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Sharing Risk Efficiently under Suboptimal Punishments for Defection

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Sharing Risk Efficiently under Suboptimal Punishments for Defection

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Abstract

The paper studies efficient risk sharing under limited enforcement (or "limited commitment") constraints determined by the threat of punishment after misbehavior. As in Kocherlakota (1996), I assume that society chooses from among those allocations implementable in subgame perfect equilibrium. Rather than assume that punishments implement the least desirable continuation equilibrium, I allow that punishments may be suboptimally specified from the point of view of enforcement. I characterize (up to a technical condition) the set of allocations that may be interpreted as efficient subject to enforcement by some punishment. The conditions rationalizing such efficiency are very weak; they are (*i*) resource exhaustion, (*ii*) satisfaction of individual rationality constraints at each continuation, and (*iii*) finiteness of the value of the allocation under the implicit decentralizing price system, the "high implied interest rates" condition of Alvarez and Jermann (2000). I show that efficient allocations may be decentralized and I state versions of the Welfare Theorems for my environment.

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1 Introduction

What characteristics of the consumption processes of a cohort of agents over time are consistent with efficient risk sharing? Under the canonical specification of preferences for consumption used in the macroeconomics literature, we can reject the hypothesis whenever we can conclude that intertemporal marginal rates of substitution are not equated across agents at each point in time. Indeed, one is not hard pressed to assemble data that contradict this implication of efficiency.[1]

Recent research has focused on deriving the observable implications for efficiency and equilibrium in economies encumbered by enforcement (or "commitment") frictions, but that are otherwise frictionless. In particular, these economies feature perfect information and no transactions costs. Of critical importance in such work is the specification of what agents may accomplish after a behavioral defection from prescribed or contracted actions; that is, specification of punishments. In this respect, Kehoe and Levine (1993) and Kocherlakota (1996) have set the paradigm adopted by the rest of the literature. Following Abreu (1988), these authors each suppose that agents are treated to the harshest punishment that is available subject to the exogenously specified "autarkic" capabilities of individuals.

A viewpoint motivating the present work is that efficient mechanisms (e.g., markets) may organize the front line (i.e., the equilibrium path) of economic behavior, but the response to a defection from the norm of prescribed behavior may be suboptimal from the point of view of enforcement. More precisely, rather than choosing

[1]Empirical investigations include Altug and Miller (1990), Mace (1991), Nelson (1994), Hayashi et al (1996), Townsend (1994), Ham and Jacobs (2000). An interesting recent contribution that tests (inter alia) for implications of efficiency under limited commitment is Ligon et al. (2002).

punishments optimally, I assume that defecting agents are punished by reversion to some arbitrary subgame perfect continuation equilibrium.

The fundamental results of the paper constitute a characterization (up to a technical condition) of the set of consumption allocations can be rationalized as efficient with respect to some (generically suboptimal) punishments. The conditions affording such an interpretation are (*i*) exhaustion of resources, (*ii*) satisfaction of individual rationality conditions at each continuation, and (*iii*) finiteness of the value of the allocation under the implicit decentralizing price system, the "high implied interest rates" condition of Alvarez and Jermann (2000). These conditions are obviously quite weak.

I also show how efficient allocations can be decentralized in Arrow-Debreu markets with "solvency constraints" that set lower limits on agents' claims positions as in Alvarez and Jermann (2000). I extend their results to the present environment by showing that, when the solvency constraints are set appropriately, equilibria of the market economy coincide with the set of efficient allocations. I also show that versions of their Welfare Theorems hold for the present environment.

There are a number of studies of economies with enforcement frictions in which defection is assumed to induce punishments other than autarkic consumption of an endowment. Important contributions include Kehoe and Perri (2002, 2004), Lustig (2004), Lustig and Van Nieuwerburgh (2005), Jeske (2006), and Krueger and Fernandez-Villaverde (2001). I regard the present work as providing some bounds on what behavior may be rationalized by modeling punishment institutions more explicitly as these papers do.

In the next section, I introduce the environment and the underlying game played by its agents. The analysis and the principal results are contained in the third section. The fourth section considers decentralization of efficient allocations in markets with solvency constraints. The final section concludes.

2 Model

2.1 Environment

Time is discrete and infinite, and is indexed by $t = 0, 1, 2, ...$ There are $I < \infty$ agents in the economy indexed by $i \in \mathcal{I} = \{1, 2, ..., I\}$. Stochastic features of the environment are summarized by a Markov process s_t taking values in a finite set $\mathcal{S}$. The probability of a transition from s to s' is denoted $\pi(s'|s)$, and I assume (except in several examples with deterministic transitions) that $\pi(s'|s) > 0$ for all $s, s' \in \mathcal{S}$. I write $s^t \in \mathcal{S}^{t+1}$ for the history of process up to date t, the *state history.* If s^τ is a feasible continuation of a state history s^t, e.g., $s^\tau \equiv (s^t, s_{t+1}, ..., s_\tau)$ and $\tau \geq t$, I will write $s^\tau \succsim s^t$. I abuse the notation by writing $\pi(s^\tau|s^t)$ for the probability that state history s^τ obtains conditional on reaching s^t. All stochastic processes in this paper are assumed to be adapted to s_t. For any such process x, I will write $x|s^t$ for the continuation of x after state history s^t; that is, $x|s^t$ is a stochastic process for initial state s_t.

There is a single (consumption) good in the economy available at each date. The aggregate endowment of the good is one unit. At each state history s^t at which the state is $s_{t,}$ agent i is endowed with a fraction $e^i(s_t) > 0$ units of the good, where

$\sum_i e^i(s_t) = 1$. A *(feasible) allocation* is a stochastic process c such that

$$c\left(s^t\right) \in \mathbb{R}_+^I \text{ and } \sum_i c^i\left(s^t\right) \leq 1 \tag{1}$$

for all $s^t \in \mathcal{S}^{t+1}$.[2]

After any state history s^t, agent i evaluates the continuation allocation $c|s^t$ according to the criterion

$$\mathcal{U}^i\left(c|s^t\right) := \sum_{\tau=t}^{\infty} \sum_{s^\tau} \beta^{\tau-t} u\left(c^i\left(s^\tau\right)\right) \pi\left(s^\tau|s^t\right),$$

where $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is differentiable, strictly concave, and strictly increasing. I also assume that the Inada condition $\lim_{c \downarrow 0} u'(c) = +\infty$ holds. For future reference, I denote the payoff from autarkic consumption as

$$\mathcal{U}_{aut}^i\left(s_t\right) := \sum_{\tau=t}^{\infty} \sum_{s^\tau} \beta^{\tau-t} u\left(e^i\left(s_\tau\right)\right) \pi\left(s^\tau|s^t\right).$$

2.2 A Game of Multilateral Transfers

The game defined here is a generalization of that studied by Kocherlakota (1996) to the case of Markov shocks and an arbitrary number of agents.

[2] The space in which allocations lie may be interpreted to be l_∞, with the supremum norm in this context defined by

$$\|c\| := \sup_{i,t,s^t} \left\|c^i\left(s^t\right)\right\|. \tag{2}$$

The set of actions available to i in state s^t is

$$A^i(s_t) = \left\{ a^i \in \mathbb{R}_+^I : \sum_j a_j^i \leq e^i(s_t) \right\},$$

which will be interpreted as the set of vectors of non-negative transfers to the agents in the game feasible from the realized endowment. A *path* is a function α from state histories s^t to profiles of actions such that $\alpha^i(s^t) \in A^i(s_t)$ for all i and s^t. Note that a path induces a consumption allocation as

$$c^i\left(s^t\right) = \gamma(\alpha)\left(s^t\right) \equiv e^i(s_t) - \sum_{j \in \mathcal{I}} \alpha_j^i\left(s^t\right) + \sum_{j \in \mathcal{I}, j \neq i} \alpha_i^j\left(s^t\right). \tag{3}$$

Also note that disposal of the good may be accomplished by agent i by setting $a_i^i > 0$.

A *game history* for the period t is a pairing of a state history s^t and a history of actions played up to date $t-1$. I denote an arbitrary game history of length $t+1$ by $h^t = (s^t, a^{t-1})$, where $a^t = (a_0, ..., a_t)$ and a_t is the profile of actions taken at t.

