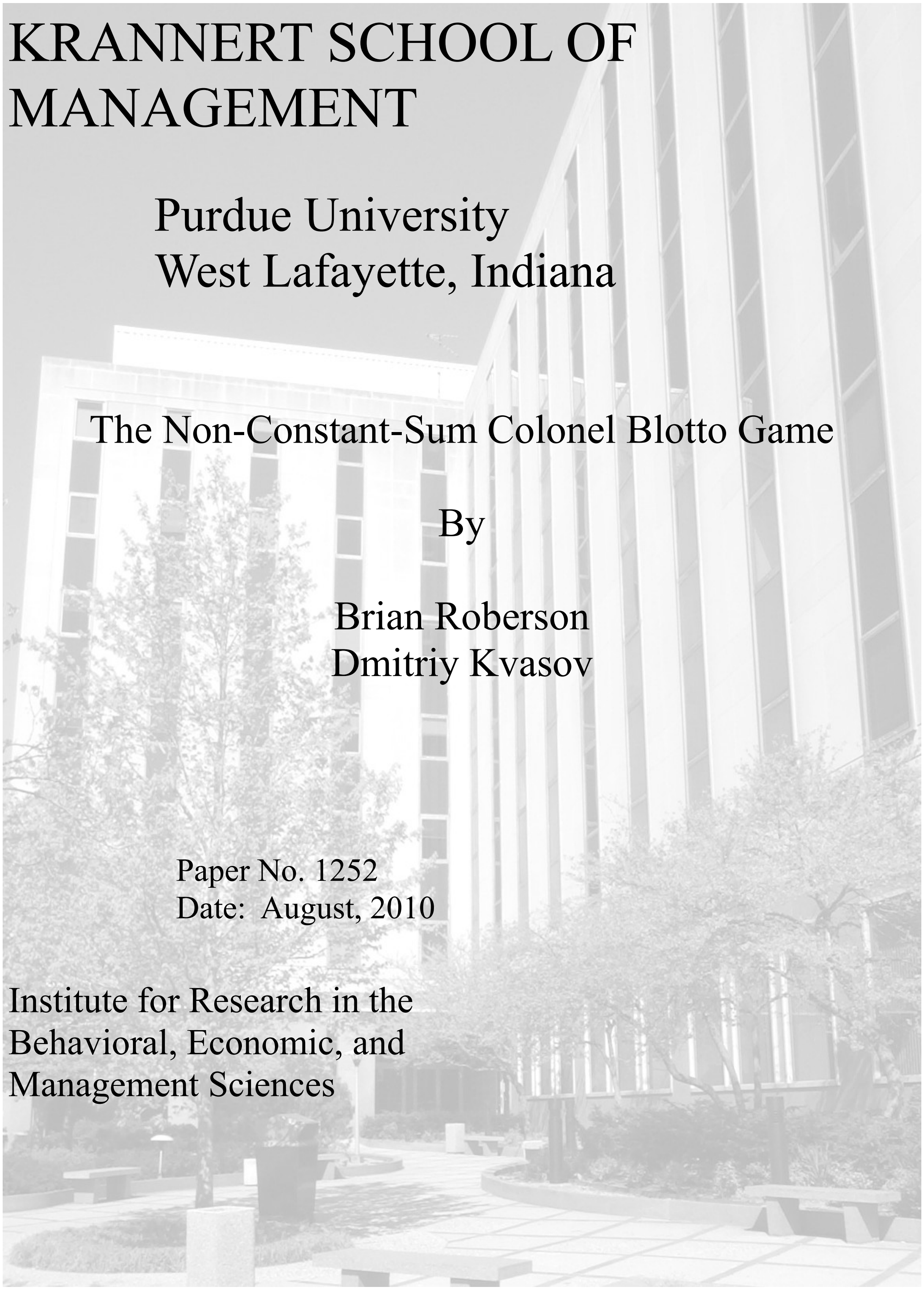

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Paper No. 1252
Date: August, 2010

Institute for Research in the
Behavioral, Economic, and
Management Sciences

The Non-Constant-Sum Colonel Blotto Game*

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Abstract

The Colonel Blotto game is a two-player constant-sum game in which each player simultaneously distributes his fixed level of resources across a set of contests. In the traditional formulation of the Colonel Blotto game, the players' resources are "use it or lose it" in the sense that any resources which are not allocated to one of the contests are forfeited. This article examines a non-constant-sum version of the Colonel Blotto game which relaxes this use it or lose it feature. We find that if the level of asymmetry between the players' budgets is below a threshold, then there exists a one-to-one mapping from the unique set of equilibrium univariate marginal distribution functions in the constant-sum game to those in the non-constant-sum game. Once the asymmetry of the players' budgets exceeds the threshold this relationship breaks down and we construct a new equilibrium.

JEL Classification: C72, D7

Keywords: Colonel Blotto Game, All-Pay Auction, Contests, Mixed Strategies

*We have benefited from the helpful comments of Dan Kovenock, Wolfgang Leininger, and participants in presentations at the University of Iowa and the CESifo Summer Institute Workshop on Advances in the Theory of Contests and its Applications.

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1 Introduction

Originating with Borel (1921), the Colonel Blotto game is a classic model of budget-constrained resource allocation across multiple simultaneous contests. Borel formulates this problem as a constant-sum game involving two players, A and B, who must each allocate a fixed amount of resources, $X_A = X_B$, over a finite number of contests. Each player must distribute their resources without knowing their opponent's distribution of resources. In each contest, the player who allocates the higher level of resources wins, and each player's payoff across all of the contests is the proportion of the wins across the individual contests.[1]

This simple model was a focal point in the early game theory literature (see, for example, Bellman 1969; Blackett 1954, 1958; Borel and Ville 1938; Gross and Wagner 1950; Shubik and Weber 1981; Tukey 1949.). The Colonel Blotto game has also experienced a recent resurgence of interest (see, for example, Golman and Page 2009; Hart 2008; Hortala-Vallve and Llorente-Saguer 2010, Kovenock and Roberson 2008; Kvasov 2007; Laslier 2002; Laslier and Picard 2002; Macdonell and Mastronardi 2010, Roberson 2006, 2008; or Weinstein 2005). One of the main appeals of the Colonel Blotto game is that it provides a unified theoretical framework which is relevant to a diverse set of environments ranging from political campaign resource allocation to military conflict. In these constant-sum applications each player has a fixed level of resources to allocate across the set of contests and any unused resources have no value.

There are also a number of closely related applications of multi-dimensional resource allocation such as research and development races, rent-seeking, lobbying, and litigation. However, these applications are non-constant sum in that any resources which are not allocated to one of the contests have value, i.e. the players' resources are not "use it or lose it." Kvasov (2007) introduces a non-constant-sum version of the Colonel Blotto game which relaxes this use it or lose it feature of the original formulation. In the case of symmetric budgets, that article establishes that there exists a one-to-one mapping from the unique set of equilibrium univariate marginal distribution functions in the constant-sum game to those in the non-constant-sum game.

