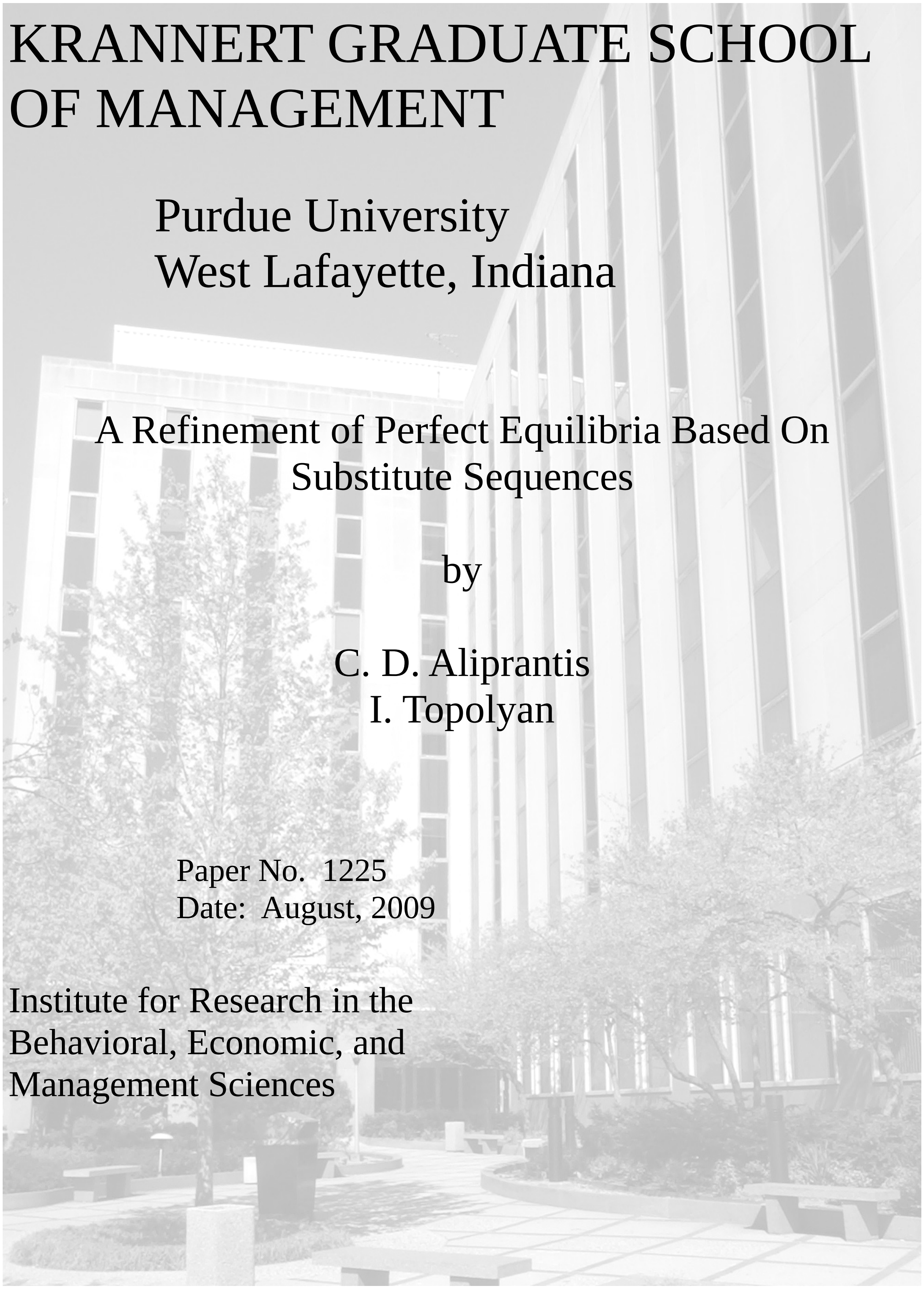

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A Refinement of Perfect Equilibria Based On Substitute Sequences

by

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A REFINEMENT OF PERFECT EQUILIBRIA BASED ON SUBSTITUTE SEQUENCES★

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Abstract. We propose an equilibrium refinement of strict perfect equilibrium for the finite normal form games, which is not known in the literature. Okada came up with the idea of strict perfect equilibrium by strengthening the main definition of a perfect equilibrium, due to Selten [14]. We consider the alternative (and equivalent) definition of perfect equilibrium, based on the substitute sequences, as appeared in Selten [14].