A (pure) *strategy* for player i is a function σ^i from game histories to actions feasible for agent i for the current state; that is, $\sigma^i(h^t) \in A^i(s_t)$. A *strategy profile* σ is a collection of strategies, one for each player. Note that a strategy profile σ induces a path, say $\alpha(\sigma)(s^t)$ describing the sequence of actions followed when players abide by σ. It follows that a strategy induces an allocation, as well; and (abusing the notation slightly) I denote this by

$$\gamma(\sigma)\left(s^t\right) \equiv \gamma(\alpha(\sigma))\left(s^t\right) = e^i(s_\tau) - \sum_{j \in \mathcal{I}} \alpha_j^i(\sigma)\left(s^t\right) + \sum_{j \in \mathcal{I}, j \neq i} \alpha_i^j(\sigma)\left(s^t\right) \tag{4}$$

When agents play according to σ, the continuation expected payoff delivered to i after game history h^t can be written as $\mathcal{U}^i\left(\gamma\left(\sigma\left(h^t,\cdot\right)\right)|s_t\right),$ where $\sigma\left(h^t,\cdot\right)$ is the strategy induced by σ for the subgame defined by starting from game history h^t and s_t is the terminal state of h^t. A *subgame perfect equilibrium (SPE)* is a strategy profile σ such that, for each i, t, h^t, and $\tilde{\sigma} := \left(\tilde{\sigma}^i, \sigma^{-i}\right)$,

$$\mathcal{U}^i\left(\gamma\left(\sigma\left(h^t,\cdot\right)\right)|s_t\right) \geq \mathcal{U}^i\left(\gamma\left(\tilde{\sigma}\left(h^t,\cdot\right)\right)|s_t\right), \tag{5}$$

where $\tilde{\sigma}^i$ is any alternative strategy for agent i. In this case, I will say that $\gamma(\sigma)$ is an *SPE allocation.* The one-deviation property of subgame perfect equilibria induces the following characterization of the set of all SPE allocations; the proof of all of the results in the paper are contained in the Appendix.

Lemma 1 *There is an SPE that implements c on the equilibrium path (i.e., c is an SPE allocation) if and only if c is feasible and $\mathcal{U}^i\left(c|s^t\right) \geq \mathcal{U}^i_{aut}\left(s_t\right)$ for all i,t, and $s^t \succsim s_0$.*

In what follows, I will write Σ for the correspondence mapping from $\mathcal{S}$ to the set of all SPEs starting from a given state.

2.3 Punishments and the Strategies of Interest

Let $f^i\left(s^t,\cdot\right)$ be a selection from Σ for each i and s^t; that is, $f^i\left(s^t,s'\right) \in \Sigma\left(s'\right)$ for each s'. In this case, I will call f an *(implementable) punishment.* A pair (α, f) of a path and a punishment induce a strategy profile, say $\sigma(\alpha, f)$, as follows. First, players are directed to choose their actions according to the path α at each game history whenever no player has defected unilaterally from the assignment at a previous history. Multilateral defections are ignored; and upon the first perpetration of a unilateral defection in the game, say by agent i at game history $h^t = (s^t, a^{t-1})$, $\sigma(\alpha, f)$ directs that play in the continuation follow $f^i(s^t, s')$.

Given f, let us say that c *is supported by* f if there exists a path α such that $c = \gamma(\alpha)$ and $\sigma(\alpha, f)$ is an SPE; in this case, write $c \in \mathcal{P}(f)$.

Given a path α, the amount of consumption that i can obtain by defecting unilaterally from the path at state history s^t is bounded above by

$$g^i(\alpha)\left(s^t\right) \equiv e^i(s_t) + \sum_{j \in \mathcal{I}, j \neq i} \alpha_i^j\left(s^t\right). \tag{6}$$

It is an important property of the restrictions placed on play after a defection in the environment is that the payoff to an agent in the period after a defection depends only on the identity of the defecting agent. This property and the one-deviation property induce the following characterization of the set of equilibria of the form $\sigma(\alpha, f)$.

Lemma 2 *Given a path α and a punishment f, $\sigma(\alpha, f)$ is an SPE if and only if*

$$\mathcal{U}^i\left(\gamma(\alpha)|s^t\right) \geq u\left(g^i(\alpha)\left(s^t\right)\right) + \beta \sum_{s'} \mathcal{U}^i\left(\gamma\left(f^i\left(s^t, s'\right)\right)|s'\right)\pi\left(s^t, s'|s^t\right) \tag{7}$$

for all i, t, and s^t.

I stress an analogy between strategies of the form $\sigma(\alpha, f)$ and those supported by Abreu's (1988) "optimal simple penal code" that imposes that any unilateral defection triggers a reversion to a particular equilibrium continuation that depends only on the identity of the defector. Each concept has the property that punishments are independent of the event that triggers them. The difference here is that the reversion need not be the worst SPE continuation; rather, I allow that a lighter punishment may be prescribed.

In much of the related literature, the properties of allocations that can be supported as SPE allocations by the threat of reversion to autarkic strategies is studied. In what follows, I will address questions that are more general in the sense that I do not take a stand on the form of the punishments, except to require that they implement equilibrium continuations. First, I will examine the properties of allocations that are optimal with respect to a specific welfare criterion subject to being supported by a given punishment for defection. Second, I will ask when it can be gleaned that a given allocation is efficient in this sense for some punishment.

3 Efficient SPE Allocations

I begin this section by showing how to construct, for a given allocation, a path that supports the allocation in way that minimizes the incentive for defection. A useful result along the lines of the discussion at the close of the previous subsection is that this may be done independently of the punishment itself.

Given an allocation c, I construct path to be denoted $\hat{\alpha}(c)$ as follows. Define $\delta^i(s^t) \equiv e^i(s_t) - c^i(s^t)$; and define $\mathcal{K}(s^t) \equiv \left\{k \in \mathcal{I} | \delta^k(s^t) > 0\right\}$, and $\bar{\mathcal{K}}(s^t) \equiv$

$\mathcal{I} \backslash \mathcal{K}(s^t)$. Let

$$\begin{aligned}\Delta\left(s^t\right) &:= \sum_{k \in \mathcal{K}(s^t)} \delta^k\left(s^t\right) \\ &= 1 - \sum_i c^i\left(s^t\right) - \sum_{k \in \bar{\mathcal{K}}(s^t)} \delta^k\left(s^t\right).\end{aligned}$$

Now for $k \in \bar{\mathcal{K}}(s^t)$, set $\hat{\alpha}_j^k(c)(s^t) = 0$ for each j. For $k \in \mathcal{K}(s^t)$, set

$$\hat{\alpha}_j^k(c)\left(s^t\right) = \begin{cases} 0 \text{ if } j \in \mathcal{K}(s^t),\ j \neq k \\ \{[1 - \sum_i c^i(s^t)] / \Delta(s^t)\} \delta^k(s^t), \text{ if } j = k \\ -\left[\delta^j(s^t) / \Delta(s^t)\right] \delta^k(s^t) \text{ if } j \in \bar{\mathcal{K}}(s^t). \end{cases}$$

It may be seen that $\hat{\alpha}(c)$ implements c with the minimal volume of transfers. In particular, an agent that consumes less that his endowment at a given state history makes transfers totalling $e^i(s_t) - c^i(s^t)$; and one that consumes more than his endowment makes no transfers. It follows that $g^i(\alpha)(s^t) = \max\{c^i(s^t), e^i(s_t)\}$. The utility of making the minimal volume of transfers is seen in the following proposition.

Proposition 1 *An allocation c is supported by a punishment f if and only if $\sigma(\hat{\alpha}(c), f)$ is an SPE.*

One interpretation of this result is that intra-temporal transfer arrangements may be chosen to optimally apply the enforcement technology; and, as long as the enforcement technology is described by reversion to a punishment chosen only as a function of the identity of the defector, the implementing path may be chosen independently of the punishments. In particular, a net clearing mechanism is bestsuited in this regard.

Corollary 1 *An allocation c is supported by a punishment f if and only if*

$$\mathcal{U}^i\left(c|s^t\right) \geq u\left(e^i\left(s_t\right)\right) + \beta \sum_{s'} \mathcal{U}^i\left(\gamma\left(f^i\left(s^t, s'\right)\right)|s'\right) \pi\left(s^t, s'|s^t\right) \tag{8}$$

and

$$\mathcal{U}^i\left(c|s^t\right) \geq u\left(c^i\left(s^t\right)\right) + \beta \sum_{s'} \mathcal{U}^i\left(\gamma\left(f^i\left(s^t, s'\right)\right)|s'\right) \pi\left(s^t, s'|s^t\right) \tag{9}$$

for all i, t, and s^t.