In this article we extend the analysis of the non-constant-sum version of the Colonel Blotto game to allow for asymmetric budget constraints. For all configurations of the asymmetric constant-sum Colonel Blotto game with three or more contests, Roberson (2006) pro-

[1]This is the plurality objective. An alternative objective [the majority or tournament objective] is for each player to maximize the probability that they win a majority of the contests. For $n > 3$ the solution to the majority game is an open question.

vides: (i) the characterization of the unique equilibrium payoffs,[2] (ii) the characterization of each player's set of equilibrium univariate marginal distributions, and (iii) the existence of joint distributions which, in addition to providing the sets of equilibrium univariate marginal distributions, expend the players' respective budgets with probability one. We find that as long as the asymmetry between the players' budgets is below a threshold, there exists a one-to-one mapping from the unique set of equilibrium univariate marginal distribution functions in the constant-sum game to those in the non-constant-sum game. Once the asymmetry of the players' budgets exceeds the threshold this relationship breaks down. For this range we construct an entirely new equilibrium for the non-constant-sum game. For all parameter configurations in which there exist unique sets of equilibrium univariate marginal distributions, we characterize these sets. For these parameter configurations we also characterize the unique equilibrium payoffs and the unique equilibrium total expected expenditures.

The non-constant-sum Colonel Blotto game is essentially a set of n independent all-pay auctions in which two players submit n-tuples of bids subject to budget constraints that hold across the entire set of auctions. Therefore, our results may also be seen as extending the analysis of the single all-pay auction with budget-constrained bidders (see Che and Gale 1998) to allow for budget constraints that apply across a finite set of auctions.

Section 2 presents the model. Section 3 provides a brief comparison of the constant-sum and non-constant-sum formulations of the Colonel Blotto game and provides intuition for the connection between the equilibria in these two games. Section 4 characterizes the equilibrium payoffs and the equilibrium sets of univariate marginal distributions for the asymmetric non-constant-sum version of the Colonel Blotto game. Section 5 concludes.

2 The Model

Two players, A and B, simultaneously enter bids in a finite number, $n \geq 2$, of independent all-pay auctions. Each all-pay auction has a common value of v for each player. Each player has a fixed level of available resources (or budget), X_i for $i = A, B$. Let $X_A \leq X_B$, and let $\mathbf{x}_i$ denote the n-tuple of bids $(x_{i,1}, \ldots, x_{i,j}, \ldots, x_{i,n})$, one bid for each auction j. If both players enter the same bid in an auction and the common bid is X_A $[X_B - (n-1)X_A]$, then it is assumed that player B [A] wins the auction. Otherwise, in the case of a tie, each player wins the auction with equal probability. As long as the asymmetry in the players' budgets

[2]The case of $n = 2$, with symmetric and asymmetric forces, is discussed by Gross and Wagner (1950). Moving from $n = 2$ to $n \geq 3$ greatly enlarges the space of feasible n-variate distribution functions, and the equilibrium strategies examined in that article differ dramatically from the case of $n = 2$.

is below a threshold [$X_B \leq (n-1)X_A$], any tie-breaking rule which avoids the need to have the stronger player B provide a bid arbitrarily close to, but above, player A's maximal bid yields similar results. However, the specification of the tie-breaking rule plays a role once the asymmetry in the players' budgets exceeds the threshold. Once $X_B > (n-1)X_A$, this tie-breaking rule avoids the need to have the weaker player A provide a bid arbitrarily close to, but above, a bid of $X_B - (n-1)X_A$ by player B when player B bids X_A in the $n-1$ other auctions. Any tie-breaking rule which eliminates this possibility yields similar results.

In each all-pay auction j the payoff to player i for a bid of $x_{i,j}$ is given by

$$\pi_{i,j}\left(x_{i,j}, x_{-i,j}\right) = \begin{cases} v - x_{i,j} & \text{if } \ x_{i,j} > x_{-i,j} \\ -x_{i,j} & \text{if } \ x_{i,j} < x_{-i,j} \end{cases}$$

where ties are handled as described above. Each player's payoff across all n all-pay auctions is the sum of the payoffs across the individual auctions.

The bid provided to each all-pay auction must be nonnegative. For player i, the set of feasible bids across the n all-pay auctions is denoted by

$$\mathfrak{B}_i = \left\{ \mathbf{x} \in \mathbb{R}_+^n \middle| \sum_{j=1}^{n} x_{i,j} \leq X_i \right\}.$$

Strategies

Given that each of the individual contests is an all-pay auction, it is not difficult to show that there are no pure strategy equilibria for this class of games. A mixed strategy, which we term a *distribution of resources*, for player i is an n-variate distribution function $P_i : \mathbb{R}_+^n \to [0,1]$ with support (denoted $\text{Supp}(P_i)$) contained in the set of player i's set of feasible bids $\mathfrak{B}_i$ and with one-dimensional marginal distribution functions $\{F_{i,j}\}_{j=1}^n$, one univariate marginal distribution function for each all-pay auction j. To avoid confusion with the support of the joint distribution, when referring to the support of a given univariate marginal distribution — the smallest closed univariate interval whose complement has probability zero — we will make a slight abuse of terminology and use the term *domain* to denote the support of the given univariate marginal distribution function. The n-tuple of player i's bids across the n all-pay auctions is a random n-tuple drawn from the n-variate distribution function P_i.

The Non-Constant-Sum Colonel Blotto game

The N-C-S Colonel Blotto game, which we label

$$NCB\{X_A, X_B, n, v\},$$

is the one-shot game in which players compete by simultaneously announcing distributions of resources subject to their budget constraints, each all-pay auction is won by the player that provides the higher bid in that auction (where in the case of a tie the tie-breaking rule described above applies), and players' receive the sum of their payoffs across the individual all-pay auctions.

3 Relationship Between the Two Formulations

Before proceeding with the equilibrium analysis, it is instructive to provide intuition for the connection between the equilibria in the constant-sum and non-constant-sum formulations of the Colonel Blotto game. The formulation of the constant-sum Colonel Blotto game differs from the non-constant-sum game in that in each contest j the payoff to each player i for a bid of $x_{i,j}$ is given by

$$\pi_{i,j}\left(x_{i,j}, x_{-i,j}\right) = \begin{cases} \frac{1}{n} & \text{if } \; x_{i,j} > x_{-i,j} \\ 0 & \text{if } \; x_{i,j} < x_{-i,j} \end{cases}$$

where ties are handled as described above. Note that, in the constant-sum game resources which are not allocated to one of the contests have no value; that is, resources are use it or lose it. Each player's payoff across all n contests is the sum of the payoffs in the individual contests.