We show that by strengthening and modifiyng this definition slightly, one can obtain a refinement stronger than strict perfectness. We call the new refinement strict substitute perfect equilibrium. The main advantage of this solution concept is that it reflects the local dominance of an equilibrium point. An example is provided to show that a strict perfect equilibrium may fail to be strict substitute perfect.

JEL classification: C7

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1. Introduction

The concept of equilibrium is central in the game theory, for it captures the essence of payoff-maximizing behavior of players faced with interactions and predicts how the game should be played. The answer to the question which outcome should be

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considered reasonable determines the solution concept. The basic idea underlying equilibrium is non-regretting, or no profitable deviation for a single player whenever others stick to their strategies, which gives rise to the definition of a Nash equilibrium.

However, some strategy profiles that pass the basic equilibrium requirement still fail to match our intuition about the way the game should be played. There were many refinements of a Nash equilibrium concept introduced in order to eliminate intuitively unreasonable equilibria. Most of them were based on robustness to slight imperfections of rationality, which was originated by Selten [14]. Whether you require stability against arbitrary deviations from rationality approaching zero or deviation satisfying certain strengthened criteria may give you refinements as proper equilibrium proposed by Myerson [7] or strictly perfect equilibrium by Okada [10].

We follow the same principle of bounded rationality requiring stability against certain perturbations of rationality, namely by requiring that a refined equilibrium should be stable against every substitute sequence for itself satisfying certain properties. We basically strengthen the alternative characterization of perfect equilibrium points, which gives even stronger refinement than strict perfection.

The paper is organized as follows. In the section Definitions and Methodology we present our framework and provide the definitions of the concepts we are using throughout this paper. In the next section we review some of the equilibrium concepts currently being used in the literature, outline their advantages and limitations, and illustrate the necessity of further refinement.

In the fourth section we are narrowing the notion of a substitute sequence, introducing so-called ultra-substitute sequences, and strengthen the definition of perfect equilibrium, requiring that the refined equilibrium should be a best reply against every substitute sequence eventually. We show that the new solution concept is actually a refinement of strict perfect and hence proper equilibria (Theorem 4.6). Example 4.7 motivates strict substitute perfect equilibria, showing that it can actually eliminate unreasonably equilibria, which proper and even strictly perfect equilibria cannot rule out.

In this Section 4 we also provide characterization of the proposed refinement in terms of behavior of best responces in a neighborhood of an equilibrium point (Lemma 4.4) and the local dominance (Theorem 4.9). The latter strengthens the motivation for the refinement, for local dominance is a desirable property that supports our intuition about a useful refinement (yet this property may be rather restrictive, which results in strict substitute perfect equilibria failing to exist in some games).

2. Definitions and Methodology

Definition 2.1. *A* ***normal form*** Γ *of a finite n-player game is a tuple* $(\Pi_1, \cdots, \Pi_n, H)$*, where* Π_i *is a finite set of pure strategies of player* i*,* $\Pi = \prod_{i=1}^{n} \Pi_i$*, and* $H : \Pi \twoheadrightarrow \mathbb{R}^n$ *is the payoff function that assigns to every* $\pi \in \Pi$ *the vector of payoffs* $H(\pi) = (H_1(\pi), \cdots, H_n(\pi))$*.*

Definition 2.2. *A* ***mixed strategy*** a_i *for player* i *is a probability distribution over* Π_i*. The set of all such probability distributions is denoted by* $A_i \equiv \Delta_{c_i}$*, where* c_i *is the cardinality of* Π_i*. The set of mixed strategies for the game* Γ *is* $A = \prod_{i=1}^{n} A_i$*.*

We can now define an expected payoff function h, which is an extension of the payoff function H to all of A.

Definition 2.3. *An* ***expected payoff function*** *is a function* $h : A \twoheadrightarrow \mathbb{R}^n$ *such that*

$$h(a) = \prod_{\pi \in \Pi} p_1(\pi) p_2(\pi) \cdots p_n(\pi) H(\pi),$$

where $p_i(\pi)$ *is the probability that* a *assigns to the* i^{th} *component of* π*, i.e., the probability with which player* i *chooses* π_i*.*

Mixed strategy a_i for player i is comletely mixed if $a_i \in \Delta^o_{p_i}$. Mixed strategy a is completely mixed if for all i, a_i is completely mixed.

