(p) \;=\; \left\{x\in E_+:\quad \begin{array}{ll} p\cdot x\le p\cdot\omega^i & \text{if } i=1\\ p\cdot x\le p\cdot\omega^i \;\;\&\;\; p\Box(x-\omega^i)\in p\Box\mathcal{M} & \text{if } i>1\end{array}\right\}.$$

Notice that the preceding definition indicates that for some consumer i (we let $i=1$ without loss of generality) her budget set $B_i(p)$ coincides with the Walrasian budget set[9] and for the remaining consumers their budget sets are subsets of their Walrasian budget sets.

Definition 5.2. *A* ***non-arbitrage equilibrium*** *is a pair (p,x) such that $p=(p_0,p_1,\dots,p_S)$ is a nonzero spot price (i.e., $0<p\in E'_+$) and $x=(x^1,x^2,\dots,x^m)$ is an allocation satisfying*

$$x^i\in B_i(p)\quad\text{and}\quad P_i(x^i)\cap B_i(p)=\emptyset$$

for each consumer i. The price p will be referred to as a spot price that ***supports*** *the allocation as a non-arbitrage equilibrium.*

Spot prices that support allocations as non-arbitrage equilibrium are strictly positive.

Lemma 5.3. *If (p,x) is non-arbitrage equilibrium, then:*

1. *the supporting spot price p is strictly positive, i.e., $p\gg 0$, and*
2. *for each consumer i we have $p\cdot x^i=p\cdot\omega^i$.*

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[8]The reader should note that we denote the dimension of the portfolio space M by $\mathfrak{J}$ and the dimension of the extended marketed space $\mathcal{M}$ by J. These dimensions need not be the same, i.e., it may happen that $\mathfrak{J}\ne J$.

[9]We can call this consumer the "Walrasian consumer."

Proof. Assume that (p, x) is non-arbitrage equilibrium.

(1) If $p = (p_0, p_1, \ldots, p_S)$ is not strictly positive, then for some state s^* the price p_{s^*} is not strictly positive. That is, there exists some vector $0 < v \in E_{s^*}$ such that $p_{s^*} \cdot v = 0$. Now if we let $u = (v, 0_{-s^*}) \in E_+$, then note that $u > 0$ and $x^i + u \in P_i(x^i) \cap B_i(p)$ for each i, which is impossible.

(2) From the definition of equilibrium, we get $p \cdot x^i \leq p \cdot \omega^i$ for each i. To see that equality holds, note that $\sum_{i=1}^m p \cdot x^i = \sum_{i=1}^m p \cdot \omega^i = p \cdot \omega$. ■

Every non-arbitrage equilibrium induces an equilibrium.

Lemma 5.4. *If $\mathcal{M} = \mathcal{M}_0 \times T(M)$, where $\mathcal{M}_0$ is a subspace of E_0 satisfying $\mathcal{M} \cap E_0^+ \neq \{0\}$, then for each non-arbitrage equilibrium (p, x) there exists an assets price q and a portfolio trade $\boldsymbol{\theta}$ such that $(p, q, x, \boldsymbol{\theta})$ is an equilibrium.*

Proof. Let (p, x) be a non-arbitrage equilibrium, where $x = (x^1, \ldots, x^m)$ is an allocation. By Lemma 5.3 we know that $p \gg 0$.

For $i = 2, \ldots, m$ choose some $z^i \in T(M)$ such that $p_s \cdot (x_s^i - \omega_s^i) = p_s \cdot z_s^i$ for each $s = 1, \ldots, S$. Put $z^1 = -\sum_{i=2}^m z^i \in T(M)$. Next, for each $i \geq 2$ choose a portfolio $\theta_i \in M$ such that $T\theta_i = z^i$ and let $\theta_1 = -\sum_{i=2}^m \theta_i$. Clearly, $T\theta_1 = z^1$ and $\sum_{i=1}^m \theta_i = 0$. In particular, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m)$ is portfolio trade. Next, define the assets price $q\colon M \to \mathbb{R}$ (i.e., a linear functional $q \in M'$) by letting

$$q \cdot \theta = \sum_{s=1}^S p_s \cdot [T\theta(s)]$$

for each $\theta \in M$. (Keep in mind that since M is finite dimensional, every linear functional on M is continuous.) We claim that $(x, p, q, \boldsymbol{\theta})$ is an equilibrium. For this, for each i we must show that:

(a) $x^i \in \beta_i(p, q, \theta_i)$, and
(b) $P_i(x^i) \cap \beta_i(p, q) = \emptyset$.

We shall verify the validity of (a) first. Start by observing that for each $s = 1, \ldots, S$ and all $i \geq 2$ we have

$$p_s \cdot (x_s^i - \omega_s^i) = p_s \cdot z_s^i = p_s \cdot [T\theta_i(s)]\,.$$

For $i = 1$ and each $s = 1, \ldots, m$ we have

$$p_s \cdot (x_s^1 - \omega_s^1) = -p_s \cdot \Big[\sum_{i=2}^m (x_s^i - \omega_s^i)\Big] = -p_s \cdot \Big(\sum_{i=2}^m z_s^i\Big) = p_s \cdot z_s^1 = p_s \cdot [T\theta_1(s)]\,.$$

Also, for $s = 0$ and each consumer i, part (2) of Lemma 5.3 yields

$$\begin{aligned} p_0 \cdot (x_0^i - \omega_0^i) &= p_0 \cdot (x_0^i - \omega_0^i) + \sum_{s=1}^S p_s \cdot (x_s^i - \omega_s^i) - \sum_{s=1}^S p_s \cdot (x_s^i - \omega_s^i) \\ &= p \cdot (x^i - \omega^i) - \sum_{s=1}^S p_s \cdot (x_s^i - \omega_s^i) = -\sum_{s=1}^S p_s \cdot (x_s^i - \omega_s^i) \\ &= -\sum_{s=1}^S p_s \cdot [T\theta_i(s)] = -q \cdot \theta_i\,. \end{aligned}$$

The above show that $x^i \in \beta_i(p, q, \theta_i)$ and the validity of (a) has been established.

Now fix $z = (z_0, z_1, \ldots, z_S) \in E_+$ and assume that $z \in \beta_i(p, q, \theta)$ for some portfolio $\theta \in M$ and some consumer i. To prove (b), it suffices to show that $z \notin P(x^i)$.

To this end, notice first that $z \in \beta_i(p, q, \theta)$ simply means that for all consumers i we have

$$p_0 \cdot z_0 \leq p_0 \cdot \omega_0^i - q \cdot \theta \quad \text{and} \quad p_s \cdot z_s \leq p_s \cdot \omega_s^i + p_s \cdot [T\theta(s)] \text{ for all } s = 0, 1 \ldots, S\,.$$

Since each p_s is strictly positive, for every state $s \in \mathcal{S}$ there exists some $u_s \in E_s^+$ such that

$$p_0 \cdot z_0 + p_0 \cdot u_0 = p_0 \cdot \omega_0^i - q \cdot \theta \quad \text{and} \quad p_s \cdot z_s + p_s \cdot u_s = p_s \cdot \omega_s^i + p_s \cdot [T\theta(s)]\,.$$

Let $y = z + u \in E_+$. From our assumption $\mathcal{M} \cap E_0^+ \neq \{0\}$, we can pick some $v \in \mathcal{M}_0$ such that $p_0 \cdot v = -q \cdot \theta$. Now note that

$$\begin{aligned} p \,\square\, (y - \omega^i) &= (p_0 \cdot v, p_1 \cdot (z_1 + u_1 - \omega_1^i), \ldots, p_S \cdot (z_S + u_S - \omega_S^i)) \\ &= p \,\square\, (v, T\theta) \in p \,\square\, (\mathcal{M}_0 \times T(M))\,. \end{aligned}$$

In addition, observe that

$$p \cdot (y - \omega^i) = \sum_{s=0}^{S} p_s \cdot (z_s + u_s - \omega_s^i) = -q \cdot \eta + \sum_{s=1}^{S} p_s \cdot [T\theta(s)] = 0\,.$$

Consequently, $y \in B_i(p)$ and since $P_i(x^i) \cap B_i(p) = \emptyset$, it follows that $y \notin P_i(x^i)$. Finally, from $y \geq z$ and the comprehensiveness property **AIII**(c), we see that $z \notin P_i(x^i)$, and the proof is finished. ■

Our next goal is to introduce the notion of a pseudo-equilibrium in a general framework. To achieve this, we need to introduce some new budget sets. For each $(p, L) \in E'_+ \times \mathbb{G}_J$ let

$$B_i(p, L) \;=\; \left\{ x \in E_+ : \begin{array}{cc} p \cdot x \leq p \cdot \omega^i & \text{if } i = 1 \\ p \cdot x \leq p \cdot \omega^i \;\; \& \;\; p \,\square\, (x - \omega^i) \in L & \text{if } i > 1 \end{array} \right\}.$$

Again, one of the consumers is "Walrasian" and the rest have budget sets that are subsets of their Walrasian budget sets.

Definition 5.5. *A* ***pseudo-equilibrium*** *is a triplet (p, x, L), where $p \in E'_+$, x is an allocation, L is a J-dimensional subspace of $\mathbb{R}^{S+1}$ (where J is the dimension of $\mathcal{M}$), such that:*

1. *For each consumer i we have $x^i \in B_i(p, L)$ and $P_i(x^i) \cap B_i(p, L) = \emptyset$.*
2. *$p \,\square\, \mathcal{M} \subseteq L$.*

A pseudo-equilibrium (p, x, L) is called ***full*** *if equality holds in (2), i.e., if $p \,\square\, \mathcal{M} = L$.*

We list an easy result.

Lemma 5.6. *If (p, x, L) is a full pseudo-equilibrium, then (p, x) is a non-arbitrage equilibrium.*

Proof. If $L = p \,\square\, \mathcal{M}$, then $B_i(p) = \beta_i(p, L)$. ■

The next property of pseudo-equilibria is also simple but very useful.