The following discussion provides a brief sketch of the relationship between the equilibria in the constant-sum and non-constant-sum formulations of the game. We begin this discussion with the disclaimer that this is not a sketch of the formal proofs of the main results [which are provided in the Appendix]. Instead, our objective for this discussion is simply to provide a few informal insights regarding some necessary conditions for equilibrium in both the constant-sum and non-constant-sum Colonel Blotto games and to highlight the relationship between these sets of necessary conditions. For $n \geq 3$ auctions, the Appendix provides the formal proof of the necessity of these conditions.[3]

[3]In the case of $n = 2$, these conditions are not necessary. See the discussion of the case of $n = 2$ at the conclusion of the next section.

Given that player $-i$'s strategy is given by the n-variate distribution function P_{-i} with the set of univariate marginal distribution functions $\{F_{-i,j}\}_{j=1}^n$, player i's expected payoff for any n-tuple of bids $\mathbf{x}_i \in \mathbb{R}_+^n$ is:

$$\pi_i \left(\mathbf{x}_i, \{F_{-i,j}\}_{j=1}^n\right) = \sum_{j=1}^n \left[vF_{-i,j}\left(x_{i,j}\right) - x_{i,j}\right]. \tag{1}$$

Observe that for a given P_{-i}, each player i's expected payoff depends only on the set of univariate marginal distribution functions $\{F_{-i,j}\}_{j=1}^n$ and not the correlation structure, utilized by player $-i$, among the univariate marginals.

Given this feature of the expected payoffs, it is useful to note that any joint distribution may be broken into a set of univariate marginal distribution functions and an n-copula, the function that maps the univariate marginal distribution functions into a joint distribution function.[4] Let $\mathcal{C}_i$ denote the collection of all sets of univariate marginal distribution functions $\{F_{i,j}\}_{j=1}^n$ which satisfy the constraint that there exists a mapping from the set of univariate marginal distributions into a joint distribution (an n-copula), C, in which the support of the resulting n-variate distribution function $C(F_{i,1}(x_1), \ldots, F_{i,n}(x_n))$ is contained in $\mathfrak{B}_i$.

Assuming that each of the univariate marginal distributions in player i's strategy is differentiable (possibly discontinuously so) and ignoring the possibility of a tie occurring with strictly positive probability, player i's optimization problem may be written as:

$$\max_{\left\{\{F_{i,j}\}_{j=1}^n \in\ \mathcal{C}_i\right\}} \sum_{j=1}^n \left[\int_0^\infty \left[vF_{-i,j}\left(x_{i,j}\right) - x_{i,j}\right] dF_{i,j}\right]. \tag{2}$$

Observe that the n-copula enters into the players' optimization problems only as a constraint and not as a strategic variable. That is, player i's optimization problem is invariant to the correlation structure among his own univariate marginal distribution functions subject to the constraint that there exists a mapping from the optimal set of univariate marginal distributions into a joint distribution that satisfies the restriction on the support.

Next, recall that the budget constraint holds with probability one. Therefore, the budget constraint must also hold in expectation, and player i's set of univariate marginal distribution functions satisfy the following constraint,

$$\sum_{j=1}^n \left[\int_0^\infty x_{i,j} dF_{i,j}\right] \leq X_i. \tag{3}$$

[4]See Nelsen (1999) or Schweizer and Sklar (1983) for an introduction to copulas.

Given that equation (3) is a constraint on only the set of univariate marginal distributions functions, it will be useful to include this constraint in player i's optimization problem. Thus, we have that player i's optimization problem from equation (2) may now be written as,

$$\max_{\left\{\{F_{i,j}\}_{j=1}^{n} \in\ \mathcal{C}_i\right\}} \sum_{j=1}^{n} \left[\int_{0}^{\infty} \left[vF_{-i,j}\left(x_{i,j}\right) - \left(1+\lambda_i\right) x_{i,j} \right] dF_{i,j} \right] + \lambda_i X_i. \tag{4}$$

This optimization problem is essentially a variational problem involving the maximization of a collection of functionals with the side constraints that there exist a sufficient n-copula and that each univariate marginal distribution is a weakly increasing function. The n Euler-Lagrange equations provide a set of necessary conditions for equilibrium. For each $j = 1, \ldots, n$ the corresponding Euler-Lagrange equation is given by

$$\frac{d}{dx} \left[vF_{-i,j}\left(x_{i,j}\right) - \left(1+\lambda_i\right) x_{i,j} \right] = 0. \tag{5}$$

Rearranging terms slightly, it becomes clear that for each auction j equation (5) is precisely the necessary condition that holds for one isolated all-pay auction without a budget constraint and in which the prize has value $v/(1+\lambda_i)$, henceforth the *implicit value of the prize*. The intuition is that the constraint on the total expenditure across all auctions implicitly imposes an opportunity cost $\lambda_i \geq 0$ of resource expenditure.[5] Therefore, the cost of allocating x_j resources to auction j entails not only the explicit cost of the bid but also the implicit opportunity cost from not being able to use those resources in another auction. An increase in the implicit opportunity cost of a bid has the dual interpretation of lowering the implicit value of the prize.

Applying a similar line of reasoning to the constant-sum Colonel Blotto game, it is straightforward to derive the set of necessary conditions for equilibrium given by the n Euler-Lagrange equations for that optimization problem. For each $j = 1, \ldots, n$ the corresponding Euler-Lagrange equation is given by

$$\frac{d}{dx} \left[\frac{1}{n} F_{-i,j}\left(x_{i,j}\right) - \lambda_i x_{i,j} \right] = 0. \tag{6}$$

In this case we see that for each contest j equation (6) is precisely the necessary condition that holds for one isolated all-pay auction without a budget constraint and in which the

[5]Note that λ_i takes the value of zero in the event that player i does not benefit from the relaxation of his budget constraint.

prize has value $1/(n\lambda_i)$.

As long as there exists a sufficient n-copula, each of the unique equilibrium univariate marginal distribution functions in the two games corresponds directly to the unique equilibrium univariate distribution function in a single two-player all-pay auction with complete information and with each player i's values for the prizes given by $v/(1+\lambda_i)$ and $1/(n\lambda_i)$ respectively [see Hillman and Riley 1989; Baye, Kovenock, and de Vries 1996]. Therefore, there exists a one-to-one mapping from the unique set of equilibrium univariate marginal distributions in the non-constant-sum game to those in the constant-sum game as long as there exists a sufficient n-copula.

Generically speaking, the constraint on the n-copula is non-binding if for each player the intersection of the hyperplane formed by the n-tuples which exhaust his respective budget and the n-box formed by the domains of each of the univariate marginal distributions for the corresponding all-pay auctions is well behaved. For example consider the case in which the n-box formed by the domains is $[0, X_A]^n$. If $X_B > (n-1)X_A$, then it is clear that there exist no n-tuples in the intersection of the hyperplane $\{\mathbf{x} \in \mathbb{R}^n_+ \big| \sum_{j=1}^n x_j = X_B\}$ and the n-box $[0, X_A]^n$ in which any $x_j = 0$. Thus, the support of player B's distribution of resources cannot be completely contained in his budget-balancing hyperplane and have univariate marginals with domain $[0, X_A]$.