Definition 2.4. *A* ***perturbed game*** $\hat{\Gamma}$ *of a normal form game* Γ *is a tuple* (Γ, η)*, where* $\eta = (\eta_1, \ldots, \eta_n)$ *is a strictly positive vector of trembles satisfying for all* i

$$\sum_{k=1}^{c_i} \eta_k^i \leq 1,$$

such that for each $i \in \{1, \ldots, n\}$ *and* $j \in \{1, \ldots, c_i\}$ *we have*

$$p_i(\pi_j^i) \geq \eta_j^i,$$

and for each i *the following holds:*

$$\sum_{j=1}^{c_i} p_i(\pi_j^i) = 1.$$

So vector η can be interpreted as a vector of minimum probabilities corresponding to each $\pi^i_j \in \Pi_i$. A perturbed game has the property that pure strategies are ruled out, that is the action set $\hat{A}_i$ for generic player i is a subset of $(A_i)^0$.

The latter restriction gives rise to the notion of maximum probability of the choice π^i_j. Observe that since no pure strategy can be played with zero probability, no pure strategy can be played with probability one either.

Definition 2.5. *A* ***maximum probability*** *of the choice π^i_j of player i is defined as*

$$\zeta(\pi^i_j) = 1 + \eta^i_j - \sum_{k=1}^{c_i} \eta^i_k < 1.$$

Definition 2.6. *Gven any topological spaces $X_1, \dots, X_m$, a* ***projection map onto*** *j* ***'s factor*** *is a function $P_j : \prod_{i=1}^m X_i \twoheadrightarrow X_j$ defined by the equation*

$$P_j(x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_m) = x_j.$$

Definition 2.7. *A strategy $\tilde{a}_i$ of player i is called a* ***weakly (strictly) dominant*** *strategy if for every $a_{-i} \in A_{-i}$ and $a_i \in A_i$, $U_i(\tilde{a}_i, a_{-i}) > U_i(a_i, a_{-i})$ $(U_i(\tilde{a}_i, a_{-i}) \geq U_i(a_i, a_{-i}))$.*

Definition 2.8. *A sequence of mixed strategies $\{a^n\}$ is called a* ***substitute sequence*** *for a mixed strategy a if $a^n \in A^o$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} a^n = a$.*

Definition 2.9. *A mixed strategy $a^*_i \in A_i$ of the player i is a* ***best reply*** *to the substitute sequence $\left\{a^n_{-i}\right\}$ for a_{-i} if a^* is in the best responce to a^n_{-i} for every $n \in \mathbb{N}$.*

Definition 2.10. *A mixed strategy $a^* \in A$ is called a* ***substitute perfect equilibrium point*** *of a normal form game Γ if a^*_i is a best reply to at least one substitute sequence for a^*, say $\{a^n\}$. That is, a^*_i is a best reply to $\left\{a^n_{-i}\right\}$ for each player i.*

Definition 2.11. *A mixed strategy $a^* \in A$ is called a* ***perfect equilibrium point*** *of a normal form game Γ if a^* is a Nash equilibrium for Γ and for some sequence of perturbed games $\hat{\Gamma}^k = (\Gamma, \eta^k)$ with $\eta^k \to 0$, there exists a Nash equilibrium point a^k of $\hat{\Gamma}^k$ for each k such that $a^k \to a^*$ as $k \to \infty$.*

Selten [14] showed that a strategy profile is perfect equilibrium if and only if it is substitute perfect equilibrium.

Definition 2.12. *A mixed strategy $a^* \in A$ is called a* ***strictly perfect equilibrium point*** *of a normal form game Γ if a^* is a Nash equilibrium for Γ and for any sequence of perturbed games $\hat{\Gamma}^k = (\Gamma, \eta^k)$ with $\eta^k \to 0$, there exists a Nash equilibrium point a^k of $\hat{\Gamma}^k$ for each k such that $a^k \to a^*$ as $k \to \infty$.*

3. Motivation and Overview of Existing Equilibrium Concepts

There have been many equilibrium concepts proposed by game theorists, each having its own advantages and limitations. The broadest definition is due to Nash, who proposed it in his pioneering work [8]. This formulation of equilibrium appeared to be weak, however it is fundamenatal in the economic literature and serves as a basis for further equilibrium refinements.

Selten [13] strengthened the definition of a Nash equilibrium, eliminating non-credible threats in the extensive form games by introducing a **subgame perfect equilibrium** (profile that induces Nash equilibrium on every subgame). He did not yet take into account the small imperfections of rationality, and there were a number of limitations, which called for revising the equilibrium concept. In fact, for the normal form games Nash and subgame perfect equilibria coincide.