I will say that c *is efficient with respect to* f if c is maximal in $\mathcal{P}(f)$ for

$$\sum_i \gamma^i \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u\left(c^i\left(s^t\right)\right) \pi\left(s^t|s_0\right) \tag{10}$$

for some γ in the I-dimensional unit simplex. From Corollary 1 and the monotonicity of $u(\cdot)$, it is equivalent to say that c is efficient with respect to f if, for some γ, c solves the programming problem of maximizing (10) subject to the feasibility constraints

$$\sum_i c^i\left(s^t\right) \leq 1 \tag{11}$$

for all t and s^t, and the inequality constraints (8) and (9). In what follows, I apply the term *efficient* generally to an allocation to mean that it is efficient with respect to some punishment.

In the risk-sharing literature, equation of agents' intertermporal marginal rates of substitution is the quintessential criterion for efficiency. In economies with limited enforcement, binding enforcement constraints may preclude that acheivement. Alvarez and Jermann (2000) show that the following definitions are useful for describing a phenomenon that is implied by efficiency in such environments. For a given allocation c, define

$$\bar{q}\left(s^{t+1}|c\right) := \max_j \left\{\beta \frac{u'\left(c^j\left(s^{t+1}\right)\right)}{u'\left(c^j\left(s^t\right)\right)} \pi\left(s_{t+1}|s_t\right)\right\} \tag{12}$$

and

$$\bar{Q}\left(s^{t+1}|c\right) := \bar{q}\left(s^1|c\right)\bar{q}\left(s^2|c\right)\cdots\bar{q}\left(s^{t+1}|c\right). \tag{13}$$

I will say that c has *high implied interest rates* (Alvarez and Jermann (2000)) or $c \in HIR$ if

$$\sum_{t=0}^{\infty}\sum_{s^t}\bar{Q}\left(s^t|c\right)\left[\sum_i c^i\left(s^t\right)\right]\pi\left(s^t\right) < \infty.$$

The following proposition establishes that having high implied interest rates is a general property of efficient consumption allocations, at least up to the restriction that $\mathcal{P}(f)$ has an interior point.

Proposition 2 *Suppose that $\hat{c}$ is efficient with respect to a given punishment f, and suppose that $\mathcal{P}(f)$ has an interior point. Then $\hat{c} \in HIR$.*

The following example applies the concepts developed above and Proposition 2 to a specific simple environment. It also shows that an efficient allocation need not exhibit high implied interest rates when $\mathcal{P}(f)$ does not have an interior point.

Example 1 Consider an two-agent economy in which agents' endowments alternate deterministically between $e_H = \frac{2}{3}$ and $e_L = 1 - e_H = \frac{1}{3}$, and let us suppose that

$u(c) = \ln c$. The utility of autarkic consumption is

$$\mathcal{V}_H(\beta) \equiv \frac{\ln\left(\frac{2}{3}\right) + \beta \ln\left(\frac{1}{3}\right)}{1-\beta^2} \tag{14}$$

for the agent with the high endowment, and

$$\mathcal{V}_L(\beta) \equiv \frac{\ln\left(\frac{1}{3}\right) + \beta \ln\left(\frac{2}{3}\right)}{1-\beta^2} \tag{15}$$

for the agent with the low one.

In this example, I consider the punishment defined at each continuation by the autarkic strategies; call this punishment f_{aut}.[3,4] Let us consider the support and efficiency of an alternating consumption plan characterized by a parameter $\bar{c}$ with respect to this punishment, where the allocation is defined such that the agent with the high endowment consumes $\bar{c}$, and the agent with the low endowment consumes $1-\bar{c}$. Call this allocation $C(\bar{c})$. Writing $\mathcal{P}_\beta(f_{aut})$ for the set of allocations that can be supported by the autarkic punishments for a given β, it can be seen that, for $\bar{c} \in \left[\frac{1}{2}, e_H\right)$, $C(\bar{c}) \in \mathcal{P}_\beta(f_{aut})$ if

$$\Delta(\bar{c}, \beta) \equiv \frac{\ln(\bar{c}) + \beta \ln(1-\bar{c})}{1-\beta^2} - \mathcal{V}_H(\beta) \geq 0. \tag{16}$$

[3] The autarkic punishment is the one that maps to the unique strategy in which no transfers are ever made by any agent after any history.

[4] In general, the ability to implement of a given (candidate) punishment as an equilibrium depends on β. Autarky is the unique SPE that is implementable for all $\beta \geq 0$.

(Note that (16) and the definitions of $\mathcal{V}_H(\beta)$ and $\mathcal{V}_L(\beta)$ imply that

$$\frac{\ln(1-\bar{c})+\beta\ln(\bar{c})}{1-\beta^2}-\mathcal{V}_L(\beta)>0 \tag{17}$$

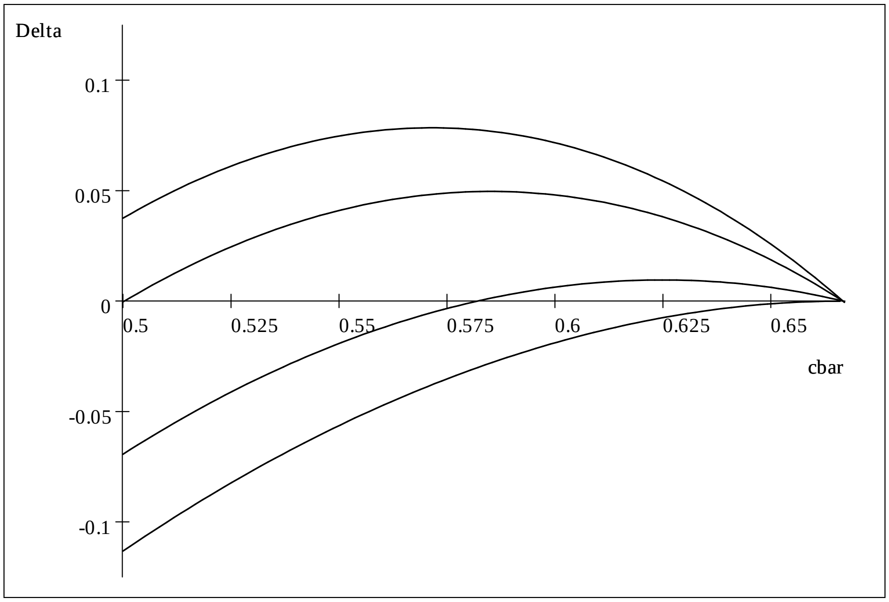
for $\bar{c}\in\left[\frac{1}{2},e_H\right]$.) Therefore, the alternating endowment $C(\bar{c})$ is efficient with respect to f_{aut} if $\bar{c}$ solves

$$\max_x \left(\frac{1}{2}\right)\frac{\ln(x)+\beta\ln(1-x)}{1-\beta^2}+\left(\frac{1}{2}\right)\frac{\ln(1-x)+\beta\ln(x)}{1-\beta^2} \tag{18}$$

subject to $\Delta(x,\beta)\geq 0$ and $x\leq\frac{2}{3}$; that is, if $C(\bar{c})$ maximizes the equally-weighted lifetime utility of the agents subject to the enforcement constraint on the agent with the high endowment in each period (and feasibility is imposed). It can be shown that $\bar{c}=\frac{1}{2}$ solves the program for $\beta\geq 0.70951$; and that $\bar{c}=\frac{2}{3}$ is the unique element of the constraint set (and thus solves the problem) when $\beta\leq\frac{1}{2}$.

The function $\Delta(\bar{c},\beta)$ is plotted in Figure 1 for $\beta\in\left\{\frac{1}{2},0.6,0.70951,\frac{3}{4}\right\}$; the higher curves correspond to higher values of β. Clearly, (16) holds for $\bar{c}=e_H=\frac{2}{3}$ for all values of β; that is, the autarkic allocation is in $\mathcal{P}_\beta(f_{aut})$. It can be verified that this is the only allocation in $\mathcal{P}_\beta(f_{aut})$ for $\beta\leq\frac{1}{2}$. For $\frac{1}{2}<\beta<0.70951$, some risk sharing is possible, but no first-best allocation is suported.[5] For $\beta=0.6$, for example, an efficient alternating allocation is given by $\bar{c}=0.58206$, the value such that the inequality (16) holds with equality. Full risk sharing (for example, the symmetric first-best allocation $\bar{c}=\frac{1}{2}$) is supported for $\beta\geq 0.70951$.