Lemma 5.7. *If (x, p, L) is a pseudo-equilibrium, then $z \in P_i(x^i)$ implies $p \,\square\, z \not\leq p \,\square\, x^i$.*

Proof. Suppose by way of contradiction that $p\,\Box\,(x^i - z) \geq 0$. We know that $p \gg 0$ and that $y \mapsto p\,\Box\,y$ maps E_+ onto $\mathbb{R}^{S+1}_+$. So we can pick a point $y \in E_+$ such that $p\,\Box\,y = p\,\Box\,(x^i - z)$. The monotonicity of preferences yields $y + z \in P_i(x^i)$ and we know that $p\,\Box\,x^i - p\,\Box\,(y + z) = 0$. Therefore, $p\,\Box\,(y + z - \omega^i) = p\,\Box\,(x^i - \omega^i) \in L$ and $p \cdot (y + z) = p \cdot x^i$, imply $y + z \in B_i(p, L)$, which is impossible. ■

The next two lemmas show that our major Theorems 3.4, 3.5, 3.7, and 3.8 hold true if we can establish the existence of a pseudo-equilibrium.

Lemma 5.8. *If (p, x, L) is a pseudo-equilibrium and $Y = \{y^1, \ldots, y^J\} \subseteq \mathcal{M}$ (where J is the dimension of $\mathcal{M}$), then there exists a linearly independent set $Z = \{z^1, \ldots, z^J\}$ of vectors in E such that for each $0 < \alpha < 1$ the triplet (p, x, L) is a full pseudo-equilibrium for the marketed space $\mathcal{M}_\alpha$ generated by*

$$\{\alpha y^1 + (1-\alpha)z^1, \ldots, \alpha y^J + (1-\alpha)z^J\}.$$

Moreover, if $y^1 = (v, 0_{-1})$ with $v > 0$ and $y^j_1 = 0$ for all $j > 1$, then Z can be chosen so that $z^1 = y^1$.

Proof. Since (p, x, L) is a pseudo-equilibrium, it must be the case that $p \gg 0$. Therefore, $x \mapsto p\,\Box\,x$ is surjective. Let

$$K = \mathrm{Span}\{p\,\Box\,y^1, p\,\Box\,y^2, \ldots, p\,\Box\,y^J\}.$$

Note that $K \subseteq L$ and if $k = \dim K$, then $k \leq J$. If $k = J$, then we are done since it must be that $K = L$ (in this case let $z^j = y^j$).

So suppose that $k < J$. We can assume without loss of generality that $\{p\,\Box\,y^1, p\,\Box\,y^2, \ldots, p\,\Box\,y^k\}$ is a basis of K.

For each $j = 1, \ldots, k$, let $h^j = p\,\Box\,y^j$. Write $L = L_K \oplus K$ and pick $J - k$ linearly independent points $\{h^{k+1}, \ldots, h^J\}$ in L_K. Clearly, the set of vectors $H = \{h^1, \ldots, h^J\}$ is linearly independent and is a basis for L. Furthermore, for each $0 < \alpha < 1$ the collection of vectors

$$\{p\,\Box\,(\alpha y^j + (1-\alpha)h^j) \colon j = 1, \ldots, J\}$$

remains a basis for L.

For $j = 1, \ldots, k$, let $z^j = y^j$. For $j > k$ pick $z^j \in E$ such that $p\,\Box\,z^j = h^j$ (This is possible since $x \mapsto p\,\Box\,x$ is surjective). Clearly, $Z = \{z^1, \ldots, z^J\}$ is linearly independent since $x \mapsto p\,\Box\,x$ is linear. Letting, $u^j = \alpha y^j + (1-\alpha)z^j$, we see that $p\,\Box\,u^j = p\,\Box\,(\alpha y^j + (1-\alpha)h^j)$. Thus, $\{p\,\Box\,u^1, \ldots, p\,\Box\,u^J\}$ spans L. This implies that (p, x, L) is a full pseudo-equilibrium for the marketed space $\mathcal{M}_\alpha = \mathrm{Span}\{u^1, \ldots, u^J\}$. For the proof of the last part repeat the arguments in the proof of Lemma 5.4. ■

Lemma 5.9. *If for every extended marketed space $\mathcal{M}$ there exists a pseudo-equilibrium, then for a $\mathfrak{J}(S - \mathfrak{J})$-strongly dense set of assets structures in $\mathcal{G}$ there exists an equilibrium.*

Proof. Let $T = (T_1, \ldots, T_{\mathfrak{J}})$ be an arbitrary assets structure, where $T_i \in \prod_{s=1}^S E_s$; see the discussion following Lemma 2.2.

Choose a $\mathfrak{J}$-dimensional subspace $\mathcal{M}_1$ of $\prod_{s=1}^S E_s$ such that $T(M) \subseteq \mathcal{M}_1$. If $T(M)$ is $\mathfrak{J}$-dimensional let $T(M) = \mathcal{M}_1$. Let $\mathcal{M}_0$ be the one dimensional subspace spanned by $\omega^1_0 > 0$ in E_1 and consider the extended marketed space $\mathcal{M} = \mathcal{M}_0 \times \mathcal{M}_1$.

By our assumption there exists a pseudo-equilibrium (p, x, L) for this extended marketed space $\mathcal{M}$, where L is $(\mathfrak{J}+1)$-dimensional. Write $\mathbb{R}^{S+1} = \mathbb{R} \times \prod_{s=1}^{S} \mathbb{R}$ and note that $L = \mathbb{R} \times L_{-0}$, where L_{-0} is a $\mathfrak{J}$-dimensional subspace of $\prod_{s=1}^{S} \mathbb{R}$. Now consider the vector space

$$H_T = \left\{ x \in \prod_{s=1}^{S} E_s : \; p_{-0} \Box x \in L_{-0} \right\}.$$

By Lemma 4.3 the space H_T is of codimension $S - \mathfrak{J}$. Therefore, $(H_T)^{\mathfrak{J}} \subseteq \left(\prod_{s=1} SE_s\right)^{\mathfrak{J}} = \mathcal{G}$ is of codimension $\mathfrak{J}(S - \mathfrak{J})$.

Notice that $T \in (H_T)^{\mathfrak{J}}$. Now define the set

$$F_T = \left\{ (R_1, R_2, \ldots, R_{\mathfrak{J}}) \in (H_T)^{\mathfrak{J}} : \; \operatorname{Span}\{p_{-0} \Box R_1, \ldots, p_{-0} \Box R_{\mathfrak{J}}\} = L_{-0} \right\}.$$

This set is algebraically open in $(H_T)^{\mathfrak{J}}$ by Lemma 4.4 and for each $(R_1, R_2, \ldots, R_{\mathfrak{J}}) \in F_T$ the triplet (p, x, L) is a full pseudo-equilibrium for the marketed space

$$\mathcal{M}_0 \times \operatorname{Span}\{R_1, R_2, \ldots, R_{\mathfrak{J}}\}.$$

So the economy with assets structure $R \in F_T$ has an equilibrium by Lemma 5.4. Now by Lemma 5.8 for any $R \in H_T$, there exists a $Q \in F_T$ such that for the assets structure $\alpha R + (1 - \alpha)Q$ with $0 < \alpha < 1$ there exists an equilibrium. To conclude the proof note that $(H_T)^{\mathfrak{J}} \setminus F_T$ is algebraically nowhere dense in $(H_T)^{\mathfrak{J}}$ and that $\bigcup_{T \in \mathcal{G}} (H_T)^{\mathfrak{J}} = \mathcal{G}$. ■

We continue with the introduction of limit economies.

Definition 5.10. *Let $\{E_\lambda, (\omega^{1,\lambda}, \ldots, \omega^{m,\lambda})\}$ be a net, where $\{E_\lambda\}$ is a net of subspaces of E and $\{(\omega^{1,\lambda}, \ldots, \omega^{m,\lambda})\}$ is a net in $(E_+)^m$, having the following properties:*

1. *The net $\{E_\lambda\}$ of subspaces is increasing and covers E, i.e., $\lambda_1 \geq \lambda_2$ implies $E_{\lambda_1} \supseteq E_{\lambda_2}$; and $E = \bigcup_\lambda E_\lambda$.*
2. *Each E_λ contains the marketed space, i.e., $\mathcal{M} \subseteq E_\lambda$, and $\omega^i \in E_\lambda$ for all i.*
3. *Each E_λ satisfies condition* **AI**.
4. *For all λ the cone $E_\lambda^+ = E_\lambda \cap E_+$ has an internal point in E_λ.*
5. *For all i, each s, and all λ, we have $\omega_s^{i,\lambda} > 0$ and $\omega^{i,\lambda} \in E_\lambda$.*
6. *For each i we have $\omega^{i,\lambda} \xrightarrow[\lambda]{} \omega^i$ in a finite dimensional vectors space and there exists some $\omega^* \in E_+$ satisfying $\sum_{i=1}^{m} \omega^{i,\lambda} \leq \omega^*$ for all λ.*

We shall denote by $\mathcal{E}_\lambda$ the economy with extended marketed space $\mathcal{M}$, an initial endowment $\omega^{i,\lambda}$ for each consumer i, and preference correspondences that are the restrictions of P_i to E_λ^+.

If the net of economies $\{\mathcal{E}_\lambda\}$ satisfies the above properties, then we shall say that $\{\mathcal{E}_\lambda\}$ ***converges*** *to our original economy $\mathcal{E}$, in symbols $\mathcal{E}_\lambda \to \mathcal{E}$.*

The next lemma shows how we can obtain a pseudo-equilibrium as a limit of pseudo-equilibria for nets of subeconomies—generalizing the limiting procedure of Bewley (1972).

Lemma 5.11. *Assume that* **A1** *and* **A2** *are satisfied and $\{\mathcal{E}_\lambda\}$ is a net of economies such that $\mathcal{E}_\lambda \to \mathcal{E}$. Further, suppose that $\{(x^\lambda, p^\lambda, L^\lambda)\}$ is a net of $E_+ \times E' \times \mathbb{G}_J$ with the same index set as the net of economies $\{\mathcal{E}_\lambda\}$ such that:*

(a) *$(x^\lambda, p^\lambda, L^\lambda) \to (x, p, L)$ holds in $(E_+, \mu)^m \times (E', \sigma(E', E)) \times \mathbb{G}_J$.*
(b) *$p \cdot \omega > 0$.*
(c) *For each index λ the triplet $(x^\lambda, p^\lambda|_{E_\lambda}, L^\lambda)$ is a pseudo-equilibrium for the economy $\mathcal{E}_\lambda$.*

Then (x, p, L) *is a pseudo-equilibrium of our economy.*

Proof. It must be the case that

$$\sum_{i=1}^{m} x^i = \omega\,, \qquad (\star)$$

since μ is a Hausdorff linear topology. We know that $p^\lambda \,\square\, x$ converges to $p \,\square\, x$ for each $x \in E$. This is in particular the case for each $x \in \mathcal{M}$. So, from part (a) of Lemma 4.6, we get $p \,\square\, \mathcal{M} \subseteq L$.