Kreps and Wilson (1982) further stregthened the concept of equilibrium and introduced **sequential equilibria** by imposing the criterion of sequential rationality (optimal play starting from every information set) and consistency. Yet, sequential and Nash equilibria coincide in case of a normal form game.

Selten realized the limitations of subgame perfect equilibria (in particular, they don't resolve all the problems related to the unreached parts of the game) and took a novel approach to defining an equilibrium. The basic idea of the equilibrium concept proposed by him is stability against arbitrary **imperfections of rationality**.

For this purpose Selten [14] introduced the so-called "Model of Slight Mistakes". According to it, for every player i there is a small probability for the breakdown of rationality $\epsilon_i > 0$. Whenever rationality breaks down, every choice $c \in \Pi_i$ is selected with some positive probability q_c^i.

Suppose that player j intends to select the choice c with probability p_c. In case of imperfect rationality the total probability of selecting the choice c is

$$\hat{p}_c = (1 - \epsilon_j)p_c + \epsilon_j q_c^j \,. \tag{3.1}$$

Notice that the mapping $g: A \to \hat{A}$ defined by the equation 3.1 for each player j is one-to-one and onto. This implies that under incomplete rstionality, the problem of choosing a strategy from A, which brings the payoff according to the composite function $h \circ g$, is equivalent to choosing a strategy from $\hat{A}$ with the corresponding payoff determined by h.

In fact, the function g is a homeomorphism from A to $\hat{A}$, since it is continuous by definition, and the inverse function $g^{-1}: \hat{A} \to A$ defined by

$$p_c = \frac{1}{1 - \epsilon_j}(\hat{p}_c - \epsilon_j q_c^j) \tag{3.2}$$

is clearly continuous.

Thus, the specified probabilities ϵ and q transform the original normal form game Γ into the perturbed game $\hat{\Gamma}$, where the strategy space of each player is restricted according to the corresponding vector of minimum probabilities

$$\eta_c^i = \epsilon_i q_c^i \,. \tag{3.3}$$

Selten treated complete rationality as a limiting case of incomplete rationality. In case of imperfect rationality players make mistakes ("hand trembles"), but when the vector of mistake probabilities ϵ approaches zero (which implies that the vector of minimum probabilities η goes to zero), we can determine whether there is any sequence of equilibrium points for perturbed games approaching a given equilibrium point of the original game. If there exists such a sequence for at least one sequence of trembles converging to zero, we call the equilibrium point of the original game perfect, or stable against certain rationality imperfections.

Then, the natural question arises whether we can find such sequence of equilibrium points for any sequence of trembles approaching zero. It turns out that in general the answer is no, which is due to Okada [10]. He found a set of conditions for games that guarantee the strengthened requirement of perfectness to hold. With this strengthened condition he introduced a refinement of perfectness concept - strictly perfect equilibria.

Myerson [7] as well strengthened the perfectness concept by imposing a restriction that more costly mistakes are made with smaller probability, introducing **proper equilibria**. It is rather a restriction of the way incomplete raionality approaches complete rationality. No surprisingly then, every proper equilibrium is strictly perfect.

It is worth mentioning that proper equilibria are attractive in the sense that, being stronger than perfect equilibria, they exist for any normal form game, while strictly perfect equilibria may fail to exist.

There is a substantial common drawback of all the known refinements based on the trembling-hand framework. It is assumed that probabilities ϵ_i and q are **common knowledge**. However, those probabilities are due to "unspecified psychological mechanism" (Selten, [14]), and the common knowledge assumption seems too restrictive.

For any player i, other players should rather have a sense of the degree of player i's irrationality (bounds on ϵ_i and q^i) rather than specific values of those probabilities. This motivated us to propose a new refinement of perfect equilibria, which relaxes the common knowledge assumption.

4. The New Solution Concept: Definition and Properties

We first introduce some new definitions and concepts that are key in formulating the new refinement, which we call strict substitute perfect equilibrium. As we mentioned, one way to obtain the refinement of perfect equilibria is to consider its main definition proposed by Selten [12, p.38], and strengthen it, requiring that a strict perfect equilibrium point is a limit equilibrium point for *every* sequence of perturbed games.

Another way to refine the perfectness concept is to consider the characterization of a substitute perfect equilibrium, due to Selten [12, p.49], which can be basically regarded as an alternative definition of a perfect equilibrium, for Selten showed that these two concepts are equivalent.