[5] Following the Kocherlakota (1996), an allocation is *first-best* if it equates agents intertemporal rates of marginal substitution at all state histories and exhausts resources. An alternating allocation $C(\bar{c})$ is first-best if and only if $\bar{c}=\frac{1}{2}$, which is the *symmetric first-best allocation*. It can be shown that there exists a first-best SPE allocation if and only if $C\left(\frac{1}{2}\right)$ is an SPE allocation.



Notice that

$$\bar{q}\left(C\left(\bar{c}\right)\right) = \frac{\beta\bar{c}}{1-\bar{c}} \tag{19}$$

for all state histories in this example.[6] For an allocation c^* exhibiting first-best risk sharing, we have $\bar{q}\left(c^*\right) = \beta < 1$, and the implied interest rates are always high. It may be verified that, for $\beta = 0.6$ and $\bar{c} = 0.58206$, $\bar{q}\left(C\left(\bar{c}\right)\right) = 0.83561 < 1$, so that $C\left(0.58206\right) \in HIR$ in this case, as well.

On the other hand, notice that $\bar{q}\left(e\right) < 1$ if and only if $\beta < \frac{1}{2}$, and autarky (e) is efficient if and only if $\beta \leq \frac{1}{2}$. When $\beta \leq \frac{1}{2}$ the constraint set of the efficiency problem is a singleton, and thus the hypotheses of Proposition 2 are not satisfied. Observing that $\bar{q}\left(e\right) < 1$ for $\beta < \frac{1}{2}$ and $\bar{q}\left(e\right) = 1$ for $\beta = \frac{1}{2}$, it is clear that the conclusion of Proposition 2 that the efficient allocation exhibits high implied interest rates may

[6] Here and in the examples below, I write $\bar{q}\left(c\right)$ rather than $\bar{q}\left(s^{t+1}|c\right)$ when the value is constant over all state histories.

apply (as for the cases with $\beta < \frac{1}{2}$) or may not apply ($\beta = \frac{1}{2}$ case). ■

The next proposition establishes a partial converse of the previous one. In the spirit of Kocherlakota's (1996) work, it addresses a question about which consumption processes can be rationalized as efficient. As in the previous proposition, the high implied interest rates condition plays an important role.

Proposition 3 *Given a feasible allocation c, suppose that (i) $\sum_i c^i(s^t) = 1$ for all t and s^t; (ii) $\mathcal{U}^i(c|s^t) \geq \mathcal{U}^i_{aut}(s_t)$ for all i, t, and s^t; and (iii) $c \in HIR$. Then there exists a punishment f such that c is efficient with respect to f.*

Example 2 Consider the environment of the first example. For the case that $\beta = \frac{3}{4}$, notice that $\bar{q}(C(\bar{c})) < 1$ whenever $\bar{c} \in \left[\frac{1}{2}, \frac{4}{7}\right)$. By Proposition 2, this implies that there is some punishment that supports the alternating allocation $C(.55)$ as efficient. It is instructive to understand what such a punishment looks like.[7]

It can be seen that $C(.55)$ is efficient with respect to a punishment $\hat{f}$ only if an agent who defects in a period when his endowment is high receives continuation

[7] It is useful to note that, under the allocation $C(.55)$, the agent with the high endowment gets continuation utility

$$\frac{\ln(.55) + .75\ln(1-.55)}{1-.75^2} = -2.7354, \tag{20}$$

and the agent with the low endowment gets

$$\frac{\ln(1-.55) + .75\ln(.55)}{1-.75^2} = -2.7726. \tag{21}$$

Similarly, the autarkic payoffs are -2.8101 and -3.2062 for the high- and low-endowment agents, respectively.

utility $V_L\left(\hat{f}\right)$ in the period following the defection, where

$$\frac{\ln\left(.55\right)+.75\ln\left(1-.55\right)}{1-.75^2}=\ln\left(\frac{2}{3}\right)+.75V_L\left(\hat{f}\right); \tag{22}$$

one may compute that $V_L\left(\hat{f}\right)=-3.1065$. An equilibrium continuation that delivers the required payoff to the agent with the low endowment is the one that delivers the consumption $\left(1-C\left(\hat{c}\right)\right)=\{1-\hat{c},\hat{c},1-\hat{c},...\}$, where $\hat{c}$ satisfies

$$\frac{\ln\left(1-\hat{c}\right)+.75\ln\left(\hat{c}\right)}{1-.75^2}=V_L\left(\hat{f}\right); \tag{23}$$

that is $\hat{c}=0.64169$. We can choose the punishments for the low-endowment agent so that they will never be binding by setting $\hat{f}$ to implement the symmetric allocation following a defection by this agent. By defecting the agent with the low endowment consumes $1-.55$ in the current period, and consumes $C\left(\hat{c}\right)$ along the path implemented by the punishment in the continuation; this results in the payoff of

$$\ln\left(1-.55\right)+\left(.75\right)\frac{\ln\left(\hat{c}\right)+.75\ln\left(1-\hat{c}\right)}{1-.75^2}=-2.8787 \tag{24}$$

for this agent.

To summarize, I have shown that the allocation in which the agent with the high endowment at $t=0$ gets the consumption

$$\{.55,.45,.55,.45,...\} \tag{25}$$

is efficient with respect to a punishment that specifies reversion to a continuation equilibrium in which the agent with the high endowment gets consumption

$$\{.64169, .35831, .64169, .35831, ...\}. \tag{26}$$

Note that the punishment is close to autarky, but is somewhat better for each agent.■

4 Decentralization

The notion of market equilibrium introduced in this section is a generalization of that studied by Alvarez and Jermann (2000).

A *portfolio* of contingent claims for agent i is a stochastic process b^i with $b^i(s^t) \in \mathbb{R}$. I write b for the profile of agents' portfolios.

An (Arrow) *price system* is a positive stochastic process p, where $p(s^t, s_{t+1})$ is interpreted as the price after (exogenous) history s^t of a claim to a unit of the good after history s^{t+1}.

A *system of solvency constraints* is a stochastic process d with $d(s^t) \in \mathbb{R}^I$.

A *competitive equilibrium with solvency constraints* is a consumption allocation c, a profile of portfolios b, a price system p, and a system of solvency constraints d such that

1. for each i, t, and s^t, $(c^i(s^t), b^i(s^t, \cdot))$ solves

$$J_t^i\left(b^i\left(s^t\right), s^t\right) = \max_{\tilde{c}^i, \tilde{b}^i(\cdot)} u\left(\tilde{c}^i\right) + \beta \sum_{s'} J_{t+1}^i\left(\tilde{b}^i(s'), \left(s^t, s'\right)\right) \pi(s'|s_t)$$

subject to

$$\tilde{c}^i + \sum_{s'} p\left(s^t, s'\right) \tilde{b}^i\left(s'\right) \leq e^i\left(s_t\right) + b^i\left(s^t\right)$$

and

$$\tilde{b}^i\left(s'\right) \geq d^i\left(s^t, s'\right)$$

for each s'; and

2. the goods and assets markets clear for each s^t,

$$\sum_i c^i\left(s^t\right) = 1$$

and

$$\sum_i b^i\left(s^t\right) = 0.$$

The following Proposition, which may be interpreted as a version of the First Welfare Theorem for the present environment, is an obvious corollary of Proposition 3.

Proposition 4 *Suppose that (c, b, p, d) is an equilibrium with solvency constraints; that $\mathcal{U}^i\left(c|s^t\right) \geq \mathcal{U}^i_{aut}\left(s_t\right)$ for all i, t, s^t; and that $c \in HIR$; then there are punishments f such that c is efficient with respect to f.*

A version of the Second Welfare Theorem that applies is the following.