We claim that for each i we have:

$$p \,\square\, (x^i - \omega^i) \in L, \quad p^\lambda \,\square\, x^{i,\lambda} \xrightarrow[\lambda]{} p \,\square\, x^i, \quad \text{and} \quad p \cdot x^i = p \cdot \omega^i\,. \qquad (\star\star)$$

To prove this claim, we shall apply Lemma 4.8. We shall prove that the conditions of that Lemma are satisfied with $u^{i,\lambda} = \omega^{i,\lambda}$, $u^i = \omega^i$, and $u^* = \omega^*$.

Notice that conditions (2), (3) and (5) of that Lemma 4.8 are automatically satisfied. For condition (1), observe that for each λ it must be the case that $p_\lambda|_{E_\lambda} \gg 0$. In particular, $p_s^\lambda \cdot \omega_s^{i,\lambda} > 0$ for all λ, all s, and all i. Concerning condition (4), for each λ and $i \geq 2$ we have $p^\lambda \,\square\, (x^{i,\lambda} - \omega^{i,\lambda}) \in L^\lambda$ by the pseudo-equilibrium assumption. Since $\sum_{i=2}^{m} (x^{i,\lambda} - \omega^{i,\lambda}) = x^{1,\lambda} - \omega^{1,\lambda}$, it must be the case that $p^\lambda \,\square\, (x^{1,\lambda} - \omega^{1,\lambda}) \in L^\lambda$ for every λ.

Finally, we show that (6) is satisfied. Choose λ large enough such that $(\omega_s, 0_{-s}) \in E_\lambda$ for all states s and $x \in (E_\lambda)^m$. We know that for each i, each s, and each $0 < \alpha < 1$ we have $(x_s^i + \alpha\omega_s, (x^i)_{-s}) \in E_\lambda$ and $(x_s^i + \alpha\omega_s, (x^i)_{-s}) \in P_i(x^i)$. By assumption **A2**, we see that for λ large enough $(x_s^i + \alpha\omega_s, (x^{i,\lambda})_{-s}) \in P_i(x^{i,\lambda})$. Therefore, by Lemma 5.7 we get

$$p^\lambda \,\square\, (x_s^i + \alpha\omega_s, (x^{i,\lambda})_{-s}) \not\leq p^\lambda \,\square\, x^{i,\lambda}$$

eventually for λ large enough. We can now apply Lemma 4.8 and establish the validity of the first two statements of $(\star\star)$. Note that $\sum_{i=1}^{m} x^{i,\lambda} = \sum_{i=1}^{m} \omega^{i,\lambda}$ and $p^\lambda \cdot x^{i,\lambda} \leq p^\lambda \cdot \omega^{i,\lambda}$ imply that $p^\lambda \cdot x^{i,\lambda} = p^\lambda \cdot \omega^{i,\lambda}$ for every λ and every i. Therefore, $p \cdot x^i = p \cdot \omega^i$ for each i.

Our next task is to show that $p \cdot \omega^1 > 0$. Since $p \cdot \omega > 0$, there exists some i and some s such that $p_s \cdot x_s^i > 0$. Now let $0 < z \in E_s$ and fix some $0 < \alpha < 1$ such that $(\alpha x_s^i + z, (x^i)_{-s}) \in P_i(x^i)$. By assumption **A2** there exists some λ large enough such that $(\alpha x_s^i + z, (x^{i,\lambda})_{-s}) \in P_i(x^{i,\lambda})$ and $(\alpha x_s^i + z, (x^{i,\lambda})_{-s}) \in E_\lambda$. Therefore, by Lemma 5.7, eventually $p_s^\lambda \cdot (\alpha x_s^i + z) > p_s^\lambda \cdot x_s^{i,\lambda}$. Since we have established that $p_s^\lambda \cdot x_s^{i,\lambda} \xrightarrow[\lambda]{} p_s \cdot x_s^i$, we see that $p_s \cdot (\alpha x_s^i + z) \geq p_s \cdot x_s^i > 0$. This tells us that $p_s \cdot z > 0$. Since z was arbitrarily chosen, this means that $p_s \gg 0$. Therefore, $p \cdot \omega^1 > 0$.

Next, we claim that

$$z \in P_1(x^1) \implies p \cdot z > p \cdot \omega^1\,.$$

To prove this claim, take $z \in P_1(x^1)$. We know by **A1** that $z \in P_1(x^{1,\lambda})$ and $z \in E_\lambda$ eventually for large enough λ. This guarantees that eventually $p^\lambda \cdot z > p^\lambda \cdot \omega^{1,\lambda}$ and that $p \cdot z \geq p \cdot \omega^1 > 0$. Since for some $0 < \alpha < 1$ we have $\alpha z \in P_1(x^1)$ it must be the case that $p \cdot z > p \cdot \omega^1$.

Now we shall show that $p \gg 0$. So, take an arbitrary $z > 0$. We know that $z + x^1 \in P_1(x^1)$ and we have already established that $p \cdot (z + x^1) > p \cdot \omega^1 = p \cdot x^1$, which implies that $p \cdot z > 0$. Since z was arbitrarily chosen we have established that $p \gg 0$.

To continue our proof, we need to establish that for all $i \geq 2$ the following holds true:

$$y \in P_i(x^i) \ \text{and} \ p \,\square\, (y - \omega^i) \in L \implies p \cdot y > p \cdot \omega^i\,. \qquad (\dagger)$$

To this end, let $y \in P_i(x)$ and $p \,\square\, (y - \omega^i) \in L$. Choose, $\hat{\lambda}$ such that

$$x^i, y, \omega^*, \omega^{i,\lambda}, (\omega_1, 0_{-1}), (\omega_1, 0_{-1}), \ldots, (\omega_S, 0_{-S}) \in E_\lambda$$

for all s, all $\lambda \geq \hat{\lambda}$ and all i. Now let $z \in P_i(x^i)$ be an internal point of $E^+_{\hat{\lambda}}$ satisfying $p \,\Box\, (z - \omega^i) \in L$. We know from part (d) of Lemma 4.6 (by passing to a subnet if necessary) $\{z^\lambda\} \subseteq E_{\hat{\lambda}}$, $z^\lambda \to z$ in a finite dimensional vector space and $p^\lambda \,\Box\, z^\lambda \in L^\lambda$. Eventually, for λ large enough $z^\lambda \in E^+_{\hat{\lambda}}$ and by **A1** we have $z^\lambda \in P_i(x^{i,\lambda})$. Therefore, eventually $p^\lambda \cdot z^\lambda \geq p^\lambda \cdot \omega^{i,\lambda}$ and since $z^\lambda \xrightarrow[\lambda]{} z$ and $\omega^{i,\lambda} \xrightarrow[\lambda]{} \omega^i$ in a finite dimensional space it must be the case that $p \cdot z \geq p \cdot \omega^i$. Consequently, we have established that:

$$z \in P_i(x), \;\; z \text{ an internal point of } E^+_{\hat{\lambda}}, \;\; \text{and } \; p \,\Box\, (z - \omega^i) \in L \implies p \cdot z \geq p \cdot \omega^i \,.$$

By Lemma 4.9 it must be the case that $p \cdot y > p \cdot \omega^i$ and that (†) holds true. The above show that (x, p, L) is indeed a pseudo-equilibrium. ■

6. Games with subspaces

Our objective in this section is to extend the ideas of Gale and Mas-Colell (1975) on the existence of an equilibrium for a market game. Unfortunately, the situation in the incomplete markets setting is far more complicated than in the Walrasian setting. In this section we use the order structure of the commodity space to rectify these difficulties.

Recall that S and J are two natural numbers such that $J \leq S + 1$.

Definition 6.1. *An* ***abstract game*** (***with subspaces***) *is a tuple* $\Gamma = (I, (\mathcal{P}_i)_{i \in I}, (X_i)_{i \in I}, \psi)$, *where:*

(1) *The set I is a finite index set of players.*
(2) *Each X_i is a nonempty set—called the* **action** (*or* **strategy**) **set** *of player i.*
(3) $\mathcal{P}_i \colon X \times \mathbb{G}_J \twoheadrightarrow X_i$ *is a correspondence, where* $X = \prod_{i \in I} X_i$.
(4) ψ *is a function from $X \times \mathbb{G}_J$ to $\mathbb{R}^{(S+1)J}$, called the* **subspace constrain function**.

The equilibrium concept of an abstract game is defined as follows.

Definition 6.2. *An* **equilibrium** *for the abstract game Γ is a point $(x, L) \in X \times \mathbb{G}_J$ satisfying $\mathcal{P}_i(x, L) = \emptyset$ for each $i \in I$ and $\psi(x, L)_j \in L$ for $j = 1, \ldots, J$.*

We shall also use the following fixed point result established by Husseini, Larsy, and Magill (1990, Theorem A, p. 50).

Lemma 6.3. *Let X be a nonempty compact convex subset of a finite dimensional vector space and let $f \colon X \times \mathbb{G}_J \to X$ and $\psi \colon X \times \mathbb{G}_J \to \mathbb{R}^{(S+1)J}$ be continuous functions. Then there exists some $(x, L) \in X \times \mathbb{G}_J$ satisfying $f(x, L) = x$ and $\psi(x, L)_j \in L$ for $j = 1, \ldots, J$.*

We can extend this fixed point like result to correspondences.