This way to obtain further refinement has not been considered in the literature so far. We show that strengthening the alternative definition in a certain way allows to obtain a refinement which is stronger than strict perfectness. We basically require that a refined equilibrium is a best reply to every substitute sequence that possesses certain properties. For these purposes we introduce an ultra-substitute sequence of strategies.

Definition 4.1. *A sequence of mixed strategies* $\{a^n\}$ *is called an* ***ultra-substitute sequence*** *for a mixed strategy* a *if the following conditions hold:*

(1) $a^n \in A^o$ *for every* $n \in \mathbb{N}$*, that is, each strategy profile* a^n *is completely mixed,*
(2) $a^n_j = a_j$ *for all* n *and for every* $j \in \{1, \dots, N\}$ *such that* $a_j \in A^o_j$*,*
(3) $\lim_{n\to\infty} a^n = a$*.*

As we can see, for any given strategy profile every ultra-substitute sequence is a substitute sequence, however the converse may not be true. Also, if a is interior, then the only substitute sequence for a is the constant sequence $\{a^n\}$ such that $a^n = a$ for all n.

Definition 4.2. *A mixed strategy* $a^*_i \in A_i$ *of player* i *is* ***eventually a best reply*** *to the substitute sequence* $\left\{a^n_{-i}\right\}$ *for* a_{-i} *if there exists* $N > 0$ *such that* a^*_i *is a best reply to* a^n_{-i} *for every* $n \geq N$*.*

The idea of the further refinement of perfect equilibrium is to require stability against any ultra-substitute sequence eventually. That is, the behavior of finite number of terms in the substitute sequence does not matter (as it does not matter for convergence of any sequence or series), what matters is the limiting behavior, or, as it turns out (see Lemma 3.4), the behavior in some neighborhood of an equilibrium profile.

Definition 4.3. *A mixed strategy profile* $a^* = (a^*_1, \dots, a^*_N) \in A$ *is called a* ***strict substitute perfect equilibrium point*** *of a normal form game* Γ *if* a^* *is eventually a best reply to every ultra-substitute sequence* a^n *for itself.*

Notice that an interior Nash equilibrium point is automatically a strict substitute perfect equilibrium point.

Let us introduce the following notation. Let I denote the index set of players for the game, that is, for the N-person game we have $I = \{1, \dots, N\}$. Given a strategy profile $a \in A$, introduce $I_1(a)$ as the set of all $i \in I$ such that a_i is on the boundary of A_i; similarly let $I_2(a)$ denote the set of all $i \in I$ such that a_i is in the interior of A_i. Since any point in a set is either in the interior of it or on the boundary, the two index sets define a partition of the index set I, that is,

$$I = I_1(a) \sqcup I_2(a) \text{ for any strategy profile } a \in A.$$

The following lemma provides characterization of a strict substitute perfect equilibrium in terms of behavior of the best responce correspondence on the neighborhoods of an equilibrium point. That is, it gives the necessary and sufficient condition for strict substitute perfectness. Essentially this condition requires the equilibrium point to be a best reply to every strategy profile in some "neighborhhod" of itself (it is not a neighborhood strictly speaking because its projection onto any index from I_2 degenerates to a singleton).

Lemma 4.4. *A strategy profile a^* is a strict substitute perfect equilibrium if and only if there exists a neighborhood V of a^* such that a^* is a best reply to a whenever $P_j(a) \in P_j(V)$ for each $j \in I_1(a^*)$ and $P_i(a) = a_i^*$ for each $i \in I_2(a^*)$.*

Proof. ($\Rightarrow$)

Claim 1. *Strict substitute perfect equilibrium at a^* implies that there exists a neighborhood V of a^* such that a^* is a best reply to a whenever $P_j(a) \in P_j(V) \cap A_j^o$ for each $j \in I_1(a^*)$ and $P_i(a) = a_i^*$ for each $i \in I_2(a^*)$.*

Assume a^* is a strict substitute perfect equilibrium for N-player normal form game Γ. Suppose by contradiction for every neighborhood V of a^* there exists a strategy profile b^V with $P_j(b^V) \in P_j(V) \cap A_j^o$ for all $j \in I_1(a^*)$, and $P_i(b^V) = a_i^*$ for all $i \in I_2(a^*)$ (call it **condition** (*) for b^V), such that a^* is not in the best responce for b^V.

Given the neighborhood V of a^*, define $diam(V) = \sup_{i\in\{1,\dots,N\}} \{\|a_i^* - a_i\| : a_i \in P_i(V)\}$.