Proposition 5 *Suppose that c is efficient with respect to the punishments f, and suppose that $c \in HIR$; then there exist portfolios b, prices p, and solvency constraints d such that (c, b, p, d) is an equilibrium with solvency constraints.*

Example 3 Let us consider again the environment of the previous examples with the parameterization of Example 2, and let us see how the efficient allocation considered there may be decentralized.[8] The price of claims to the good one period in the future (in this deterministic environment) is constant across time at

$$\bar{p} = \bar{q}\left(C\left(.55\right)\right) = \frac{\beta\left(.55\right)}{\left(1-.55\right)} = 0.91667. \tag{27}$$

From the budget constraints, we have

$$.55 + \left(0.91667\right)b^L = \frac{2}{3} + b^H \tag{28}$$

and

$$\left(1-.55\right) + \left(0.91667\right)b^H = \frac{1}{3} + b^L, \tag{29}$$

where b^H (b^L, respectively) is the quantity of claims held at the beginning of a period by the agent who has the high (low) endowment in the period. Solving the two budget constraints, we compute that $b^L = -b^H = 0.06087$. From the previous example, we have seen that the support constraint must bind for the agent with the high endowment; thus we set $d^H = b^H = -0.06087$. It was shown that the support constraint was not binding for the agent with the low endowment, and it follows that any value less than or equal to b^L will serve as d^L for the purpose of decentralization. ■

Alvarez and Jermann (2000) show that the solvency constraints supporting an

[8] As above, I have found it convenient to drop the time subscripts and index agents by their current endowments rather than fixed indices where no confusion is likely.

allocation efficient with respect to the autarkic punishments may be chosen so that

$$J_t^i\left(d^i\left(s^t\right), s^t\right) = \mathcal{U}_{aut}^i\left(s_t\right) \tag{30}$$

for all i, t, s^t. This is an intuitive benchmark for an economy in which agents are able only to consume their own endowments in a period in which they defect. Then it can be interpreted that an agent with wealth $d^i\left(s^t\right)$ at s^t is indifferent between abiding the continuation of the competitive equilibrium and consuming his endowment in the current and in each subsequent period. In the present environment, under an SPE supported by arbitrary punishments, this interpretation is invalid, since defection after s^t may earn an agent a payoff larger than $\mathcal{U}_{aut}^i\left(s_t\right)$.

On the other hand, solvency constraints defined by (30) are sufficient for the decentralization of an allocation efficient with respect to autarkic punishments. The conditions (30) serve in this case, because (uniquely for the autarkic punishments) (8) implies (9). Thus, it can be seen that the support constraints will never bind for i in a period in which $c^i\left(s^t\right) > e^i\left(s_t\right)$, so that the associated solvency constraints will not be binding either. More generally, this need not be the case.

Alvarez and Jermann (2000) show that (30) implies that the solvency constraints are non-positive, and thus they may be interpreted naturally as constraints only on the amount of state-contingent "debt" agents may take on. That this must be so can be seen from the fact that autarkic consumption for all time after s^t satisfies the budget constraints after s^t for an agent with exactly zero wealth; this implies that $J_t^i\left(0, s^t\right) \geq \mathcal{U}_{aut}^i\left(s_t\right)$. Thus, if the solvency constraints are consistent with (30), it must be that $J_t^i\left(0, s^t\right) \geq J_t^i\left(d^i\left(s^t\right), s^t\right)$; monotonicity of the function $J_t^i\left(\cdot, s^t\right)$ implies the result. This property, too, may fail for more general punishments.

5 Conclusion

The paper studies efficient multilateral risk sharing subject to enforcement (or "commitment") constraints determined by the threat of punishment after misbehavior. As in Kocherlakota (1996), I assume that the society chooses from among those allocations implementable under subgame perfect equilibria. The novelty of the present analysis is that, while most of the existing literature studies efficiency when the least desirable continuation equilibrium is assumed to obtain after a defection, I allow that punishments may coordinate play on a continuation equilibrium that is suboptimal from the point of view of enforcement. I take a relatively agnostic view about what may constitute a punishment convention, imposing only the following restriction on the enforcement technology: following Abreu (1988), the punishments are defined by the choice of a subgame perfect equilibrium continuation to be followed after a defection, and the selection depends only on the exogenous history and the identity of the defector.

The main results of the paper constitute a characterization (up to a technical condition) of the set of allocations that may be interpreted as efficient with respect to some punishment. The observable restrictions imposed by efficiency are very weak; these are (*i*) exhaustion of resources, (*ii*) satisfaction of an individual rationality constraint at each continuation, and (*iii*) finiteness of the value of the aggregate endowment under an implicit decentralizing price system, the "high implied interest rates" condition of Alvarez and Jermann (2000). Proposition 2 shows that an allocation that is efficient with respect to some punishments has properties (i)-(iii) whenever the constraint set has an interior point. Proposition 3 shows that an allocation that has these properties is efficient with respect to some punishment (constructed in the proof of the Proposition).

A heuristic motivation for studying such an environment is the notion that, while front line institutions may facilitate the arrival of the economy at a locally efficient outcome, coordination of play off the equilibrium path may be accomplished less effectively. For example, participation in markets is the focal device for coordinating this "front line", and much theory and evidence establishes their efficiency properties; but there is less agreement about how the institutions that govern the treatment of bankruptcy are determined. In economies with limited enforcement, these punishment institutions determine which paths may be sustained in equilibrium. Thus, the determination of these features is obviously important.

There are a number of papers that study market economies with limited enforcement in which the determination of the punishments is carefully modeled, and even endogenous. Kehoe and Perri (2002, 2004) study international risk sharing in a model with capital accumulation in which a country's capital may not be seized, so that "autarkic" production and consumption depends on the quantity of capital the country has accumulated. Lustig (2004) studies an economy in which "bankruptcy" results only in seizure of a collateral asset, with bankrupt agents resuming their participation in the markets after their default. Krueger and Fernández-Villaverde (2001) and Lustig and Van Nieuwerburgh (2005) study economies in which housing acts as collateral, and bankruptcy results only in the seizure of that asset.

A contribution of the present paper is to establish a benchmark bound on the set of consumption allocations that may be implemented as outcomes of markets facing the sorts of frictions modeled in these economies. The intuition I would like to advance is that limited enforcement is a broadly applicable modeling tool, and that a theory of economic behavior derives mostly from the specification of the institutions that define the punishments. This point of view suggests that more careful modeling of these institutions is the key to understanding the behaviorial implications of limited commitment for consumption in the real world.

The analysis suggests a number of questions open for future research. First, Proposition 1 suggests the value of efficiently structured institutions for accomplishing a desired set of transfers when enforcement is limited. It may be of interest to investigate the degree to which real world "clearing mechanisms" fulfill this theoretical objective. Relatedly, it would be interesting to see how these optimal transfers change as the restriction on the form of the punishments is relaxed; for example, Under what circumstances and in what ways will the optimal transfer algorithm depend on the punishments? Second, it might be interesting to study the implications of applying a restriction like renegotiation-proofness to the set of admissible equilibria.[9] Of related interest are criteria under which punishment continuations can be sustained in markets, and the possibility that punishment continuations may be interpreted to obtain on the equilibrium path.[10] Finally, it may of focal interest to study economies in which punishments are applied randomly after a deviation. Such

[9] See Evans and Maskin (1989), for example.

[10] Lustig (2004) offers an interpretation under which bankruptcy actually occurs in equilibrium.

a model may offer a simple parameterization of the "degree of enforcement" available in an economy useful in calibrations and empirical work.

Appendix

Proof of Lemma 1. Necessity is obvious from the definition of an SPE, and the fact that $\sigma_j^i(h^t) = 0$ for all j and h^t defines a strategy that gives at least $\mathcal{U}_{aut}^i(s_t)$ to i at each history h^t.

To show sufficiency, I construct a "simple" (Abreu (1988)) pure strategy profile σ that implements c on its path. I then show that σ is an equilibrium. I will write $h(s^t)$ for the game history that obtains when players abide by the path for exogenous history s^t; and I let $\bar{H}$ be the set of all other histories.[11]

I begin by describing play following a defection: let $\sigma(h^t) = 0$ for all $h^t \in \bar{H}$. That such play describes an equilibrium for the subgame following from h^t is obvious, since all $h^\tau \succsim h^t$ are in $\bar{H}$, and any unilateral defection can be seen to hurt the deviating player.

Let play along the path of σ be as prescribed by the function $\hat{\alpha}(c)$ in Section 3 of the text; that is, set $\sigma(h(s^t)) = \hat{\alpha}(c)(s^t)$ for all s^t.