Lemma 6.4. *Let X be a nonempty compact convex subset of a finite dimensional vector space, let $f \colon X \times \mathbb{G}_J \twoheadrightarrow X$ be an upper hemicontinuous nonempty convex-valued correspondence, and let $\psi \colon X \times \mathbb{G}_J \to \mathbb{R}^{(S+1)J}$ be a continuous function. Then there exists some $(x, L) \in X \times \mathbb{G}_J$ satisfying $x \in f(x, L)$ and $\psi(x, L)_j \in L$ for $j = 1, \ldots, J$.*

Proof. According to Lemma C.2 in Appendix C, for each $\varepsilon > 0$ there exists a continuous function $f_\varepsilon \colon X \times \mathbb{G}_J \to X$ such that

$$\sup\{d(y, \operatorname{Gr} f) \colon \; y \in \operatorname{Gr} f_\varepsilon\} < \varepsilon \,,$$

where d is a compatible metric on $X \times \mathbb{G}_J \times X$. By Lemma 6.3 there exists a point $(x_\varepsilon, L_\varepsilon)$ such that $f(x_\varepsilon, L_\varepsilon) = x_\varepsilon$ and $\psi(x_\varepsilon, L_\varepsilon)_j \in L_\varepsilon$ for $j = 1, \dots, J$. Take a sequence $\{\varepsilon_n\}$ such that $\varepsilon_n \downarrow 0$.

By passing to a subsequence if necessary, we can assume that there exists some $(x, L) \in X \times \mathbb{G}_J$ satisfying $(x_{\varepsilon_n}, L_{\varepsilon_n}) \to (x, L)$. Since $\operatorname{Gr} f$ is closed it must be the case that $((x, L), x) \in \operatorname{Gr} f$. Therefore, $x \in f(x, L)$. Now $\psi(x_{\varepsilon_n}, L_{\varepsilon_n})_j \in L_{\varepsilon_n}$ for $j = 1, \dots, J$. Since ψ is continuous and the function $L \mapsto L$, from $\mathbb{G}_J$ to $\mathbb{R}^{S+1}$, has a closed graph, we get $\psi(x, L)_j \in L$ for $j = 1, \dots, J$. ■

The next lemma presents conditions that guarantee the existence of an equilibrium for an abstract game. The proof is a straightforward extension of the argument in Gale and Mas-Colell (1975). We shall say that the correspondence $\mathcal{P}_i \colon X \times \mathbb{G}_J \twoheadrightarrow X_i$ is ***irreflexive*** if for each $(x, L) \in X \times \mathbb{G}_J$ we have $x^i \notin \mathcal{P}_i(x, L)$.

Lemma 6.5. *Let $\Gamma = (I, (\mathcal{P}_i)_{i\in I}, (X_i)_{i\in I}, \psi)$ be an abstract game. Assume that each X_i is a nonempty compact and convex subset of a finite dimensional vector space and that each correspondence $\mathcal{P}_i$ is irreflexive, convex-valued, and lower hemicontinuous. If the subspace constrain function ψ is continuous, then the game Γ has an equilibrium.*

Proof. For every i let $\mathcal{Y}_i = \{(x, L) \in X \times \mathbb{G}_J \colon \ \mathcal{P}_i(x, L) \neq \emptyset\}$. Clearly, the restriction $\mathcal{P}_i|_{\mathcal{Y}_i}$ is lower hemicontinuous having convex nonempty values and $\mathcal{Y}_i$ is open in $X \times \mathbb{G}_J$.

Therefore, by Michael (1956, Theorem 3.1‴) (see also Lemma C.1 in Appendix C) the correspondence $\mathcal{P}_i|_{\mathcal{Y}_i}$ admits a continuous selection, say $f_i \colon \mathcal{Y}_i \to X_i$. Next, we define the correspondence $F_i \colon X \times \mathbb{G}_J \twoheadrightarrow X_i$ by

$$F_i(x, L) = \begin{cases} f_i(x, L) & \text{if } (x, L) \in \mathcal{Y}_i\,, \\ X_i & \text{otherwise}\,. \end{cases}$$

This correspondence is convex nonempty-valued and upper hemicontinuous. Letting $F = \prod_{i\in I} F_i$, it follows from Lemma 6.4 that there exists some $(x, L) \in X \times \mathbb{G}_J$ such that $x \in F(x^i)$ and $\psi(x, L)_j \in L$ for $j = 1, \dots, J$. By construction, this is an equilibrium for the game Γ. ■

Our next goal is to induce a market game from our economic model. To do this, we need to introduce some more notation. We define the convex sets:

$$\begin{aligned} \Delta &= \{p \in E'_+ \colon \ p \geq 0 \text{ and } p \cdot \omega = 1\}\,, \\ \Delta_{++} &= \{p \in E'_+ \colon \ p \gg 0 \text{ and } p \cdot \omega = 1\}\,, \\ (0, x) &= \{y \in E'_+ \colon \ 0 \ll y \ll x\}\,, \\ E_{++} &= \{x \in E'_+ \colon \ x \gg 0\}\,. \end{aligned}$$

With this notation in mind, for each i define the correspondences $\gamma_i, \eta_i \colon \Delta \times \mathbb{G}_J \to E'_+$ by

$$\gamma_i(p, L) = \left\{ x \geq 0 \colon \ \exists\, z \in [0, 2\omega] \text{ such that } \begin{array}{c} x \leq z \\ p \cdot (z - \omega^i) \leq 0 \\ p \,\Box\, (z - \omega^i) \in L \end{array} \right\},$$

$$\eta_i(p, L) = \left\{ x \gg 0 \colon \ \exists\, z \in (0, 2\omega) \text{ such that } \begin{array}{c} x \ll z \\ p \cdot (z - \omega^i) < 0 \\ p \,\Box\, (z - \omega^i) \in L \end{array} \right\},$$

and the correspondences $g_i, e_i \colon \Delta \to E'_+$ by

$$g_i(p) = \big\{x \in [0, 2\omega]\colon\ p \cdot (z - \omega^i) \le 0\big\} \quad \text{and} \quad e_i(p) = \big\{x \in (0, 2\omega)\colon\ p \cdot (z - \omega^i) < 0\big\}\,.$$

Notice that γ_i is a nonempty-valued correspondence since $\omega^i \in \gamma_i(p, L)$ for each $(p, L) \in \Delta \times \mathbb{G}_J$. Now if ω^i is an interior point of E_+, then $2\omega \gg \omega \ge \omega^i$ and so each $\gamma_i(p, L)$ contains the nonempty open set $(0, \omega^i)$. However, the correspondence η_i may take some empty-values even if $p \gg 0$.

The basic properties of the correspondences γ_i and η_i are included in the next result.

Lemma 6.6. *The following statements hold true:*

1. *Both correspondences γ_i and η_i are convex-valued and for each $(p, L) \in \Delta \times \mathbb{G}_J$ we have $\eta_i(p, L) \subseteq \gamma_i(p, L)$.*
2. *If E is finite dimensional, then γ_i has a closed graph in $\Delta \times \mathbb{G}_J \times [0, 2\omega]$.*
3. *If E is finite dimensional and ω is in the interior of E_+, then the (restriction) correspondence $\eta_i \colon \Delta_{++} \times \mathbb{G}_J \twoheadrightarrow (0, 2\omega)$ has an open graph in $\Delta_{++} \times \mathbb{G}_J \times (0, 2\omega)$.*
4. *If ω^i is an interior point of E_+, $p \gg 0$, and $p \,\Box\, \mathcal{M} \subseteq L$, then*

$$\overline{\eta_i(p, L)} = \gamma_i(p, L)\,.$$

In particular, in this case $\eta_i(p, L)$ is nonempty.

Proof. (1) & (2) Observe that for each $(p, L) \in \Delta \times \mathbb{G}_J$ we have

$$\gamma_i(p, L) = \Big(\big[\{x \in E_+ : p \cdot (x - \omega^i) \le 0\} \cap \{x \in E_+ : p \,\Box\, (x - \omega^i) \in L\}\big] - E_+\Big) \cap [0, 2\omega]\,. \qquad (\star)$$

Therefore, for each i the correspondence γ_i is convex-valued and if E is finite dimensional, then it has a closed graph (according to Lemma 4.6). Moreover, for each $(p, L) \in \Delta \times \mathbb{G}_J$ the following holds true:

$$\eta_i(p, L) = \Big(\big[\{x \in E_+ :\ p \cdot (x - \omega^i) < 0\} \cap \{x \in E_+ :\ p \,\Box\, (x - \omega^i) \in L\}\big] - E_{++}\Big) \cap (0, 2\omega)\,. \qquad (\star\star)$$

Therefore, η_i is a convex-valued correspondence and $\eta_i(p, L) \subseteq \gamma_i(p, L)$ for each $(p, L) \in \Delta \times \mathbb{G}_J$. It is now easy to see that (1) and (2) are true statements.

(3) Since E is finite dimensional and ω is an interior point of E_+, it follows that every $x \in E_+$ satisfying $x \gg 0$ is an interior point of E_+. In particular, $(0, 2\omega)$ is an open subset of E.

To see that the correspondence $\eta_i \colon \Delta_{++} \times \mathbb{G}_J \twoheadrightarrow (0, 2\omega)$ has an open graph, let (p, L, x) be a point in the graph of η_i. The openness of the graph of $\eta_i \colon \Delta_{++} \times \mathbb{G}_J \twoheadrightarrow (0, 2\omega)$ will be established, if $(p_n, L_n, x_n) \to (p, L, x)$ in $\Delta_{++} \times \mathbb{G}_J \times (0, 2\omega)$ implies $x_n \in \eta_i(p_n, L_n)$ for infinitely many n.

So, assume $(p_n, L_n, x_n) \to (p, L, x)$ in $\Delta_{++} \times \mathbb{G}_J \times (0, 2\omega)$. From the definition of η_i there exists some $z^* \in (0, 2\omega)$ satisfying $p \cdot (z^* - \omega^i) < 0$, $p \,\Box\, (z^* - \omega^i) \in L$, and $z^* \gg x^*$. Fix any vector $y^* \in E_+$ such that $z^* \gg y^* \gg x^*$ and assume without loss of generality that $x_n \ll y^*$ for all n. We know from part (d) of Lemma 4.6 (by passing to a subsequence if necessary) that there exists a sequence $\{z_n\} \subseteq E_+$ with $z_n \to z^*$ and such that $p \,\Box\, (z_n - \omega^i) \in L$ for all n. Clearly, there exists some k such that if $n \ge k$, then $z_n \in (0, 2\omega)$, $p \cdot (z_n - \omega^i) < 0$, and $y^* \ll z_n$. Therefore, for all $n \ge k$ we have $x_n \ll y^* \ll z_n$. Consequently, for all $n \ge k$ we have $x_n \in \eta_i(p_n, L_n)$, which proves that correspondence $\eta_i \colon \Delta_{++} \times \mathbb{G}_J \twoheadrightarrow (0, 2\omega)$ has an open graph.