Consider the sequence of neighborhoods of a^*, $\{V^n\}_{n=1}^\infty$ such that $diam(V^n) \to 0$ as $n \to \infty$. Let $\{b^n\}_{n=1}^\infty$ be the corresponding sequence of strategy profiles such that condition (*) holds. Then $\{b^n\}$ is an ultra-substitute sequence for a^*, for we have $\sup_{i\in\{1,\dots,N\}} \|a_i^* - b_i^n\| \to 0$ as $n \to \infty$. But this is a contradiction, since a^* is a strict substitute perfect equilibrium. This proves our claim.

Now we prove the desired implication. Let a^* be a strict substitute perfect equilibrium, and fix $a \in A$ such that $P_j(a) \in P_j(V) \cap \partial A_j$ for each $j \in I_1(a^*)$ and $P_i(a) = a_i^*$ for each $i \in I_2(a^*)$. Let $\{a^n\}$ be any ultra-substitute sequence approaching a such that $a^n \in V$ for each n. a^* is a best responce to a^n for each n. Then by upper-hemicontinuity of the best responce correspondence, which follows from Berge's maximum theorem, a^* is a best responce to a (since $\lim_{n\to\infty} a^n = a$). Hence a^* is a best responce to a whenever $P_j(a) \in P_j(V) \cap A_j^o$ for each $j \in I_1(a^*)$ and $P_i(a) = a_i^*$ for each $i \in I_2(a^*)$.

($\Leftarrow$) Let V be such neighborhood of a^* that a^* ia a best reply to a whenever $P_j(a) \in P_j(V)$ for all $j \in I_1(a^*)$ and $P_i(a) = a_i^*$ for all $i \in I_2(a^*)$. Suppose by contradiction a^* is not a strict substitute perfect equilibrium. Then there exists an ultra-substitute sequence for a^*, $\{b^n\}_{n=1}^\infty$ satisfying

$$\forall m \in \mathbb{N}\ \exists\, k \geq m \text{ such that } a^* \text{ is not a best reply for } b^k.$$

Let m be such index that $P_j(b^m) \in P_j(V)$ for all $j \in I_1(a^*)$, and $P_j(b^k) \in P_j(V)$ for all $k \geq m$ and for all $j \in I_1(a^*)$. Such m exists since b^n is an ultra-substitute sequence for a^*. But then there exists $k \geq m$ such that a^* is not a best responce for b^k, which together with $P_j(b^k) \in P_j(V)$ for all $j \in I_1(a^*)$ drives to a contradiction. ■

Theorem 4.5. *For any finite normal form game Γ a strict substitute perfect equilibrium is a Nash equilibrium for Γ.*

The proof of this theorem is trivial, for strict substitute perfect equilibrium is a substitute perfect equilibrium (which is impled by the fact that for any $a \in A$, any ultra-substitute sequence for a is a substitute sequence for a). It was proven by Selten [14] that every substitute perfect equilibrium is a Nash equilibrium.

The following theorem shows that the equilibrium concept introduced in this paper is a refinement of strict perfection. Obviously it is stronger than Theorem 3.5, however in its proof we used the fact that every strict substitute perfect equilibrium is a Nash equilibrium. Using the characterization of strict substitute perfect equilibria expressed in Lemma 3.4. makes the proof of the following theorem tractable.

Theorem 4.6. *If a strategy profile a^* is strict substitute perfect equilibrium, then it is strict perfect equilibrium.*

Proof. Assume that $a^* \in A$ is strict substitute perfect equilibrium. Let $\tilde{V}$ be the neighborhood of a^* satisfying conditions of Lemma 3.4, that is, a^* is a best reply to a whenever $P_j(a) \in P_j(\tilde{V})$ for all $j \in I_1(a^*)$ and $P_i(a) = a_i^*$ for all $i \in I_2(a^*)$.

Fix $\epsilon > 0$, and consider the neighborhood of a^* given by $V = \tilde{V} \cap B_\epsilon(a^*)$, where $B_\epsilon(a^*)$ is the ϵ-ball about a^*. Fix a sequence of perturbed games $\hat{\Gamma}^n$, and let $\hat{A}^n$ be the corresponding sequence of strategy spaces for $\hat{\Gamma}^n$. Since $\lim_{n\to\infty} \hat{A}^n = A$, then

$$\exists m \in \mathbb{N}\ \ni\ \forall n \geq m\ :\ \hat{A}^n \cap P_j(V) \neq \emptyset,\ \forall\, j \in I_1(a^*).$$

Claim: for every $k \geq m$ there exists a strategy profile b^k satisfying $b_j^k \in \partial \hat{A}_j^k \cap V_j$ for all $j \in I_1(a^*)$ and $b_i^k = a_i^*$ for all $i \in I_2(a^*)$, such that b^k is an equilibrium point of $\hat{\Gamma}^k$.