I have now described $\sigma(h^t)$ for all histories on and off the path. To see that σ

[11] It would be more precise to construct the path of σ and the function $h(\cdot)$ by recursions, and then define σ for histories off the path. This could be done as follows. First let $h(s_0) = s_0$ and then define $\sigma(h(s_0))$. Then successive values $h(s_0, s')$ and $\sigma(h(s_0, s'))$ can be assigned recursively by setting

$$h\left(s^{t-1}, s'\right) = \left(h\left(s^{t-1}\right), \sigma\left(h\left(s^{t-1}\right)\right), s'\right)$$

for each s', and then defining $\sigma(h(s^t))$. That such an algorithm is available for the strategy profile I describe is obvious.

constitutes an SPE for histories on the path, note that a defection at any history h^t in which $c^i(s^t) \geq e^i(s_t)$ gets i at most

$$u\left(c^i\left(s^t\right)\right) + \beta \sum_{s'} \mathcal{U}^i_{aut}\left(s'\right) \leq \mathcal{U}^i\left(c|s^t\right); \tag{31}$$

and by deviating at a history with $c^i(s^t) < e^i(s_t)$, i gets at most

$$u\left(e^i\left(s_t\right)\right) + \beta \sum_{s'} \mathcal{U}^i_{aut}\left(s'\right) = \mathcal{U}^i_{aut}\left(s_t\right) \leq \mathcal{U}^i\left(c|s^t\right).$$

Thus, the one-deviation property implies that there can be no profitable defection from σ, Q.E.D. ■

Proof of Lemma 2. Suppose that $\sigma(\alpha, f)$ is an SPE; and suppose that

$$\mathcal{U}^i\left(\gamma\left(\alpha\right)|s^t\right) < u\left(g^i\left(\alpha\right)\left(s^t\right)\right) + \beta \sum_{s'} \mathcal{U}\left(\gamma\left(f^i\left(s^t, s'\right)\right)|s'\right)\pi\left(s^t, s'|s^t\right) \tag{32}$$

for some i, t, and s^t.

Now if $\alpha^i_j(s^t) > 0$ for some j, then the agent can consume exactly $g^i(\alpha)(s^t)$ at s^t by taking the action defined by $a^i_j = 0$ for all j. The inequality above shows that this is a profitable defection when the continuation will be governed by $f^i(s^t, s')$. If $\alpha^i_j(s^t) = 0$, on the other hand, i can deviate by setting $\alpha^i_1 = \varepsilon$ (for example). This induces consumption of $g^i(\alpha)(s^t) - \varepsilon$ at s^t and a continuation payoff of

$$u\left(g^i\left(\alpha\right)\left(s^t\right) - \varepsilon\right) + \beta \sum_{s'} \mathcal{U}\left(\gamma\left(f^i\left(s^t, s'\right)\right)|s'\right)\pi\left(s^t, s'|s^t\right). \tag{33}$$

Th inequality above shows that this defection is profitable for $\varepsilon > 0$ small enough. The existence of a profitable defection is a contradiction; thus, the (7) must hold whenever $\sigma(\alpha, f)$ is an SPE.

For the converse, suppose that (7) holds for all s^t. First note that these conditions and the fact that $g^i(\alpha)(s^t) \geq e^i(s_t)$ for all s^t imply that $\mathcal{U}^i(\gamma(\alpha)|s^t) \geq \mathcal{U}^i_{aut}(s_t)$ for all s^t; thus (by Lemma 1) there is an SPE (continuation) that delivers payoff $\mathcal{U}^i(\gamma(\alpha)|s^t, s')$ for each (s^t, s'). Since $f^i(s^t, s')$ is a selection from the set of (continuation) equilibria feasible from state s', it follows that $\sigma(\alpha, f)$ describes an equilibrium for each subgame off of the path α. Moreover, there can be no profitable defection along the path α, since a defection at a history $h^t = (s^t, \gamma(\alpha)(s^{t-1}))$ gets i at most the payoff on the right-hand side of (7). Thus, $\sigma(\alpha, f)$ is an SPE, Q.E.D. ■

Proof of Proposition 1. The "if" part is obvious from the definition of $\mathcal{P}(f)$. For the "only if" part, suppose that c is supported by a punishment f, so that $\sigma(\alpha, f)$ is an SPE for some path α. Then Lemma 2 implies that

$$\mathcal{U}^i\left(c|s^t\right) \geq u\left(g^i(\alpha)(s_t)\right) + \beta \sum_{s'} \mathcal{U}\left(\gamma\left(f^i\left(s^t, s'\right)\right)|s'\right) \pi\left(s^t, s'|s^t\right) \tag{34}$$

Now note that, by construction,

$$g^i(\alpha)\left(s^t\right) \geq g^i(\hat{\alpha}(c))\left(s^t\right) \tag{35}$$

for all paths α that induce allocation c. Thus, (34) implies that

$$\mathcal{U}^i\left(c|s^t\right) \geq u\left(g^i(\hat{\alpha}(c))(s_t)\right) + \beta \sum_{s'} \mathcal{U}\left(\gamma\left(f^i\left(s^t, s'\right)\right)|s'\right) \pi\left(s^t, s'|s^t\right). \tag{36}$$

The result then follows from Lemma 2. ■

Proof of Corollary 1. From the proof of Proposition 1, the result follows after noting that

$$g^{i}\left(\hat{\alpha}\left(c\right)\right)\left(s^{t}\right)=\max\left\{c^{i}\left(s^{t}\right),e^{i}\left(s_{t}\right)\right\}. \tag{37}$$

■

Proof of Proposition 2. It can be seen that $\hat{c}$ solves a programming problem of the form described in the text for some γ in the I-dimensional unit simplex. Write the Lagrangian for the problem as

$$\begin{aligned}
\mathcal{L}\left(c,\lambda,\eta\right) \quad : \quad &= \sum_{i}\gamma^{i}\sum_{t=0}^{\infty}\sum_{s^{t}}\beta^{t}u\left(c^{i}\left(s^{t}\right)\right)\pi\left(s^{t}|s_{0}\right) \\
&+\sum_{t=0}^{\infty}\sum_{s^{t}}\beta^{t}\lambda\left(s^{t}\right)\pi\left(s^{t}|s_{0}\right)\left[1-\sum_{i}c^{i}\left(s^{t}\right)\right] \\
&+\sum_{i}\sum_{t=0}^{\infty}\sum_{s^{t}}\beta^{t}\eta_{1}^{i}\left(s^{t}\right)\pi\left(s^{t}|s_{0}\right)\left\{\sum_{\tau=t}^{\infty}\sum_{s^{\tau}}\beta^{\tau-t}u\left(c^{i}\left(s^{\tau}\right)\right)\pi\left(s^{\tau}|s_{t}\right)\right. \\
&\left.-\left[u\left(e^{i}\left(s^{t}\right)\right)+\beta\sum_{s'}\mathcal{U}^{i}\left(\gamma\left(f^{i}\left(s^{t},s'\right)\right)|s'\right)\pi\left(s'|s_{t}\right)\right]\right\} \\
&+\sum_{i}\sum_{t=0}^{\infty}\sum_{s^{t}}\beta^{t}\eta_{2}^{i}\left(s^{t}\right)\pi\left(s^{t}|s_{0}\right)\left\{\sum_{\tau=t+1}^{\infty}\sum_{s^{\tau}}\beta^{\tau-t}u\left(c^{i}\left(s^{\tau}\right)\right)\pi\left(s^{\tau}|s^{t}\right)\right. \\
&\left.-\beta\sum_{s'}\mathcal{U}^{i}\left(\gamma\left(f^{i}\left(s^{t},s'\right)\right)|s'\right)\pi\left(s'|s_{t}\right)\right\},
\end{aligned} \tag{38}$$

where $\beta^{t}\lambda\left(s^{t}\right)\pi\left(s^{t}\right)$, and $\beta^{t}\eta_{k}^{i}\left(s^{t}\right)\pi\left(s^{t}\right)$ for $k\in\{1,2\}$ are non-negative Lagrange multipliers on the constraints. Necessary (Kuhn-Tucker) conditions include the firstorder conditions