(4) Fix any (p, L) such that $p \gg 0$ and $p \,\Box\, \mathcal{M} \subseteq L$. Let $K = \{y \in E\colon\ p \,\Box\, (y - \omega^i) \in L\}$. We first show that there exists $y' \in (0, 2\omega) \cap K$ such that $p \cdot y' < p \cdot \omega^i$. To this end, fix some $v \in \mathcal{M}$ with $v > 0$. Clearly $p \cdot (-v) < 0$. Letting $y = \omega^i - v$, we see that $p \cdot y < p \cdot \omega^i$. Since $p \,\Box\, (-v) \in L$ it must be the case that $y \in K$. Now for $0 < \alpha < 1$ we have $p \cdot ((1 - \alpha)y - \alpha\omega^i) < p \cdot \omega^i$. Moreover,

since $\omega^i \in (0, 2\omega)$, we can choose $0 < \alpha < 1$ that satisfies $(1-\alpha)y - \alpha\omega^i \in (0, 2\omega)$. Clearly, $(1-\alpha)y - \alpha\omega^i \in K$ since K is convex and $\omega^i \in K$. So, letting $y' = (1-\alpha)y - \alpha\omega^i$, we see that

$$y' \in (0, 2\omega) \cap K, \quad \text{and} \quad p \cdot y < p \cdot \omega^i . \tag{†}$$

Now let $\gamma_i(p, L)^\circ$ denote the interior of $\gamma_i(p, L)$ in E, which is nonempty by (†). Since $\gamma_i(p, L)$ is convex with a nonempty interior, we see that $\overline{\gamma_i(p, L)^\circ} = \gamma_i(p, L)$. But $\eta_i(p, L) \subseteq \gamma_i(p, L)$, and from this we see that our proof will be finished if we can show that $\gamma_i(p, L)^\circ \subseteq \overline{\eta_i(p, L)}$.

To this end, let $x \in \gamma_i(p, L)^\circ$. Since x is an interior point of $\gamma_i(p, L)$, it must be the case that $x \in (0, 2\omega)$. We also know from the definition of γ_i that there exists some $z \in [x, 2\omega] \cap K$ such that $p \cdot (z - \omega^i) \le 0$. Now, for each $0 < \alpha < 1$ let $w^\alpha = (1-\alpha)z + \alpha y'$, where y' satisfies (†). Clearly, $w^\alpha \in K$, $w^\alpha \in (0, 2\omega)$ and $p \cdot w^\alpha < p \cdot \omega^i$.

Next, once again, for each $0 < \alpha < 1$ let $u^\alpha = [(1-\alpha)(x - z) + \alpha(-\omega)] + w^\alpha$. Since $x - z \le 0$, $-\omega \ll 0$ and $w^\alpha \ge 0$, we see that $w^\alpha \gg u^\alpha$. Furthermore, if $\alpha \downarrow 0$, then $w^\alpha \to z$ and $u^\alpha \to x$. Therefore, for α close enough to zero, we have $u^\alpha \in (0, 2\omega)$. A glance at the definition of $\eta_i(p, L)$ guarantees that for any such α we have $u^\alpha \in \eta_i(p, L)$ and from this we get $x \in \overline{\eta_i(p, L)}$. ■

We are now ready to introduce an abstract game that is induced by our economy $\mathcal{E}$.

Definition 6.7. *The* **market game** $\Gamma_{\mathcal{E}} = (\mathcal{I}, (\mathcal{P}_i)_{i\in\mathcal{I}}, (X_i)_{i\in\mathcal{I}}, \psi)$ *induced by our economy* $\mathcal{E}$ *is the abstract game with the following characteristics.*

(1) $\mathcal{I} = \{0, 1, \dots, m\}$.
(2) $X_0 = \Delta_{++}$ *and* $X_i = [0, 2\omega]$ *for* $i = 1, \dots, m$.
(3) *The function* $\psi\colon \prod_{i=0}^m X_i \times \mathbb{G}_J \to \mathbb{R}^{(S+1)J}$ *is defined by* $\psi(p, x, L) = [p \Box b_1, \dots, p \Box b_J]$ *for each* $(p, x, L) \in \prod_{i=0}^m X_i \times \mathbb{G}_J = \Delta_{++} \times \prod_{i=1}^m X_i \times \mathbb{G}_J$, *where* $\{b_1, \dots, b_J\}$ *is a fixed basis of* $\mathcal{M}$.
(4) *For* $i = 0$ *we define* $\mathcal{P}_0\colon X \times \mathbb{G}_J \twoheadrightarrow X_0$ *by*

$$\mathcal{P}_0(p, x, L) = \Big\{q \in \Delta_{++}\colon\ q \cdot \Big(\sum_{i=1}^m x^i - \omega\Big) > p \cdot \Big(\sum_{i=1}^m x^i - \omega\Big)\Big\},$$

for $i = 1$ *we define* $\mathcal{P}_1\colon X \times \mathbb{G}_J \twoheadrightarrow X_1$ *by*

$$\mathcal{P}_1(p, x, L) = \begin{cases} e_1(p) & \text{if } x^1 \notin g_1(p), \\ P_1(x^1) \cap e_1(p) & \text{otherwise,} \end{cases}$$

for $i = 2, \dots, m$ *we define* $\mathcal{P}_i\colon X \times \mathbb{G}_J \twoheadrightarrow X_i$ *by*

$$\mathcal{P}_i(p, x, L) = \begin{cases} \eta_i(p, L) & \text{if } x^i \notin \gamma_i(p, L), \\ P_i(x^i) \cap \eta_i(p, L) & \text{otherwise.} \end{cases}$$

The importance of the market game will be established in the next two results.

Lemma 6.8. *If* E *is finite dimensional, the vector* ω^i *is an interior point of* E_+ *for each* i, *and* (p, x, L) *is an equilibrium for the market game* $\Gamma_{\mathcal{E}}$, *then* (x, p, L) *is a pseudo-equilibrium for the economy* $\mathcal{E}$.

Proof. Let (p, x, L) be an equilibrium for the market game. That is, $\mathcal{P}_i(p, x, L) = \emptyset$ holds true for each i and $\psi(p, x, L)_j = p \Box b_j \in L$ for each j. In particular, $p \Box \mathcal{M} \subseteq L$.

Since $p \gg 0$ each ω^i is an interior point of E_+, and $p \,\square\, \mathcal{M} \subseteq L$, it follows from Lemma 6.6 (4) that each $\eta_i(p, L)$ is nonempty. This implies $p \cdot x^i \leq p \cdot \omega^i$ for each i.

From $\mathcal{P}_0(p, x, L) = \emptyset$, it follows that $q \cdot \left(\sum_{i=1}^m x^i - \omega\right) \leq p \cdot \left(\sum_{i=1}^m (x^i - \omega^i)\right) \leq 0$ holds for all $q \in \Delta_{++}$. Since E_+ in this case is closed, it follows that $\sum_{i=1}^m x^i \leq \omega$. In particular, we have $x^i \leq \omega \ll 2\omega$ for all i.

Since $P_i(x^i) \cap \eta_i(p, L) = \emptyset$ and (by Lemma 6.6) $\overline{\eta_i(p, L)} = \gamma_i(p, L)$ hold for each $i \geq 2$, it follows that $P_i(x^i) \cap \gamma_i(p, L) = \emptyset$ (recall that $P_i(x^i)$ is open in E_+ for the Euclidean topology of E by **AII**(d)). By the same argument, we see that $P_1(x^1) \cap g_1(p, L) = \emptyset$.

Next, we shall show that $p \,\square\, (x^i - \omega^i) \in L$ for $i \geq 2$. Suppose by way of contradiction that this is not the case for some $i \geq 2$. So, according to the definition of γ_i, there exists some $z \in [x^i, 2\omega]$ satisfying $p \,\square\, (z - \omega^i) \in L$, and $p \cdot (z - \omega^i) \leq 0$. Clearly, $z \neq x^i$ and thus $z \in P_i(x^i)$. But $z \in \gamma_i(p, L)$, which is a contradiction. Therefore,

$$p \,\square\, (x^i - \omega^i) \in L \quad \text{for } i \geq 2\,.$$

We want to show that $p \cdot x^i = p \cdot \omega^i$ for $i = 1, \ldots, m$. Consider a vector $v > 0$ with $v \in \mathcal{M}$. It follows that

$$p \,\square\, ((v + x^i) - \omega^i)) = p \,\square\, (x^i - \omega^i) + p \,\square\, v \in L\,,$$

for each $i = 2, \ldots, m$. Furthermore, for $i = 2, \ldots, m$ we have

$$\alpha(v + x^i) + (1 - \alpha)x^i \in P_i(x^i) \quad \text{and} \quad p \,\square\, (\alpha(v + x^i) + (1 - \alpha)x^i - \omega^i) \in L\,,$$

for each $0 < \alpha < 1$.

Therefore, suppose by way of contradiction that $p \cdot x^i < p \cdot \omega^i$ holds true. For α close to zero we have $\alpha(v + x^i) + (1 - \alpha)x^i \in [0, 2\omega]$ (since $x^i \ll 2\omega$) and $p \cdot (\alpha(v + x^i) + (1 - \alpha)x^i) < p \cdot \omega^i$. This means $\alpha(v + x^i) + (1 - \alpha)x^i \in \gamma_i(p, L)$ for some α, which is impossible. Therefore, $p \cdot x^i = p \cdot \omega^i$ for $i = 2, \ldots, m$. A similar argument shows that $p \cdot x^1 = p \cdot \omega^1$. The fact that $p \gg 0$ and $p \cdot x^i = p \cdot \omega^i$ for $i = 1, \ldots, m$ shows that $\sum_{i=1}^m x^i = \omega$. Thus, x is an allocation.