Indeed, fix $k \geq m$, and note that $\hat{A}^k \cap P_j(V) \neq \emptyset$, for all $j \in I_1(a^*)$. Fix $j \in I_1(a^*)$, and let s_j be the face of the unit simplex A_j that is the best reply to any a_{-j} satisfying condition (*). Notice that since a^* is a Nash equilibrium, we have $a_j^* \in s_j$. Let $\hat{s}_j^k$ be the corresponding face of the unit simplex $\hat{A}_j^k$, then by Lemma 3.5 from [3], $\hat{s}_j^k$ is the best reply to any $a_j \in \hat{A}_j^k$ satisfying condition (*) (taking V as a neighborhood of a^*).

Also $\hat{s}_j^k \cap V_j \neq \emptyset$ since $\hat{A}_j^k \cap V_j \neq \emptyset$. Pick any $b_j^k \in \hat{s}_j^k \cap V_j$ for each $j \in I_1(a^*)$, and let $b_i^k = a_i^*$ for each $i \in I_2(a^*)$. Then the tuple $b^k = (b_1^k, b_2^k, \ldots, b_N^k)$ is an equilibrium point of $\hat{\Gamma}^k$. Indeed, $b_j^k \in V_j$ for every $j \in I_1(a^*)$ and $b_i^k = a_i^*$ for each $i \in I_2(a^*)$, so b^k satisfies condition (*). We also have $b_j^k \in \hat{s}_j^k$ for every $j \in I_1(a^*)$. Therefore b^k is a best reply to itself. Observation that $b_j^k \in \hat{A}^k$ shows that b^k is a Nash equilibrium for $\hat{\Gamma}^k$.

Thus we just proved existence of an equilibrium point for eventually all $\hat{\Gamma}^k$ within ϵ-neighborhood of a^*, for any $\epsilon > 0$. Letting $\epsilon \downarrow 0$, we conclude that a^* is strict perfect equilibrium. ■

We try to provide justification for the equilibrium refinement introduced in this paper and show the ways it captures the intuition about how the game is to be played. Introducing the perfect equilibria, Selten required it to be stable against at least one sequence of perturbations of rationality, that is to be a limit point of equilibria for some sequence of perturbed games. Okada strengthened this concept, requiring strict perfect eqiuilibria to be stable against any sequence of trembles approaching zero.

The idea of our equilibrium refinement is based upon so-called "trembling-hand miopy". We can describe it as follows: consider the standard trembling-hand framework, where there is a slight chance of rationality break-down. The main reason for introducing the ultra-substitute sequences and using it to obtain our equilibrium refinement is that the interior strategy profiles are not disturbed for sufficiently small trembles. At the same time strategy profiles that are on the boundary of the strategy set cannot be implemented due to trembles, and they are approximated by the sequence of completely mixed strategy profiles.

When player i is choosing best reply against some strategy profile a_{-i}, if some a_j such that $j \neq i$ is on the boundary of A_j, player i knows that a_j cannot be implemented due to trembles, however rather some *approximating* profile from certain neighborhood of a_j is to be played. Since the moves are made simultaneously, player

1/2	u	d
A	1, -1	0, 3
B	1, 2	0, -2
C	0, 0	0, 0

TABLE 1

i cannot distinguish between the strategy profiles from that neighborhood (which we call **"trembling-hand miopy"**), and to ensure stability of the equilibrium point, we require that a_i is a best reply no matter which point from the sufficiently small neighborhood of a_j is chosen.

The following example provides a further motivation for our refinement.

Example 4.7. Consider the two-person normal form game depicted in the Table 1.

There is a continuum of Nash equilibria (among which the pure strategies equilibria are (A, d), (B, u) and (C, d)):

(1) strategy profiles satisfying $p_d = 1$, $p_A \geq p_B$
(2) profiles with $p_A = p_B = 0.5$, $0 < p_d < 1$, and
(3) strategy profiles satisfying $p_A \leq p_B$, $p_u = 1$.