In the constant-sum game, the constraint on the existence of a sufficient n-copula is non-binding as long as $(2/n) < (X_A/X_B) \leq 1$. Within this region, which is illustrated in Panel (i) of Figure 1, Theorem 2 of Roberson (2006) characterizes the unique sets of equilibrium univariate marginal distribution functions and Theorem 4 of that article provides the proof of the existence of a sufficient n-copula for this range.

[Insert Figure 1 here]

Before tracing out the corresponding region for the non-constant-sum formulation of the game, observe that Panel (i) of Figure 1 also delineates the regions of the parameter space that correspond to Theorems 3 and 5 of Roberson (2006), labelled regions 3 and 5 respectively. In these regions, in which $(1/n) < (X_A/X_B) \leq (2/n)$, there exists a corresponding parameter region in the non-constant-sum game over which the equilibrium univariate marginal distribution functions in the two games are related. However, this relationship is not necessarily one-one. The issue is that the constraint on the existence of a sufficient

n-copula comes into play and the sets of univariate marginal distributions must be adjusted accordingly. In the two games these adjustments may vary.

For the constant-sum game's remaining parameter configuration $X_A \leq (X_B/n)$ the players are at the extreme end of the asymmetry spectrum. Over this parameter region, the stronger player (B) has a sufficient level of resources to win each of the n contests with certainty, and, due to the use it or lose it feature of the constant-sum formulation, that game becomes trivial. In this region there is no relationship between the two games. Due to the relaxation of the use it or lose it feature, the non-constant-sum game is never trivial, and in this range we construct entirely new equilibrium distributions of resources for the non-constant-sum game.

We now introduce what we term the *modified budgets* for the non-constant-sum game with $n \geq 3$. In the expressions for the modified budgets we define the sets Tk for $k = 1, 2, 3, 5$ to denote the portion of the parameter space that is covered by the corresponding theorem number $k(= 1, 2, 3, 5)$ in the following section.[6] These regions are delineated as follows.

T1: $\Big\{(X_A, X_B) \in \mathbb{R}^2_+ \big| (\frac{2}{n}) \min\{v, X_B\} < X_A \leq X_B\Big\}$

T2: $\Big\{(X_A, X_B) \in \mathbb{R}^2_+ \big| X_B/(n-1) \leq X_A \leq (\frac{2}{n}) \min\{v, X_B\} \quad \text{or} \quad X_A = \frac{2v}{n} \quad \text{and} \quad X_B > v(2 - \frac{2}{n})\Big\}$

T3: $\Big\{(X_A, X_B) \in \mathbb{R}^2_+ \big| X_A < (\frac{2v}{n}) \ \text{ and } \ X_A \leq \max\Big\{\frac{X_B - \frac{2v}{n}}{n-2}, \frac{X_B}{n}\Big\}\Big\}$

T5: $\Big\{(X_A, X_B) \in \mathbb{R}^2_+ \big| \max\Big\{\frac{X_B - \frac{2v}{n}}{n-2}, \frac{X_B}{n}\Big\} < X_A < \frac{X_B}{n-1}\Big\}$

Recall that the floor function $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. Player A's modified budget is given by

$$M_{X_A}(X_A, X_B) = \begin{cases} \min\Big\{X_A, \frac{nv}{2}\Big\} & \text{if } \ (X_A, X_B) \in \text{T1} \\ X_A & \text{if } \ (X_A, X_B) \in \text{T2} \\ \frac{n(X_A)^2}{2v} & \text{if } \ (X_A, X_B) \in \text{T3} \\ X_A - \frac{(1 - \frac{nX_A}{2v})(nX_A - X_B)}{\lfloor \frac{X_A}{X_B - (n-1)X_A} \rfloor + 1} & \text{if } \ (X_A, X_B) \in \text{T5} \end{cases}$$

[6]Theorem 4 is a proof of the existence of a pair of n-variate joint distributions which satisfy the conditions specified in Theorem 3.

and player B's modified budget is given by

$$M_{X_B}(X_A, X_B) = \begin{cases} \min\left\{X_B, \frac{nv}{2}, \left(\frac{nvX_A}{2}\right)^{1/2}\right\} & \text{if } (X_A, X_B) \in \text{T1} \\ \min\left\{X_B, v\left(2 - \frac{2}{n}\right)\right\} & \text{if } (X_A, X_B) \in \text{T2} \\ n\left(X_A - \frac{X_A^2}{2v}\right) & \text{if } (X_A, X_B) \in \text{T3} \\ \frac{nX_A(nX_B-(n-1)^2X_A)}{2v} + \left(1 - \frac{n(X_B-(n-2)X_A)}{2v}\right)\frac{\left(\left\lfloor\frac{X_A}{X_B-(n-1)X_A}\right\rfloor+2\right)X_A}{\left\lfloor\frac{X_A}{X_B-(n-1)X_A}\right\rfloor+1} & \text{if } (X_A, X_B) \in \text{T5} \end{cases}$$

It will be useful to define the set of n-tuples which exhaust the modified budgets M_{X_A} and M_{X_B}. Let $\overline{\mathfrak{B}}_i$ denote this set, defined as

$$\overline{\mathfrak{B}}_i = \left\{\mathbf{x} \in \mathbb{R}^n_+ \middle| \sum_{j=1}^{n} x_{i,j} = M_{X_i}(X_A, X_B)\right\},$$

and note that $\overline{\mathfrak{B}}_i \subset \mathfrak{B}_i$.

The players' modified budgets, which are illustrated in (X_A, X_B)-space as the shaded regions in Panel (ii) of Figure 1, are the equilibrium total expected expenditures for each of the equilibria examined in the following section [i.e., for player i, $M_{X_i} = \sum_j E_{F_{i,j}}(x_{i,j})$]. As shown in the Appendix [see Lemma 2], in the T1 and T2 parameter regions with $X_A \neq (2v/n)$, these equilibrium total expected expenditures are unique. In the remaining parameter regions, there exist other payoff non-equivalent equilibria.

Note that given a pair of resource levels X_A and X_B which satisfy $(X_A, X_B) \in$ T1 there are three possible cases: (a) neither player uses all of their available resources [i.e., $M_{X_A} = nv/2$ and $M_{X_B} = nv/2$], (b) only (the weaker) player A uses all of his available resources [i.e., $M_{X_A} = X_A$ and $M_{X_B} = (nvX_A/2)^{1/2}$], and (c) both players A and B use all of their available resources [i.e., $M_{X_A} = X_A$ and $M_{X_B} = X_B$]. The regions corresponding to each of these cases appears in Panel (ii) of Figure 1 as 1a, 1b, and 1c respectively. Given that in the constant-sum game resources are use it or lose it, such considerations do not arise in that game.