Now for $i \geq 2$ take any $y \in P_i(x^i)$ with $p \,\square\, (y - \omega^i) \in L$. We want to show that $p \cdot y > p \cdot \omega^i$. Since $x^i \ll 2\omega$ we have $\alpha y + (1 - \alpha)x \ll 2\omega$ for α close enough to zero. Therefore, $\alpha y + (1 - \alpha)x \notin \gamma_i(p, L)$ and $p \cdot (\alpha y + (1 - \alpha)x^i) > p \cdot \omega^i = p \cdot x^i$, which implies $p \cdot y > p \cdot x^i = p \cdot \omega^i$. A similar argument shows that if $y \in P_1(x^1)$, then $p \cdot y > p \cdot \omega^1$. These facts in connection with the above proven properties show that (p, x, L) is a pseudo-equilibrium for the economy $\mathcal{E}$. ■

Lemma 6.9. *If E is finite dimensional, ω^1 is an interior point of E_+, and each P_i is lower hemicontinuous, then there exists an equilibrium for the market game $\Gamma_{\mathcal{E}}$.*

Proof. Notice that from parts (1), (2), and (3) of Lemma 6.6, we can verify easily that each $\mathcal{P}_i$ is lower hemicontinuous, convex-valued, and irreflexive. Now take an increasing sequence $\{\Delta_n\}$ of nonempty compact subsets of Δ_{++} such that $\bigcup_{n=1}^\infty \Delta_n = \Delta_{++}$. For each n define the abstract game Γ_n which is the restriction of $\Gamma_{\mathcal{E}} = (\mathcal{I}, \mathcal{P}_i, X_i, \psi)$ to the set $\Delta_n \times X_1 \times \cdots X_m \times \mathbb{G}_J$. Each Γ_n has an equilibrium (p_n, x_n, L_n), by Lemma 6.5. By taking a subsequence, we can assume that $(p_n, x_n, L_n) \to (p, x, L)$, where $p \in \Delta$. The proof will be completed if we can show that $p \gg 0$.

To see this, observe first that $\sum_{i=1}^m x^i \leq \omega$. Therefore, $x^1 \leq \omega \ll 2\omega$. One can also verify that $x^1 \in g_1(p)$ and $P_1(x^1) \cap g_1(p) = \emptyset$. It follows that $x^1 > 0$. Since for some $\alpha > 1$ we have $\alpha x^1 \in [0, 2\omega]$, it is the case that $p \cdot x^1 = p \cdot \omega^1 > 0$.

Finally, taking an arbitrary $y > 0$, we see that for $0 < \alpha < 1$ close enough to one we have $\alpha x^1 + (1 - \alpha)(y + x^1) \ll 2\omega$, and that for any such α we have $p \cdot (\alpha x^1 + (1 - \alpha)(y + x^1)) > p \cdot x^1$. This implies $p \cdot y > 0$ and so $p \gg 0$, and the proof is finished. ■

7. The existence of pseudo-equilibrium

We now have enough machinery at our disposal to prove our major results. For completeness, we state them again in this section and provide their proofs.

Theorem 3.4. *If E is finite dimensional and for each i we have $\omega^i \gg 0$ and P_i is lower hemicontinuous, then for a $\mathfrak{J}(S-\mathfrak{J})$-strongly dense set of assets structures in $\mathcal{G}$ there exists an equilibrium.*

Proof. This result is a consequence of Lemmas 5.9, 6.8, and 6.9. ■

We turn to the case where E is infinite dimensional.

Lemma 7.1. *If* **A1** *and* **A2** *are satisfied and each ω^i is an interior point of E_+, then there exists a pseudo-equilibrium.*

Proof. Take an increasing net $\{E_\lambda\}$ of finite dimensional subspaces of E and such that $\mathcal{M} \subseteq E_\lambda$ and $\omega^i \in E_\lambda$ for each λ and each i. For each λ define the economy $\mathcal{E}_\lambda$ to be the the economy with marketed space $\mathcal{M}$, initial endowments the ω^i and preferences the restrictions of the P_i to E_λ.

From Lemma 6.9 we know that each $\mathcal{E}_\lambda$ has a pseudo-equilibrium $(x^\lambda, p^\lambda, L^\lambda)$. Since E_+ has an interior point we can assume without loss of generality that $p^\lambda \in E'_+$ and that $p^\lambda \cdot \omega = 1$ for each λ. Therefore, by passing to a subnet if necessary, we can assume that $(x^\lambda, p^\lambda, L^\lambda) \to (x, p, L)$ holds true in $(E_+, \mu)^m \times (E'_+, \sigma(E', E)) \times \mathbb{G}_J$ with x being an allocation and $p \cdot \omega = 1$. By Lemma 5.11, (x, p, L) is a pseudo-equilibrium. ■

Lemma 7.2. *If ω is an interior point of E_+ and conditions* **A1** *and* **A2** *are satisfied, then pseudo-equilibria exist.*

Proof. For each $0 < \lambda < 1$ let $\omega^{i,\lambda} = \lambda\omega^i + \frac{(1-\lambda)}{m}\omega$. Clearly, each $\omega^{i,\lambda}$ is an interior point of E_+. Notice that $\sum_{i=1}^m \omega^{i,\lambda} = \omega$ for all $0 < \lambda < 1$. Therefore, $\{(\omega^{i,\lambda})_{i=1}^m\}$ is bounded from above by ω and (by passing to a subnet), we can assume that $\{(\omega^{i,\lambda})_{i=1}^m\}$ is converging to $\{(\omega^i)_{i=1}^m\}$ in a finite dimensional space. Let $\mathcal{E}_\lambda$ be the economy with $\{\omega^{i,\lambda}\}$ as initial endowments. By Lemma 7.1, each such economy has a pseudo-equilibrium $(x^\lambda, p^\lambda, L^\lambda)$. We can assume that $(x^\lambda, p^\lambda, L^\lambda) \to (x, p, L)$ in $(E_+, \mu)^m \times (E'_+, \sigma(E', E)) \times \mathbb{G}_J$ with $p \cdot \omega = 1$. Once again, Lemma 5.11 guarantees that (x, p, L) is a pseudo-equilibrium. ■

An immediate consequence of the preceding result and Lemma 5.9 is our second major result.

Theorem 3.5. *If ω is an interior point of E_+ and conditions* **A1** *and* **A2** *are satisfied, then for a $\mathfrak{J}(S-\mathfrak{J})$-strongly dense set of assets structures in $\mathcal{G}$ there exists an equilibrium.*

Let us now move to the case where E is a locally convex-solid Riesz space.

Lemma 7.3. *If* **A1, A2**, *and* **A3** *are satisfied, E is a locally convex-solid Riesz space, and ω is an internal point of E_+, then there exists a pseudo-equilibrium (x, p, L). Furthermore, p is strictly positive on any set of the form $mv + \prod_{s=1}^S N_s$, where $v \in E_+$ and for all s:*

1. *N_s is a convex and τ-solid neighborhood of zero in E_s, and*
2. *for all i, the pointwise properness cone $C_s^{i,x}$ in state s at x^i satisfies $v_s + N_s \subseteq C_s^{i,x}$.*

Proof. The norm induced by the gauge of $[-\omega, \omega]$ generates a linear topology ρ that is finer than τ, since order intervals are τ-bounded. Therefore, all assumptions of Lemma 7.2 are satisfied and so there exists a pseudo-equilibrium (x, p, L) with p being ρ-continuous.

We need to show that p is τ-continuous. To this end, fix s and let N_s be a convex and solid τ_s-neighborhood of zero in E_s, and let $v \in E_+$ be a vector satisfying properties (1) and (2). Such a neighborhood N_s and a vector v always exist. (Indeed, let $v = \omega$ and choose N_s such that $\omega_s + N_s \subseteq \bigcap_{i=1}^m C_s^{i,x}$.)

Fix a vector $z \in N_s$. We know that for $r > 0$ large enough $\sum_{i=1}^m x_s^i = \omega_s \geq \frac{1}{r}|z|$. Therefore, by the Riesz decomposition property, there exist $y_1, \ldots, y_m \in E_s^+$ satisfying $y_i \leq x_s^i$ for each i and $\sum_{i=1}^m y_i = \frac{1}{r}|z|$.

For each i, let $u^i = \big(x_s^i + \frac{1}{r}v_s - y_i, (x^i)_{-s}\big)$. Clearly, $u_i \geq 0$. From $ry_i \leq |z|$ and the solidness of N_s, it follows that $y_i \in \frac{1}{r}N_s$ and $-y_i \in \frac{1}{r}N_s$. Therefore, $y_i + \frac{1}{r}v_s \in C_s^{i,x}$, since $\frac{1}{r}N_s + \frac{1}{r}v_s \subseteq C_s^{i,x}$. By **A3**, it must be the case that $u^i \in P_i(x^i)$. By Lemma 5.7, we have $p_s \cdot u_s^i > p_s \cdot x_s^i$. Now

$$\frac{1}{r}z \leq \frac{1}{r}|z| = \sum_{i=1}^m y_i = \sum_{i=1}^m x^i - \sum_{i=1}^m u_s^i + \frac{m}{r}v_s\,.$$

Therefore,

$$\frac{1}{r}p_s \cdot z \leq \sum_{i=1}^m p_s \cdot y_i = \sum_{i=1}^m p_s \cdot (x_s^i - u_s^i) + \frac{m}{r}p \cdot v_s < \frac{m}{r}p_s \cdot v_s\,,$$

and so for all $z \in N_s$ we have

$$mp_s \cdot v_s + p_s \cdot (-z) > 0\,.$$

Since N_s (as being solid) is circled, we see $z \in N_s$ implies $-z \in N_s$. So, for all $z \in N_s$ we have

$$mp_s \cdot v_s + p_s \cdot z > 0\,.$$

This implies that p_s is τ_s-continuous for each state s, i.e., p is τ-continuous. Finally, since s is arbitrary, we have verified also that p is strictly positive on $mv + \prod_{s=1}^S N_s$. ■

In the next theorem E_+ has an internal point that need not be the total endowment.

Lemma 7.4. *If* **A1**, **A2**, *and* **A4** *are satisfied, each* E_s *is a locally convex-solid Riesz space and* E_+ *has an internal point, then there exists a pseudo-equilibrium* (x, p, L). *Moreover,* p *is positive on any set of the form* $m\omega + \prod_{s=1}^S N_s$, *where for all* s:

(1) N_s *is a convex solid* τ_s*-neighborhood of zero in* E_s, *and*
(2) *for all* i, *the uniform properness cone* C_s^i *in state* s *satisfies* $\omega_s + N_s \subseteq C_s^i$.

Proof. Let v be an internal point of E_+ such that $mv_s \in \bigcap_{i=1}^m C_s^i$ for each s.