An equilibrium point with $p_C = p_d = 1$ looks inferior, for it gives the smallest equilibrium payoff to either player. No surprise, it does not even satisfy the perfectness criterion. However even strict perfect equilibria may turn out to be unreasonable.

Notice that an equilibrium point a such that $p_A(a) = p_B(a) = 0.5$, $p_d(a) = 1$ is strictly perfect and hence proper. However, we argue that this equilibrium point hardly can be considered reasonable, that is we would not expect it to be an outcome whenever the game is played. Calculating expected payoffs at a, we get $U_1(a) = 0$ and $U_1(a) = 0.5$. At the same time an equilibrium point b with $p_B(b) = 1$, $p_u(b) = 1$ yields higher payoffs to both players - 1 and 2, respectively. Even an equilibrium profile c such that $p_A(c) = 1$, $p_d(c) = 1$ brings higher payoff to the second player, leaving player 1 as well-off as before. In this case we say that profile b Pareto dominates profile a, while c weakly Pareto dominates a.

Equilibrium profile a fails to be strict substitute perfect, while it is proper and even strictly perfect.

From the previous example one would hope that the concept of strict substitute perfect equilibrium rules out unreasonable equilibria in the sence of Pareto dominance. Unfortunately, this is not true in general: the game depicted in Table 2

1/2	u	d
A	0.5, -1	2, 3
B	1, 2	0, -2
C	0, 0	0, 0

TABLE 2

illustrates this assertion. We perturb our previous example slightly, so that now profile c Pareto dominates profile b. However, it can be easily seen from the graph of the best responces that an equilibrium profile b is strict sibstitute perfect.

Nevetherless, it turns out that our equilibrium refinement is attractive in the sense that it captures the idea of strategical dominance. In game theory we usually deal with the concept of "global" dominance, as described in Definition 2.7. It may happen, however, that for generic player i some strategy a_i is dominant only in some region - subset of A_{-i}. We then introduce the idea of **local dominance** (or dominance on a subset).

Definition 4.8. *A strategy $\tilde{a}_i$ of player i is called a **weakly (strictly) dominant strategy** on a subset $V \subseteq A_{-i}$ if for every $a_{-i} \in V$ and $a_i \in A_i$, $U_i(\tilde{a}_i, a_{-i}) \geq U_i(a_i, a_{-i})$ (correspondingly, $U_i(\tilde{a}_i, a_{-i}) > U_i(a_i, a_{-i})$).*

It follows from Lemma 3.4 that given a strict substitute perfect equilibrium point a^*, for each $j \in I$ we can find a region - subset of A_{-j} containing a^*_{-j}, on which a^*_j is weakly dominant. This region may not be a neighborhood of a^*_{-j} because some a^*_i for $i \neq j$ could be in the interior of A_i. It may even happen that this region degenerates to a single point (as in the case of the classical Matching Pennies game with two players, where $a^*_1 = \frac{1}{2}$ is a best reply to a_2 if and only if $a_2 = \frac{1}{2}$).

However, if a^*_j is not interior for every $j \in I$, then a^* is dominant on some neighborhood of itself. The above considerations are summarized in the following theorem, which results directly from Lemma 3.4.

Theorem 4.9. *An equilibrium profile a^* is strict substitute perfect if and only if for each player i, a^*_i is weakly dominant on a set $D^i \subseteq A_{-i}$ such that $P_j(D^i)$ is some neighborhood of a^*_j for each $j \in I_1(a^*) \setminus \{i\}$ and $P_k(D^i) = a^*_k$ for each $k \in I_1(a^*) \setminus \{i\}$.*

5. Conclusions

We introduced strict substitute perfect equilibria as a refinement of perfectness concept, which turns out to be stronger than strict perfect or proper equilibria. The main advantage of the new solution concept is that it reflects the local dominance of a strategy profile. This is a desirable property since according to our intuition, the dominating strategy profiles should be considered desirable equilibria.

Our refinement is robust to arbitrary imperfections of rationality under the so-called "trembling-hand miopy" (in contrast, strict perfect equilibria are stable with respect to arbitrary, and perfect equilibria - with respect to certain imperfections of rationality, which may fail under the "trembling-hand miopy").

Among the weaknesses of strict substitute perfectness are the failure to rule out all Pareto inferior perfect equilibria and the nonexistence property - in general such equilibrium may not exist. This happens because even perfect equilibrium point doesn't have to satisfy local dominance. The local dominance is a very desirable property, however for some games it's never satisfied, so there is no strict substitute perfect equilibrium for such games.

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