$$\left[\gamma^i + \sum_{\tau=0}^{t} \eta_1^i\left(s^\tau\right) + \sum_{\tau=0}^{t-1} \eta_2^i\left(s^\tau\right)\right] \beta^t u'\left(\hat{c}^i\left(s^t\right)\right)\pi\left(s^t\right) - \beta^t \lambda\left(s^t\right)\pi\left(s^t\right) = 0 \tag{39}$$

for each i and s^t. These conditions imply that $\lambda\left(s^t\right) > 0$ for all s^t, and that

$$\begin{aligned} \frac{\beta\lambda\left(s^{t+1}\right)\pi\left(s^{t+1}|s^t\right)}{\lambda\left(s^t\right)} &= \frac{\left[\gamma^i + \sum_{\tau=0}^{t}\left[\eta_1^i\left(s^\tau\right)+\eta_2^i\left(s^\tau\right)\right] + \eta_1^i\left(s^{t+1}\right)\right]\beta u'\left(\hat{c}^i\left(s^{t+1}\right)\right)\pi\left(s^{t+1}|s^t\right)}{\left[\gamma^i + \sum_{\tau=0}^{t-1}\left[\eta_1^i\left(s^\tau\right)+\eta_2^i\left(s^\tau\right)\right] + \eta_1^i\left(s^t\right)\right]u'\left(\hat{c}^i\left(s^t\right)\right)} && (40) \\ &\geq \max_j\left\{\beta\frac{u'\left(\hat{c}^j\left(s^{t+1}\right)\right)}{u'\left(\hat{c}^j\left(s^t\right)\right)}\pi\left(s^{t+1}|s^t\right)\right\}. && (41) \end{aligned}$$

Now it follows from (12) and (13) that

$$\bar{Q}\left(s^{t+1}|\hat{c}\right) \leq \frac{\beta^{t+1}\lambda\left(s^{t+1}\right)\pi\left(s^{t+1}\right)}{\lambda\left(s_0\right)}, \tag{42}$$

so that

$$\begin{aligned} \sum_{t=0}^{\infty}\sum_{s^t}\bar{Q}\left(s^t|\hat{c}\right)\left[\sum_i \hat{c}^i\left(s^t\right)\right]\pi\left(s^t\right) &\leq \sum_{t=0}^{\infty}\sum_{s^t}\bar{Q}\left(s^t|\hat{c}\right)\pi\left(s^t\right) && (43) \\ &\leq \sum_{t=0}^{\infty}\sum_{s^t}\frac{\beta^t\lambda\left(s^t\right)\pi\left(s^t\right)}{\lambda\left(s_0\right)}. && (44) \end{aligned}$$

Finiteness of the last expression follows from the fact that $\left(\beta^t\lambda\left(s^t\right)\pi\left(s^t\right)\right)_{t=0}^{\infty}$ is a summable sequence (i.e., an element of l_1) by Theorem 1 on page 249 of Luenberger (1968).[12] ■

[12]In the Theorem 1 on page 249 of Luenberger, sequences $\left(\beta^t\hat{\lambda}\left(s^t\right)\pi\left(s^t\right)\right)_{t=0}^{\infty}$ and $\left(\beta^t\hat{\eta}^i\left(s^t\right)\pi\left(s^t\right)\right)_{t=0}^{\infty}$ are the l_1 components of elements in the non-negative orthant of the norm-dual of l_∞. This space can be interpreted to be $l_1 + fa$, where fa is the space of finitely additive

Before giving the proof of Proposition 3, I present several auxiliary results useful in the proof of the main one.

In what follows, I define $\Omega(s)$ as the set of payoff vectors $w \in \mathbb{R}^I$ such that, for some $\sigma \in \Sigma(s)$, $w^i = \mathcal{U}^i(\gamma(\sigma)|s)$ for each i; that is, $\Omega(s)$ is the set of payoff vectors available under equilibria starting from state s.

Lemma 3 *$\Omega(s)$ is convex for each s.*

Proof. The set of allocations that can be supported by strategies constructed as in the proof of Lemma 1 is easily seen to be convex. The convexity of $\Omega(s)$ is then easy to establish from the continuity and concavity of $\mathcal{U}(\cdot|s)$ in allocations, and the fact that the action set admits the possibility of free-disposal of the good. ■

Lemma 4 *If c is a feasible allocation, and $\mathcal{U}^i(c|s^t) \geq \mathcal{U}^i_{aut}(s_t)$ for all s^t, then $u(c^i(s^t))$ is bounded.*

Proof. Clearly, $u(c^i(s^t)) \leq u(1)$, so $\mathcal{U}^i(c|s^t) \leq u(1)/(1-\beta)$. Thus $\mathcal{U}^i(c|s^t) \geq \mathcal{U}^i_{aut}(s_t)$ implies that

$$\begin{aligned} u\left(c^i\left(s^t\right)\right) &\geq u\left(e^i\left(s_t\right)\right) + \beta \sum_{s'} \left[\mathcal{U}^i_{aut}(s') - \mathcal{U}^i\left(c|s^t, s'\right)\right] \pi(s'|s_t) \\ &\geq u\left(e^i\left(s_t\right)\right) + \beta \sum_{s'} \left[\mathcal{U}^i_{aut}(s') - u(1)/(1-\beta)\right] \pi(s'|s_t) \\ &> -\infty. \end{aligned}$$

measures. For the proof of the present proposition, it is sufficient to restrict attention to properties exhibited by the l_1 component of the multipliers.

Note that the regularity of the maximum required by Luenberger's Theorem are guaranteed by the existence of an interior point and the fact that the constraint set is convex.

The result follows from the fact that $\mathcal{S}$ is finite. $\blacksquare$

Proof of Proposition 3. First set $f^i(s_0, s') = \sigma_{aut}$ for each i.[13] The value of f^i at other points will be set according to the following algorithm.

Fix i and $s^t \succ s_0$. First, if

$$\frac{u'(c^i(s^t))}{u'(c^i(s^{t-1}))} = \max_j \left\{ \frac{u'(c^j(s^t))}{u'(c^j(s^{t-1}))} \right\}, \tag{45}$$

set $f^i(s^t, s') = \sigma_{aut}$. Second, if instead

$$\frac{u'(c^i(s^t))}{u'(c^i(s^{t-1}))} < \max_j \left\{ \frac{u'(c^j(s^t))}{u'(c^j(s^{t-1}))} \right\} \tag{46}$$

and $c^i(s^t) \leq e^i(s^t)$, then it follows that

$$\begin{aligned} & u\left(e^i(s^t)\right) + \beta \sum_{s'} \mathcal{U}^i\left(c|s^t, s'\right) \pi(s'|s_t) && (47) \\ \geq\ & \mathcal{U}^i\left(c|s^t\right) && (48) \\ \geq\ & u\left(e^i(s^t)\right) + \beta \sum_{s'} \mathcal{U}^i_{aut}(s') \pi(s'|s_t). && (49) \end{aligned}$$

From Lemma 1 and Lemma 3, we can select $f^i(s^t, s') \in \Sigma(s')$ for each s' so that

$$\mathcal{U}^i\left(c|s^t\right) = u\left(e^i(s^t)\right) + \beta \sum_{s'} \mathcal{U}^i\left(\gamma\left(f^i\left(s^t, s'\right)\right)|s^t, s'\right) \pi(s'|s_t). \tag{50}$$

[13] The punishment will be constructed so that the enforcement constraints do not bind at s_0.

Finally, if (46) holds and $c^i(s^t) > e^i(s^t)$, we need to set $f^i(s^t, \cdot)$ so that

$$\sum_{s'} \mathcal{U}^i\left(c|s^t, s'\right)\pi(s'|s_t) = \sum_{s'} \mathcal{U}^i\left(\gamma\left(f^i\left(s^t, s'\right)\right)|s^t, s'\right)\pi(s'|s_t). \tag{51}$$

Since $\mathcal{U}^j(c|s^t, s') \geq \mathcal{U}^j_{aut}(s')$ for all j, t, and s^t, there is an equilibrium continuation that delivers payoff $\mathcal{U}^i(c|s^t, s')$ to i for each s'; select $f^i(s^t, s')$ to implement such an equilibrium for each s'. Repeating this procedure for each i and $s^t \succ s_0$ completes the definition of f.