For each $0 < \lambda < 1$ let $\omega^{i,\lambda} = \lambda\omega^i + (1-\lambda)v$. Clearly, each $\omega^{i,\lambda}$ is an internal point of E_+. Moreover, $\sum_{i=1}^m \omega^{i,\lambda} = \lambda\omega + (1-\lambda)mv$ for all $0 < \lambda < 1$ and so $\sum_{i=1}^m \omega_s^{i,\lambda} \in \bigcap_{i=1}^m C_s^i$ for each s.

Let $\mathcal{E}_\lambda$ be the economy with $(\omega^{i,\lambda})_{i=1}^m$ as initial endowments. Assumptions **A1, A2, A4** hold for each such economy. By Lemma 7.3, each such economy has a pseudo-equilibrium $(x^\lambda, p^\lambda, L^\lambda)$. Furthermore, the net $\{p^\lambda\}$ can be chosen to be in the $\sigma(E', E)$-compact set

$$\Delta^* = \Big\{p \in E'_+ :\ p \cdot \omega = 1 \ \text{ and } \ p \cdot \big(m\omega + \textstyle\prod_{s=1}^S N_s\big) \subseteq \mathbb{R}_+\Big\}\,,$$

where each N_s is a convex solid τ_s-neighborhood of zero in E_s satisfying

$$\omega_s + N_s \subseteq \bigcap_{i=1}^{m} C_s^i\,.$$

So, we can assume that $(x^\lambda, p^\lambda, L^\lambda) \xrightarrow[\lambda]{} (x, p, L)$ in $(E_+, \mu)^m \times (E', \sigma(E', E)) \times \mathbb{G}_J$ with $p \in \Delta^*$. By Lemma 5.11 the tuple (x, p, L) is a pseudo-equilibrium. ■

Our third major result will be proven next.

Theorem 3.7. *If* **A1, A2,** *and* **A3** *are satisfied, each E_s is a locally convex-solid Riesz space, $\omega \gg 0$ and $\mathcal{M} \subseteq E_\omega$, then there exists a pseudo-equilibrium. In particular, for a $\mathfrak{J}(\mathfrak{S} - \mathfrak{J})$-strongly dense set of assets structures in $\mathcal{G}$ there exists an equilibrium.*

Proof. Notice that the principal ideal $E_\omega = \bigcup_{n=1}^\infty n[-\omega, \omega]$ contains $\mathcal{M}$ and each ω^i. Restricting the economy to E_ω, it is easy to verify that all conditions of Lemma 7.3 are satisfied. Therefore, there exists a pseudo-equilibrium (x, p, L) for this restricted economy with p being a τ-continuous positive functional on E_ω. Thus p can be extended to a positive continuous linear functional on all of E, which we shall denote it by p again. We show that (x, p, L) is a pseudo-equilibrium for the unrestricted economy.

Since (x, p, L) is a pseudo-equilibrium on E_ω, it must be the case that $p \gg 0$ on E_ω. Since ω is a quasi-interior point of E and τ is locally solid it must be the case that $p \gg 0$ on E.

Now take any $z \in E_+$ such that $z \in P_i(x^i)$ and $p \,\square\, (z - \omega^i) \in L$. We shall use Lemma 4.9 to show that $p \cdot z > p \cdot \omega^i$. To this end, consider the principal ideal $E_{\omega+z}$. Clearly, $z \in E_{\omega+z}$ and $\omega + z$ is an internal point of $E^+_{\omega+z}$. Now take an internal point $y \in E^+_{\omega+z}$ such that $y \in P_i(x^i)$ and $p \,\square\, (y - \omega^i) \in L$. We see that $p \,\square\, y \gg 0$.

There exists a net $\{y^\lambda\}$ of positive vectors in E^+_ω that converges to y. Since p is τ-continuous, it must be the case that $p \,\square\, y^\lambda$ converges to $p \,\square\, y \gg 0$. Therefore, for λ large enough $p \,\square\, y^\lambda \gg 0$. For these λ define the vectors

$$h^\lambda = \Big(\frac{p_0 \cdot y_0}{p_0 \cdot y_0^\lambda}\, y_0^\lambda, \frac{p_1 \cdot y_1}{p_1 \cdot y_1^\lambda}\, y_1^\lambda, \ldots, \frac{p_S \cdot y_S}{p_S \cdot y_S^\lambda}\, y_S^\lambda\Big)\,.$$

Clearly, $h^\lambda \in E^+_\omega$ and $p \,\square\, (h^\lambda - \omega^i) = p \,\square\, (y - \omega^i) \in L$. Notice also that $h^\lambda \to y$, and thus for large enough λ we have $h^\lambda \in P_i(x^i)$. This implies that for large enough λ we have $p \cdot h^\lambda > p \cdot \omega^i$. This in turn implies that $p \cdot y \geq p \cdot \omega^i$. In view of Lemma 4.9, it must be the case that $p \cdot z > p \cdot \omega^i$. Therefore, (x, p, L) is a pseudo-equilibrium. Once again, the final claim of the lemma follows from Lemma 5.9. ■

We are now ready to prove fourth and last of our major results.

Theorem 3.8. *If* **A1, A2,** *and* **A4** *are satisfied, each E_s is a locally convex-solid Riesz space, then there exists a pseudo-equilibrium. In particular, for a $\mathfrak{J}(\mathfrak{S} - \mathfrak{J})$-strongly dense set of assets structures in $\mathcal{G}$ there exists an equilibrium.*

Proof. Consider the net of principal ideals $\{E^\lambda\}$ directed by inclusion. Truncate this net and assume that each E^λ contains each ω^i and $\mathcal{M}$. For each such ideal consider the economy $\mathcal{E}_\lambda$ which is the restriction of the economy to E^λ. We know from Lemma 7.4 that each such economy has a pseudo-equilibrium $(x^\lambda, p^\lambda, L^\lambda)$. Now for each s, let N_s be a convex and solid τ_s-neighborhood of zero in E_s such that $\omega_s + N_s \subseteq \bigcap_{i=1}^m C_s^i$.

We can assume that $p^\lambda \cdot \omega = 1$. We see from Lemma 7.4 that each p_s^λ is positive on $(\omega_s + N_s) \cap E^\lambda$. This means that each p^λ can be extended to a τ-continuous linear functional on E that lies in the $\sigma(E', E)$-compact set

$$\Delta^* = \left\{ p \in E'_+ \colon\ p \cdot \omega = 1 \ \text{ and } \ p \cdot \left(m\omega + \prod_{s=1}^{S} N_s \right) \subseteq \mathbb{R}_+ \right\},$$

Therefore, we can suppose that $p^\lambda \in E'$ and that (by moving to a subnet) $(x^\lambda, p^\lambda, L^\lambda)$ converges in $(E_+, \mu)^m \times (E'_+, \sigma(E', E)) \times \mathbb{G}_J$ to (x, p, L) with $p \cdot \omega = 1$. By Lemma 5.11, we see that (x, p, L) is a pseudo-equilibrium. The proof of the final claim follows from Lemma 5.9. ■

Appendix A. Infinitely many states

Our theorems have interesting applications to models with infinitely many states of the world. Consider for instance an economy over three periods $t = 0, 1, 2$. In this model the time line is as follows:

(1) Consumers consume in periods zero and two.
(2) Assets pay in period one according to the expected value of an asset conditional upon the information in that period.

There are infinitely many final period states of the world. These are given by a probability space $(\Omega, \mathcal{F}_2, \pi)$. We are also given a subalgebra $\mathcal{F}_1$ denoting the information that is available in period one. We assume that $\mathcal{F}_1$ is generated by a finite partition $\mathcal{S} = \{\mathcal{S}_1, \ldots, \mathcal{S}_S\}$ of Ω with S elements.

There are ℓ commodities traded in periods zero and two. Thus, the commodity space is

$$E = \mathbb{R}^\ell \times L_2(\mathbb{R}^\ell, \Omega, \mathcal{F}_2, \pi),$$

where $L_2(\mathbb{R}^\ell, \Omega, \mathcal{F}_2, \pi)$ is the space of all π-square integrable functions from Ω to $\mathbb{R}^\ell$. Notice that $E = \mathbb{R}^\ell \times \prod_{s=1}^{S} L_2(\mathbb{R}^\ell, \Omega, \mathcal{S}_s, \pi)$. So let $E_0 = \mathbb{R}^\ell$ and $E_s = L_2(\mathbb{R}^\ell, \Omega, \mathcal{S}_s, \pi)$.

There are J assets given by an operator $T\colon M \to \prod_{s=1}^{S} L_2(\mathbb{R}^\ell, \Omega, \mathcal{S}_s, \pi)$. So, for each p in $\mathbb{R}^\ell \times \prod_{s=1}^{S} L_2(\mathbb{R}^\ell, \Omega, \mathcal{S}_s, \pi)$, each portfolio $\theta \in M$ and each assets price $q \in M'$ the budget sets for this model are as follows:

$$\beta_i(p, q, \theta) \;=\; \left\{ x \in E_+ \colon \begin{bmatrix} p_0(x_0 - \omega_0^i) \\ \int_{\mathcal{S}_1} p(x_1 - \omega_1^i)\, d\pi \\ \vdots \\ \int_{\mathcal{S}_S} p(x_S - \omega_S^i)\, d\pi \end{bmatrix} \leq \begin{bmatrix} -q(\theta) \\ \int_{\mathcal{S}_1} (T\theta^1)\, d\pi \\ \vdots \\ \int_{\mathcal{S}_S} (T\theta^S)\, d\pi \end{bmatrix} \right\}.$$

Clearly, our theorems imply the existence of equilibrium for this incomplete markets model with infinitely many states of the world.

Appendix B. The Grassmanian manifold

In this appendix, we shall introduce briefly the Grassmanian manifold and list its basic properties. For our work, the **Grassmanian manifold** is the collection of all J-dimensional vector subspaces of $\mathbb{R}^{S+1}$. For simplicity this we shall take the space $\mathbb{R}^{S+1}$ as given and denote this collection by $\mathbb{G}_J$, i.e.,

$$\mathbb{G}_J = \left\{ L \subseteq \mathbb{R}^{S+1} \colon\ L \text{ is a } J\text{-dimensional vector subspace of } \mathbb{R}^{S+1} \right\}.$$

There is a natural topology on $\mathbb{G}_J$ that makes it a compact metrizable space. For a comprehensive study of the Grassmanian manifold we refer the reader to the monograph Abraham et al. (1988).