It is important to observe that when X_A and X_B satisfy the condition that $X_A \in ((2/n)\min\{v, X_B\}, X_B]$ [i.e., regions 1a, 1b, and 1c of Panel (ii) of Figure 1], the modified budgets satisfy the corresponding condition that $(2/n) < (M_{X_A}/M_{X_B}) \leq 1$. As we will show, there exists a one-to-one correspondence between the sets of equilibrium univariate marginal distribution functions that arise in this region and those that arise in the constant-sum game for the region $(2/n) < (X_A/X_B) \leq 1$. This characterization is formally stated in Theorem 1 of the next section.

Similarly, for X_A and X_B which lie in regions 2 and 5 [which correspond to Theorems 2 and 5] of Panel (ii) of Figure 1, the modified budgets satisfy the condition that $(1/n) < (M_{X_A}/M_{X_B}) \leq (2/n)$. In these regions the sets of equilibrium univariate marginal distribution functions are related to those arising in the constant-sum game for the parameter range $(1/n) < (X_A/X_B) \leq (2/n)$. But as mentioned before, this relationship is not necessarily one-one.

For all budget configurations (X_A, X_B) which lie in region 3 of panel (ii), we construct an entirely new set of equilibrium distributions of resources [see Theorem 3]. Note that this region covers not only the portion of the parameter space which corresponds to the trivial region of the constant-sum game [i.e., $X_A \leq (X_B/n)$], but also a portion of the parameter space in which the constant-sum game is non-trivial. Again, this breakdown in the relationship between the equilibria in the two games occurs in sufficiently asymmetric regions of the parameter space because of the discrepancy in the value of unused resources in the two formulations.

To summarize, whereas there is a one-to-one relationship between the unique equilibrium sets of univariate marginal distribution functions in the constant-sum and and non-constant-sum versions of the game — when the asymmetry between the players' budgets is below a threshold — this relationship is non-linear with respect to the players' budgets but is linear with respect to the players' modified budgets.

4 Equilibrium Distributions of Resources

The following Theorems examine the equilibrium distributions of resources for all parameter configurations of the non-constant-sum Colonel Blotto game with $n \geq 3$ auctions. This section concludes with the case of $n = 2$ auctions. In the Theorem 1 parameter range we characterize each player's unique set of equilibrium univariate marginal distributions. In the Theorem 2 parameter range with $X_A \neq (2v/n)$ we characterize the unique set of equilibrium univariate marginal distributions for player A and provide an equilibrium distribution of resources for player B. Over this range player B does not have a unique set of equilibrium univariate marginal distribution functions.[7] In the Theorem 3 and 5 parameter ranges we provide an equilibrium distribution of resources for each player. Over this range neither player has a unique set of univariate marginal distribution functions.[8] For the Theorem

[7]An alternative set of equilibrium univariate marginal distribution functions is provided in the discussion following Lemma 7 in the Appendix.

[8]For information on the non-uniqueness of the univariate marginals over the Theorem 5 [3] parameter range, see the discussion preceding Theorem 5 [at the conclusion of the Appendix].

1 and 2 parameter ranges with $X_A \neq (2v/n)$ the equilibrium expected payoffs and the equilibrium total expected expenditures are unique [see Lemma 2 in the the Appendix].

Three or more Auctions

For the game $NCB\{X_A, X_B, n, v\}$ with $n \geq 3$, Theorem 1 examines all parameter configurations which lie in the 1a, 1b, and 1c regions of panel (ii) of Figure 1. Recall that in these regions the resulting modified budgets satisfy the condition $(2/n) < (M_{X_A}/M_{X_B}) \leq 1$.

Theorem 1. *Let X_A, X_B, v, and $n \geq 3$ satisfy $(2/n)\min\{v, X_B\} < X_A \leq X_B$ (equivalently $(2/n) < (M_{X_A}/M_{X_B}) \leq 1$). The pair of n-variate distribution functions P_A^* and P_B^* constitute a Nash equilibrium of the game $NCB\{X_A, X_B, n, v\}$ if and only if they satisfy the following two conditions: (1) For each player i, $\mathrm{Supp}(P_i^*) \subset \mathfrak{B}_i$ and (2) P_i^*, $i = A, B$, provides the corresponding unique set of univariate marginal distribution functions $\{F_{i,j}^*\}_{j=1}^n$ outlined below.*

$$\forall\ j \in \{1,\ldots,n\} \qquad F_{A,j}^*\left(x_j\right) = \left(1 - \tfrac{M_{X_A}}{M_{X_B}}\right) + \tfrac{x_j}{(2/n)M_{X_B}}\left(\tfrac{M_{X_A}}{M_{X_B}}\right) \qquad \text{for} \quad x_j \in \left[0, \tfrac{2}{n}M_{X_B}\right].$$

$$\forall\ j \in \{1,\ldots,n\} \qquad F_{B,j}^*\left(x_j\right) = \tfrac{x_j}{(2/n)M_{X_B}} \qquad \text{for} \quad x_j \in \left[0, \tfrac{2}{n}M_{X_B}\right].$$

The unique equilibrium expected payoff for player A is $(nvM_{X_A}/2M_{X_B}) - M_{X_A}$, and the unique equilibrium expected payoff for player B is $nv\left(1 - (M_{X_A}/2M_{X_B})\right) - M_{X_B}$. The unique equilibrium total expected expenditure for player A is $M_{X_A}(X_A, X_B) = \min\{X_A, (nv/2)\}$, and the unique equilibrium total expected expenditure for player B is $M_{X_B}(X_A, X_B) = \min\{X_B, (nv/2), (nvX_A/2)^{1/2}\}$.

The existence of a pair of n-variate distribution functions which satisfy conditions (1) and (2) of Theorem 1 is provided in Roberson (2006). In particular, Theorem 4 of Roberson (2006) establishes the existence of n-variate distribution functions for which $\mathrm{Supp}(P_i^*) \subset \overline{\mathfrak{B}}_i$ and that provide the necessary sets of univariate marginal distribution functions given in Theorem 1. The proof of the uniqueness of the equilibrium sets of univariate marginal distribution functions, equilibrium payoffs, and equilibrium total expected expenditures is given in the Appendix.