Now it is sufficient to show that c solves a programming problem of the form described in the text. From the hypotheses and the construction of f, it follows that (8) and (9) hold for each i, t, and s^t. Thus, c is in the constraint set of the problem. The Lagrangian function for this problem has the form in (38). To show that c solves such a programming problem it suffices (by Theorem 2 on p. 221 of Luenberger (1968)) to find γ and multipliers (λ, η) such that (c, λ, η) constitutes a saddle point of $\mathcal{L}(c, \lambda, \eta)$.[14] I begin by defining appropriate weights and multipliers.

Define $\gamma \in \Delta^I$ by

$$\gamma^i u'\left(c^i(s_0)\right) = \gamma^j u'\left(c^j(s_0)\right)$$

for all i and j.

The multipliers η^i_1 and η^i_2 will be defined recursively as follows. First, let $\eta^i_1(s_0) = 0$. Now for $t \geq 0$, suppose that $\eta^i_1(s^t)$ and $\eta^i_2(s^{t-1})$ have been defined (interpreting $\eta^i_2(s^{-1})$ as the value 0). If $c^i(s^{t+1}) \leq e^i(s^{t+1})$, then set $\eta^i_2(s^t) = 0$ and set $\eta^i_1(s^{t+1})$

[14] Note that, for the purpose of the Theorem of Luenberger, the Lagrange multipliers are the sequences whose elements are $\beta^t\lambda(s^t)\pi(s^t|s_0)$ and $\beta^t\eta^i(s^t)\pi(s^t|s_0)$. It will follow from condition (iii) of the hypothesis of the Proposition that each of the sequences constructed below is summable, so that each sequence defines an element of the norm dual space of l_∞.

so that

$$
\begin{aligned}
&\frac{\left\{\gamma^i+\sum_{\tau=0}^{t}\left[\eta_1^i\left(s^\tau\right)+\eta_2^i\left(s^\tau\right)\right]+\eta_1^i\left(s^{t+1}\right)\right\}\beta u'\left(c^i\left(s^{t+1}\right)\right)\pi\left(s^{t+1}|s^t\right)}{\left\{\gamma^i+\sum_{\tau=0}^{t-1}\left[\eta_1^i\left(s^\tau\right)+\eta_2^i\left(s^\tau\right)\right]+\eta_1^i\left(s^t\right)\right\}u'\left(c^i\left(s^t\right)\right)} \qquad (52)\\
= \quad &\max_j \frac{u'\left(c^j\left(s^t,s'\right)\right)}{u'\left(c^j\left(s^t\right)\right)}.
\end{aligned}
$$

Notice that $\eta_1^i\left(s^{t+1}\right)\geq 0$, and that $\eta_1^i\left(s^{t+1}\right)=0$ whenever (45) holds, or whenever[15]

$$
\mathcal{U}^i\left(c|s^t\right)>u\left(c^i\left(s^t\right)\right)+\beta\sum_{s'}\mathcal{U}^i\left(\gamma\left(f^i\left(s^t,s'\right)\right)|s^t,s'\right)\pi\left(s'|s_t\right). \qquad (53)
$$

Second, if $c^i\left(s^{t+1}\right)>e^i\left(s^{t+1}\right),$ then set $\eta_1^i\left(s^{t+1}\right)=0$ and set $\eta_2^i\left(s^t\right)$ so that (52) holds. Notice that $\eta_2^i\left(s^t\right)\geq 0$, and that $\eta_2^i\left(s^t\right)=0$ whenever (45) holds, or whenever[16]

$$
\sum_{s'}\mathcal{U}^i\left(c|s^t,s'\right)\pi\left(s'|s_t\right)>\sum_{s'}\mathcal{U}^i\left(\gamma\left(f^i\left(s^t,s'\right)\right)|s^t,s'\right)\pi\left(s'|s_t\right). \qquad (54)
$$

Finally, define

$$
\lambda\left(s^t\right)=\left\{\gamma^i+\sum_{\tau=0}^{t-1}\left[\eta_1^i\left(s^\tau\right)+\eta_2^i\left(s^\tau\right)\right]+\eta_1^i\left(s^t\right)\right\}u'\left(c^i\left(s^t\right)\right);
$$

note that the expression on the RHS is independent of i by construction.

Now by construction, the multipliers (λ,η) can be seen to minimize $\mathcal{L}(c,\cdot,\cdot)$ over all non-negative alternatives. It remains to verify that c maximizes $\mathcal{L}(\cdot,\lambda,\eta)$. From

[15] To see the second claim in this sentence, note the following. We've already seen that the weak inequality must hold. If $\eta_1^i\left(s^{t+1}\right)>0$, then by construction it must be that $c^i\left(s^{t+1}\right)\leq e^i\left(s^{t+1}\right)$ and (46) holds. In such cases, $f^i\left(s^t,\cdot\right)$ has been defined by (50), so (53) cannot hold.

[16] The second claim in this sentence follows by logic similar to that in footnote 15 .

Lemma 4, $\left|u\left(c^i\left(s^t\right)\right)\right|$ is bounded. It follows that the sums

$$\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\eta_1^i\left(s^t\right)\pi\left(s^t|s_0\right)\left\{\sum_{\tau=t}^{\infty}\sum_{s^\tau}\beta^{\tau-t}u\left(c^i\left(s^\tau\right)\right)\pi\left(s^\tau|s^t\right)\right\}$$

and

$$\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\eta_2^i\left(s^t\right)\pi\left(s^t|s_0\right)\left\{\sum_{\tau=t+1}^{\infty}\sum_{s^\tau}\beta^{\tau-t}u\left(c^i\left(s^\tau\right)\right)\pi\left(s^\tau|s^t\right)\right\}$$

converge absolutely, since (taking the first sum, for example)

$$\begin{aligned}&\left|\beta^t\eta_1^i\left(s^t\right)\pi\left(s^t|s_0\right)\left\{\sum_{\tau=t}^{\infty}\sum_{s^\tau}\beta^{\tau-t}u\left(c^i\left(s^\tau\right)\right)\pi\left(s^\tau|s_t\right)\right\}\right|\\ \leq\;&\beta^t\eta_1^i\left(s^t\right)\pi\left(s^t|s_0\right)\left\{\sum_{\tau=t}^{\infty}\sum_{s^\tau}\beta^{\tau-t}\left|u\left(c^i\left(s^\tau\right)\right)\right|\pi\left(s^\tau|s_t\right)\right\}.\end{aligned}$$

Thus (e.g., by Theorem 3.55 of Rudin (p.78)) terms in the expressions may be re-arranged without changing the value of the sums. Now showing that c maximizes $\mathcal{L}\left(\cdot,\lambda,\eta\right)$ may be seen as equivalent to showing that

$$\begin{aligned}&\sum_i\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\left\{\left\{\gamma^i+\sum_{\tau=0}^{t-1}\left[\eta_1^i\left(s^\tau\right)+\eta_2^i\left(s^\tau\right)\right]+\eta_1^i\left(s^t\right)\right\}u\left(c^i\left(s^t\right)\right)-\lambda\left(s^t\right)c^i\left(s^t\right)\right\}\\ &-\sum_i\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\left\{\left\{\gamma^i+\sum_{\tau=0}^{t-1}\left[\eta_1^i\left(s^\tau\right)+\eta_2^i\left(s^\tau\right)\right]+\eta_1^i\left(s^t\right)\right\}u\left(\tilde{c}^i\left(s^t\right)\right)-\lambda\left(s^t\right)\tilde{c}^i\left(s^t\right)\right\}\end{aligned}$$

is non-negative for all allocations $\tilde{c}$. Now using the definition of $\lambda\left(s^t\right)$, and combining and rearranging the terms, this expression is seen to equal

$$\sum_i \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \quad \left\{ \left\{ \gamma^i + \sum_{\tau=0}^{t-1} \left[\eta_1^i \left(s^\tau \right) + \eta_2^i \left(s^\tau \right) \right] + \eta_1^i \left(s^t \right) \right\} \right. \\ \left. \times \left\{ u \left(c^i \left(s^t \right) \right) - u' \left(c^i \left(s^t \right) \right) \left[\tilde{c}^i \left(s^t \right) - c^i \left(s^t \right) \right] - u \left(\tilde{c}^i \left(s^t \right) \right) \right\} \right\}.$$

By the concavity of u, this expression is non-negative, Q.E.D. ■

Proof of Proposition 4. The result follows immediately from Proposition 3. ■

Proof of Proposition 5. The result may be established eactly as in the Proof of Proposition 4.1 of Alvarez and Jermann (2000). ■

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