First we discuss the notion of a topological manifold. Let X be an arbitrary set and let E be a Banach space. An ***atlas*** on X over the Banach space E is a family of pairs $(U_i, \psi_i)_{i\in I}$ such that:

(1) Each U_i is a non-empty subset of X and the family $(U_i, \psi_i)_{i\in I}$ covers X, i.e., $X = \bigcup_{i\in I} U_i$.
(2) Each ψ_i is a one-to-one mapping from U_i onto an open subset of E.
(3) For each pair of indices i and j the set $\psi_i(U_i \cap U_j)$ of E is an open subset of E.
(4) The mapping $\psi_j \circ \psi_i^{-1} \colon \psi_i(U_i \cap U_j) \to \psi_j(U_i \cap U_j)$ is a homeomorphism.

The basic result regarding atlases is the following.

Lemma B.1. *If $(U_i, \psi_i)_{i\in I}$ is an atlas of a set X over a Banach space E, then there is a unique topology τ on X (called the topology **induced** by the atlas $(U_i, \psi_i)_{i\in I}$ on X) for which each U_i is open and each ψ_i is a homeomorphism.*

If the topology τ induced on a set X by an atlas is Hausdorff and X is second countable, then the topological space (X, τ) is called a **topological manifold** over on E. If $E = \mathbb{R}^n$, then (X, τ) is referred to as an **n-dimensional topological manifold**.

We note the following result whose proof can be found in Conlon (2001, Theorem 1.5.5.).

Lemma B.2. *If M is a compact n-dimensional topological manifold, then there is an integer $k > n$ and a topological embedding $i \colon M \hookrightarrow \mathbb{R}^k$.*

From now on $E = \mathbb{R}^{S+1}$ and we are interested in the canonical local "Euclidean" structure on the Grassmanian manifold $\mathbb{G}_J$ that makes it a $J(S+1-J)$-dimensional manifold.

Fix a J-dimensional vector subspace $F \in \mathbb{G}_J$. For each complement G of F (i.e., $E = F \oplus G$) we define a set U_G^F by

$$U_G^F = \{H \in \mathbb{G}_J \colon \ E = H \oplus G\}.$$

Notice that $F \in U_G^F$ and that the family of sets

$$\{U_G^F \colon \ F \in \mathbb{G}_J, \ G \in \mathbb{G}_{S+1-J} \text{ and } E = F \oplus G\}$$

is a cover of $\mathbb{G}_J$.

We now define a one-to-one mapping $\psi_{F,G} \colon U_G^F \to \mathcal{L}(F, G)$, the vector space of all linear operators from F to G. Fix a set U_G^F. Let $\pi_G \colon E \to G$ and let $\pi_F \colon E \to F$ denote the projections induced by the direct sum decomposition $E = F \oplus G$. Furthermore, for each $H \in U_G^F$ let $\pi_{H,G} = \pi_G|H$ and $\pi_{H,F} = \pi_F|H$. Notice that $\pi_{H,F}$ is a linear isomorphism between H and F. Next, define the mapping $\psi_{F,G} \colon U_G^F \to \mathcal{L}(F, G)$ via the formula

$$\psi_{F,G}(H) = \pi_{H,G} \circ \pi_{H,F}^{-1}.$$

We have the following.

Lemma B.3. *Each mapping $\psi_{F,G} \colon U_G^F \to \mathcal{L}(F, G)$ is one-to-one and its range is an open subset of $\mathcal{L}(F, G)$. Moreover, if $H \in U_G^F$, then $\psi_{F,G}(H)$ as an operator from F to G whose graph in $F \oplus G$ is H. In particular, $\psi_{F,G}(F) = 0$.*

Identifying (appropriately) $\mathcal{L}(F, G)$ with $\mathbb{R}^{J(S+1-J)}$, we can state now a basic result as follows; for a proof see Abraham et al. (1988, Chapter 3.).

Lemma B.4. *The family*

$$\{U_G^F\colon\ F \in \mathbb{G}_J,\ \ G \in \mathbb{G}_{S+1-J}\ \text{ and } E = F \oplus G\}$$

is an atlas on $\mathbb{G}_J$ *over* $\mathcal{L}(F,G) = \mathbb{R}^{J(S+1-J)}$ *under which* $\mathbb{G}_J$ *is a compact metrizable and connected* $J(S+1-J)$*-dimensional topological manifold.*

Convergence in $\mathbb{G}_J$ is characterized as follows.

Lemma B.5. *The following statements hold true:*

(1) *A net* $\{F_\alpha\}$ *satisfies* $F_\alpha \to F$ *in* $\mathbb{G}_J$ *if and only if:*
   (a) $F_\alpha \in U_G^F$ *holds for all* α *eventually large, and*
   (b) $\psi_{F,G}(F_\alpha) \to \psi_{F,G}(F) = 0$ *in* $\mathcal{L}(F,G)$.

(2) *If* $F_\alpha \to F$ *holds in* $\mathbb{G}_J$*, then for each* $x \in F$ *there exists a net* $\{x_\alpha\} \subseteq \mathbb{R}^{S+1}$ *satisfying* $x_\alpha \in F_\alpha$ *for all* α *and* $x_\alpha \to x$ *in* $\mathbb{R}^{S+1}$.

(3) *The correspondence* $\lambda\colon \mathbb{G}_J \twoheadrightarrow \mathbb{R}^{S+1}$*, defined by* $\lambda(F) = F$*, has a closed graph.*

Proof. (1) If $F_\alpha \to F$ in $\mathbb{G}_J$, then eventually $F_\alpha \in U_G^F$ for each G satisfying $E = F \oplus G$ since $F \in U_G^F$ and the set U_G^F is open. Moreover, since $\psi_{F,G}$ is a homeomorphism, it follows that $\psi_{F,G}(F_\alpha) \to \psi_{F,G}(F) = 0$ in $\mathcal{L}(F,G)$.

Conversely, assume that $F_\alpha \in U_G^F$ for all α eventually large and $\psi_{F,G}(F_\alpha) \to \psi_{F,G}(F) = 0$ in $\mathcal{L}(F,G)$. Let N be an open neighborhood of F. Pick G satisfying $E = F \oplus G$ and note that U_G^F is an open neighborhood of F. Therefore, $N \cap U_G^F$ is an open neighborhood of F and $\psi_{F,G}[N \cap U_G^F]$ is an open neighborhood of $\psi_{F,G}(F) = 0$ in $\mathcal{L}(F,G)$. Therefore, eventually we have $\psi_{F,G}(F_\alpha) \in \psi_{F,G}[N \cap U_G^F]$ and $F_\alpha = \psi_{F,G}^{-1}(F_\alpha)$ is eventually in $N \cap U_G^F$. This implies $F_\alpha \to F$.

(2) Let $F_\alpha \to F$ in $\mathbb{G}_J$ and fix $x \in F$. Pick G such that $E = F \oplus G$. Since U_G^F is open and $F \in U_G^F$, we can assume without loss of generality that $F_\alpha \in U_G^F$ for all α.

Clearly, $\psi_{F,G}(F_\alpha) \to \psi_{F,G}(F) = 0$. Let $y_\alpha = \psi_{F,G}(F_\alpha)(x)$ and note that $x_\alpha = x + y_\alpha \in F_\alpha$. Then $y_\alpha \to 0$ and so $x_\alpha \to x$. Therefore, the net $\{x_\alpha\}$ satisfies the desired properties.

(3) Let $(F_n, x_n) \to (F, x)$ in $\mathbb{G}_J \times \mathbb{R}^{S+1}$ such that $x_n \in F_n$ for all n. We need to show that $x \in F$. Pick a subspace G such that $\mathbb{R}^{S+1} = F \oplus G$. Since U_G^F is open and $F \in U_G^F$, we can assume without loss of generality that $F_n \in U_G^F$ for all n.

According to part (1) we have $\psi_{F,G}(F_n) \to \psi_{F,G}(F) = 0$ in $\mathcal{L}(F,G)$. In particular, it follows that $\psi_{F,G}(F_n)(x_n) \to \psi_{F,G}(F)(x) = 0$. This implies $x = x + 0 \in F$. ∎

Appendix C. Continuous and approximate selections

Let S be a metric space and Y a normed space. Define the following sets:

$$\begin{aligned} \mathfrak{K}(Y) &= \{X \subseteq Y\colon X \text{ nonempty and convex}\}. \\ \mathfrak{D}(Y) &= \{X \in \mathfrak{K}(Y)\colon X \text{ is finite dimensional or closed or has an interior point}\}. \end{aligned}$$

Recall that a correspondence $P\colon S \twoheadrightarrow Y$ ***admits a continuous selection*** if there exists a function a continuous $f\colon S \to Y$ such that

$$f(x) \in P(x)$$

for every $x \in S$.

Lemma C.1 (Michael (1956)). *If Y is a separable Banach space and $P\colon S \to \mathfrak{D}(Y)$ is lower hemicontinuous, then P admits a continuous selection.*

Let A be a metric space. An open ball of radius ε about a subset X of A is denoted by $B(X,\varepsilon)$. The range of a mapping $P\colon S \twoheadrightarrow Y$ is denoted by $\mathcal{R}(P)$ and we denote by $\mathrm{co}(\mathcal{R}(P))$ the convex hull of $\mathcal{R}(P)$.

Lemma C.2 (Cellina (1969)). *Let S be a compact metric space and d let be the product metric on $S \times Y$. If $P\colon S \to \mathfrak{K}(Y)$ is an upper hemicontinuous correspondence, then for each $\varepsilon > 0$ there exists a continuous function $f\colon S \to B(\mathcal{R}(P),\varepsilon) \cap co(\mathcal{R}(P))$ such that*

$$\sup\{d(y, GrP)\colon y \in Grf\} < \varepsilon\,.$$

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C. D. Aliprantis, Department of Economics, Krannert School of Management, Purdue University, W. Lafayette IN 47907-1310, USA

E-mail address: aliprantis@mgmt.purdue.edu

R. Tourky, Department of Economics, University of Melbourne, Melbourne, Victoria 3010, AUSTRALIA

E-mail address: rtourky@unimelb.edu.au