Although it is straightforward to show that any pair of n-variate distribution functions which satisfy conditions (1) and (2) of Theorem 1 form an equilibrium, it is useful to provide the intuition for this result. We begin with the expected payoffs for each player. Let P_B^* denote a feasible n-variate distribution function for player B with the univariate marginal distributions $\{F^*_{B,j}\}_{j=1}^n$ given in Theorem 1. If player B is using P^*_B, then player A's expected payoff π_A, when player A chooses any n-tuple of bids $\mathbf{x}_A \in \mathfrak{B}_A \bigcap [0, (2/n)M_{X_B}]^n$ [i.e., one bid for each of the n all-pay auctions such that $\sum_j x_{A,j} \leq X_A$ and $x_{A,j} \in [0, (2/n)M_{X_B}]$ for each auction j], is

$$\pi_A(\mathbf{x}_A, P^*_B) = \sum_{j=1}^{n} \left[vF^*_{B,j}(x_{A,j}) - x_{A,j} \right].$$

Recall that for all j, $F^*_{B,j}(x_j) = \frac{x_j}{(2/n)M_{X_B}}$ for $x_j \in [0, (2/n)M_{X_B}]$. Simplifying yields

$$\pi_A(\mathbf{x}_A, P^*_B) = \left(\frac{nv}{2M_{X_B}} - 1 \right) \sum_{j=1}^{n} x_{A,j}. \tag{7}$$

Similarly, the expected payoff π_B to player B from any n-tuple of bids $\mathbf{x}_B \in \mathfrak{B}_B \bigcap (0, (2/n)M_{X_B}]^n$ — when player A uses a feasible n-variate distribution P^*_A with the univariate marginal distributions $\{F^*_{A,j}\}_{j=1}^n$ given in Theorem 1 — follows directly,

$$\pi_B(\mathbf{x}_B, P^*_A) = nv\left(1 - \frac{M_{X_A}}{M_{X_B}}\right) + \left(\frac{nvM_{X_A}}{2M^2_{X_B}} - 1\right) \sum_{j=1}^{n} x_{B,j}. \tag{8}$$

Observe that neither player can bid below 0 and that bidding above $(2/n)M_{X_B}$ is suboptimal. Thus, for the Theorem 1 parameter range equations (7) and (8) provide the maximal payoffs (for player A and player B respectively) for any feasible n-tuple of bids across the n all-pay auctions.

Recall that there are three possible cases: (a) neither player uses all of his available resources, (b) only (the weaker) player A uses all of his available resources, and (c) both players A and B use all of their available resources. These three regions are shown graphically in panel (ii) of Figure 1 as regions 1a, 1b, and 1c respectively. Suppose that we are in case (a) in which neither player uses all of his available resources. Case (a) corresponds to the situation in which the total value of the n auctions nv is low enough relative to the players' budgets that neither player has incentive to commit all of his resources. In the Theorem 1 parameter range player A's modified budget is given by $M_{X_A} = \min\{X_A, nv/2\}$. If player A does not use all of his budget, then it must be that $X_A > (nv/2)$ and so $M_{X_A} = (nv/2)$. Similarly from player B's modified budget in the Theorem 1 range [$M_{X_B} = \min\{X_B, nv/2, (nvX_A/2)^{1/2}\}$], it follows that if player A (the weaker player) is not using all of his budget then $M_{X_B} = (nv/2)$. Because $M_{X_A} = M_{X_B} = (nv/2)$, the expected payoffs given in (7) and (8) are $\pi_A(\mathbf{x}_A, P^*_B) = 0$ and $\pi_B(\mathbf{x}_B, P^*_A) = 0$ respectively. Observe that in case (a) neither player has incentive to change their total resource expenditure, $\sum_j x_{i,j}$, across the n all-pay auctions. That is, because $M_{X_A} = M_{X_B} = (nv/2)$ and the opponent is using the equilibrium strategy, the expected payoff to player i, given in equations (7) and (8), is zero for all $\mathbf{x}_i \in [0, v]^n$ regardless of player i's total expenditure, $\sum_j x_{i,j}$, in the n all-pay auctions.

Now suppose that we are in case (b) in which only player A uses all of his budget. Case (b) corresponds to the situation in which the total value of the n all-pay auctions nv is high enough that the weaker player optimally commits all of his resources but not so high that the stronger player must also commit all of his resources to the n all-pay auctions. From the preceding discussion it follows that $X_A \leq (nv/2)$ and thus $M_{X_A} = X_A$. If player B is not using all of his budget then from $M_{X_B} = \min\{X_B, nv/2, (nvX_A/2)^{1/2}\}$, it must be that $X_B > (nvX_A/2)^{1/2}$ and so $M_{X_B} = (nvX_A/2)^{1/2}$. Inserting M_{X_A} and M_{X_B} into equations (7) and (8) and simplifying yields

$$\pi_A\left(\mathbf{x}_A, P_B^*\right) = \left(\left(\frac{nv}{2X_A}\right)^{1/2} - 1\right)\sum_{j=1}^{n} x_{A,j} \tag{9}$$

and

$$\pi_B\left(\mathbf{x}_B, P_A^*\right) = nv\left(1 - \left(\frac{2X_A}{nv}\right)^{1/2}\right). \tag{10}$$

Recall that in case (b) $X_A \leq (nv/2)$ and so $((nv/2X_A)^{1/2} - 1) \geq 0$. From equation (9) we see that player A is indifferent with regards to which all-pay auctions to commit resources to, but has incentive to increase his total resource expenditure across the n all-pay auctions [i.e., $\sum_j x_{A,j}$]. However in case (b), player A's equilibrium distribution of resources P_A^* expends his budget with probability one [i.e., at each point $b_A \in \text{Supp}(P_A^*)$, $\sum_j x_{A,j} = X_A$].[9] From equation (10) we see that, when $M_{X_A} = X_A$ and $M_{X_B} = (nvX_A/2)^{1/2}$ are inserted into player B's expected payoff given in equation (8), player B's expected payoff is the same for all n-tuples $\mathbf{x}_B \in (0, (2nvX_A)^{1/2}]^n$. That is player B's expected payoff is independent of his total expenditure $\sum_j x_{B,j}$ [so long as $\mathbf{x}_B \in (0, 2(nvX_A/2)^{1/2}]^n$], and so player B does not have incentive to change his total resource expenditure across the n all-pay auctions.

Finally, suppose that we are in case (c) in which each player is at his respective budget constraint. Case (c) corresponds to the situation in which the total value of the n all-pay

[9]Recall that Roberson (2006) establishes the existence of n-variate distribution functions for which $\text{Supp}(P_i^*) \subset \overline{\mathfrak{B}}_i$, and that in this case $M_{X_A} = X_A$. It follows directly that player A expends his budget with probability one.

auctions nv is high enough that both players optimally commit all of their resources to the n all-pay auctions. Thus, $M_{X_A} = X_A$ and $M_{X_B} = X_B$. From equations (7) and (8) it follows that

$$\pi_A(\mathbf{x}_A, P_B^*) = \left(\frac{nv}{2X_B} - 1\right)\sum_{j=1}^{n} x_{A,j} \tag{11}$$

and

$$\pi_B(\mathbf{x}_B, P_A^*) = nv\left(1 - \frac{X_A}{X_B}\right) + \left(\frac{nvX_A}{2X_B^2} - 1\right)\sum_{j=1}^{n} x_{B,j}. \tag{12}$$

In case (c), $X_A < (nv/2)$ and $X_B < (nvX_A/2)^{1/2} < (nv/2)$. Observe in equation (11) that $((nv/2X_B) - 1) > 0$ and, thus, player A has incentive to increase his total resource expenditure across the n all-pay auctions, but in his equilibrium distribution of resources P_A^* he is already at his budget constraint with probability one [i.e., at each point $\mathbf{x}_A \in \text{Supp}(P_A^*)$, $\sum_j x_{A,j} = X_A$]. Similarly, in equation (12) $((nvX_A/2X_B^2) - 1) > 0$ and, thus, player B has incentive to increase his total resource expenditure across the n all-pay auctions, but in his equilibrium distribution of resources P_B^* he is already at his budget constraint with probability one [i.e., at each point $\mathrm{x}_B \in \text{Supp}(P_B^*)$, $\sum_j x_{B,j} = X_B$].

Because Roberson (2006) demonstrates the existence of a pair of n-variate distributions $\{P_{A,}^*, P_{A,}^*\}$ in which $\text{Supp}(P_i^*) \subset \overline{\mathfrak{B}}_i$ for $i = A, B$ and that provides the sets of univariate marginal distributions specified in condition (2) of Theorem 1, it follows from the arguments given above that such a pair of n-variate distribution functions constitute an equilibrium in all three cases (a), (b), and (c). The proof of the uniqueness of the sets of univariate marginal distributions is given in the Appendix.

Once $(M_{X_A}/M_{X_B}) = (2/n)$ both the uniqueness of player B's set of equilibrium univariate marginal distributions and the relationship with the two-player all-pay auction with complete information fail to hold. The reason for this breakdown is that once $X_B/(n-1) \leq X_A \leq (2/n)\min\{v, X_B\}$, or equivalently$(1/(n-1)) \leq (M_{X_A}/M_{X_B}) \leq (2/n)$, it is possible for player B's set of equilibrium univariate marginals to have atoms that lie strictly within the interior and at the upper bound of the domain and player B's equilibrium total expected expenditure is not unique.[10] In Theorem 2 we provide the unique set of equilibrium univariate marginal distributions for player A and provide an equilibrium set of univariate marginal distributions for player B.

Theorem 2. *Let X_A, X_B, v, and $n \geq 3$ satisfy $X_B/(n-1) \leq X_A \leq (2/n)\min\{v, X_B\}$ or $X_A = (2v/n)$ and $X_B > v(2-(2/n))$ [equivalently $1/(n-1) \leq (M_{X_A}/M_{X_B}) \leq (2/n)$]. The*

[10]See the discussion at the conclusion of the Appendix.

n-variate distribution function P_A^* is a Nash equilibrium strategy for player A in the game $NCB\{X_A, X_B, n, v\}$ if and only if it satisfies the following two conditions: (1) $\mathrm{Supp}(P_A^*) \subset \mathfrak{B}_A$ *and (2) P_A^* provides the corresponding set of univariate marginal distribution functions $\{F_{A,j}^*\}_{j=1}^n$ outlined below.*

$$\forall\ j \in \{1, \ldots, n\} \qquad F_{A,j}^*(x_j) = \left(1 - \tfrac{2}{n}\right) + \tfrac{x_j}{X_A}\left(\tfrac{2}{n}\right) \qquad \text{for} \quad x_j \in [0, X_A]\,.$$

Sufficient conditions for P_B^* to be a Nash equilibrium strategy include: $\mathrm{Supp}(P_B^*) \subset \mathfrak{B}_B$ *and that P_B^* provides the corresponding set of univariate marginal distribution functions $\{F_{B,j}^*\}_{j=1}^n$ outlined below.*

$$\forall\ j \in \{1, \ldots, n\} \qquad F_{B,j}^*(x_j) = \begin{cases} \frac{2x_j\left(X_A - \frac{M_{X_B}}{n}\right)}{(X_A)^2} & \text{for} \quad x_j \in [0, X_A) \\ 1 & \text{for} \quad x_j \geq X_A \end{cases}.$$

In equilibria satisfying these conditions on P_A^* and P_B^*, the expected payoff for player A is $2v(1-(M_{X_B}/nX_A))-X_A$, the expected payoff for player B is $nv-2v(1-(M_{X_B}/nX_A))-M_{X_B}$, the total expected expenditure for player A is $M_{X_A}(X_A, X_B) = X_A$, and the total expected expenditure for player B is $M_{X_B}(X_A, X_B) = \min\{X_B, v(2-(2/n))\}$.

If $X_A \neq (2v/n)$, then the equilibrium expected payoffs and total expected expenditures are unique. In the event that $X_A = (2v/n)$ player B's equilibrium total expected expenditure is not unique. As a direct consequence player A's equilibrium expected payoff is not unique when $X_A = (2v/n)$.

The existence of a pair of n-variate distribution functions which satisfy Theorem 2's necessary and sufficient condition for player A and sufficient condition for player B is provided in Theorem 4 of Roberson (2006). For $X_A \neq (2v/n)$, the proof of uniqueness for the equilibrium payoffs, the equilibrium total expected expenditures, and player A's set of univariate marginal distributions is given in the Appendix. If $X_A = (2v/n)$, then there exist equilibria in which player B uses strategies P_B in which $\sum_j E_{F_{B,j}}(x_{B,j}) \neq M_{X_B}(X_A, X_B)$, where for $(X_A, X_B) \in T2$, $M_{X_B}(X_A, X_B) = \min\{X_B, v(2-(2/n))\}$. In fact, there exist a continuum of equilibria in which P_B satisfies a modified form of the sufficient conditions given in Theorem 2. The modification to the sufficient conditions for P_B^* is that the term M_{X_B} in the univariate marginal distributions given above may be replaced by any value in the set $[v, \min\{X_B, v(2-(2/n))\}]$. In this case it is clear that the equilibrium payoffs are not unique. Player B's set of equilibrium univariate marginal distributions is, also, not unique, and an alternative set of equilibrium univariate marginal distributions for player B is given in the discussion at the conclusion of the Appendix.

To sketch the proof that a pair of n-variate distributions that satisfy the conditions of Theorem 2 form an equilibrium, let P_B^* denote a feasible n-variate distribution for player B with the univariate marginal distributions $\{F_{B,j}^*\}_{j=1}^n$ given in Theorem 2. If player B is using P_B^*, then player A's expected payoff π_A, when player A chooses any n-tuple of bids $\mathbf{x}_A \in \mathfrak{B}_A \bigcap [0, X_A]^n$, is

$$\pi_A\left(\mathbf{x}_A, P_B^*\right) = \left(\frac{2v\left(X_A - (M_{X_B}/n)\right)}{X_A^2} - 1\right) \sum_{j=1}^{n} x_{A,j}. \tag{13}$$

Note that $(2v/X_A^2)(X_A - (M_{X_B}/n)) - 1 \geq 0$ is equivalent to $M_{X_B} \leq (n - (nX_A/2v))X_A$. If $X_A < (2v/n)$, it follows from equation (13) that player A has incentive to choose n-tuples $\mathbf{x}_A \in [0, X_A]^n$ such that $\sum_j x_{A,j} = X_A$. When $X_A = (2v/n)$, player A's expected payoff from any n-tuple $\mathbf{x}_A \in [0, X_A]^n$ is zero.

Similarly, the expected payoff π_B to player B from any n-tuple of bids across the n all-pay auctions $\mathbf{x}_B \in \mathfrak{B}_B \bigcap (0, X_A]^n$, when player A uses a feasible n-variate distribution P_A^* with the univariate marginal distributions $\{F_{A,j}^*\}_{j=1}^n$ given in Theroem 2, is

$$\pi_B\left(\mathbf{x}_B, P_A^*\right) = nv\left(1 - \frac{2}{n}\right) + \left(\frac{2v}{nX_A} - 1\right) \sum_{j=1}^{n} x_{B,j}. \tag{14}$$

Because $X_A \leq (2v/n)$ it follows that $(2v/nX_A) - 1 \geq 0$. If $X_A < (2v/n)$, player B has incentive to choose n-tuples $\mathbf{x}_B \in (0, X_A]^n$ such that $\sum_j x_{B,j} = X_B$. If $X_A = (2v/n)$, then any n-tuple $\mathbf{x}_B \in (0, X_A]^n$ provides player B with an expected payoff of $nv(1 - (2/n))$.

Seeing that Roberson (2006) demonstrates the existence of a pair of n-variate distributions that result in the sets of univariate marginal distributions given in Theorem 2 and that satisfy the respective budget restrictions with probability one [i.e., for $i = A, B$ at each point $b_i \in \text{Supp}(P_i^*)$, $\sum_j x_{i,j} = M_{X_i}$], it follows from the arguments given above that such a pair of n-variate distribution functions constitute an equilibrium. The proof of uniqueness of player A's set of univariate marginal distributions is given in the Appendix.

The following Theorem constructs entirely new equilibrium distributions of resources for the highly asymmetric portion of the parameter space in which the relationship between the constant-sum and non-constant-sum versions of the game breaks down.

Theorem 3. *Let X_A, X_B, v, and $n \geq 3$ satisfy $X_A < (2v/n)$ and $X_A \leq \max\{(X_B -$*

$\frac{2v}{n})/(n-2), X_B/n\}$. *The pair of n-variate distribution functions P_A^* and P_B^* constitute a Nash equilibrium of the game $NCB\{X_A, X_B, n, v\}$ if they satisfy the following two conditions: (1) For each player i, $\text{Supp}(P_i^*) \subset \mathfrak{B}_i$ and (2) P_i^*, $i = A, B$, provides the corresponding set of univariate marginal distribution functions $\{F_{i,j}^*\}_{j=1}^n$ outlined below.*

$$\forall\ j \in \{1, \ldots, n\} \qquad F_{A,j}\left(x_j\right) = \left(1 - \tfrac{X_A}{v}\right) + \tfrac{x_j}{v} \qquad \text{for} \ \ x_j \in [0, X_A]\,.$$

$$\forall\ j \in \{1, \ldots, n\} \qquad F_{B,j}\left(x_j\right) = \begin{cases} \frac{x_j}{v} & \text{for} \ \ x_j \in [0, X_A) \\ 1 & \text{for} \ \ x_j \geq X_A \end{cases}\,.$$

In equilibria satisfying these conditions on P_A^* and P_B^*, the expected payoff for player A is 0, the expected payoff for player B is $nv(1-(X_A/v))$, the total expected expenditure for player A is $(X_A)^2(n/2v)$, and the total expected expenditure for player B is $n(X_A - (X_A)^2/2v)$.

We begin with a sketch of the proof that a pair of n-variate distribution functions which satisfy the conditions of Theorem 3 form an equilibrium, and then move on to the proof of existence of such a pair of n-variate distribution functions.

To see that these two sets of univariate marginal distributions form an equilibrium in the Theorem 3 parameter region, let P_B^* denote a feasible n-variate distribution for player B with the univariate marginal distributions $\{F_{B,j}^*\}_{j=1}^n$ given in Theorem 3. If player B is using P_B^*, then player A's expected payoff π_A, when player A chooses any n-tuple of bids $\mathbf{x}_A \in \mathfrak{B}_A$ is

$$\pi_A\left(\mathbf{x}_A, P_B^*\right) = 0. \tag{15}$$

From equation (15), player A does not have incentive to increase or decrease his total expenditure in the n all-pay auctions.

Similarly, the expected payoff π_B to player B from any n-tuple of bids across the n all-pay auctions $\mathbf{x}_B \in \mathfrak{B}_B \bigcap (0, X_A]^n$, when player A uses a feasible n-variate distribution P_A^* with the univariate marginal distributions $\{F_{A,j}^*\}_{j=1}^n$ given in Theorem 3, is

$$\pi_B\left(\mathbf{x}_B, P_A^*\right) = nv\left(1 - \frac{X_A}{v}\right). \tag{16}$$

Thus, player B also has the same expected payoff for each $\mathbf{x}_B \in (0, X_A]^n$ and therefore has no incentive to increase or decrease his total expenditure in the n all-pay auctions.

Assuming that there exists a pair of n-variate distribution functions which satisfy conditions (1) and (2) of Theorem 3, it follows from the arguments given above that such a pair of n-variate distribution functions constitute an equilibrium. We now establish the existence of sufficient n-variate distributions for the Theorem 3 parameter range.

Theorem 4. *For each set of equilibrium univariate marginal distribution functions, $\{F_{i,j}\}_{j=1}^{n}$, given in Theorem 3, there exists an n-copula, C, such that the support of the n-variate distribution function $C(F_{i,1}(x_1), \ldots, F_{i,n}(x_n))$ is contained in $\mathfrak{B}_i$.*

We begin with the proof for player A. The construction of a sufficient n-variate distribution function for player A and $X_A \geq (v/n)$ is outlined as follows [recall that in the Theorem 3 parameter region $X_A < (2v/n)$]. The remaining case that $X_A < (v/n)$ is addressed directly following this case.

1. Player A selects $n-2$ of the all-pay auctions, each all-pay auction chosen with equal probability, and bids zero in each of those all-pay auctions.

2. On the remaining 2 all-pay auctions, player A randomizes uniformly on three line segments: (i) $\{(x_1, x_2) \in \mathbb{R}_+^2 |\ x_1 + x_2 = 2X_A - (2v/n)\}$, (ii) $\{(x_1, x_2) |\ x_1 = 0 \text{ and } 2X_A - (2v/n) \leq x_2 \leq X_A\}$, and (iii) $\{(x_1, x_2) |\ x_2 = 0 \text{ and } 2X_A - (2v/n) \leq x_1 \leq X_A\}$. This support is shown in panel (i) of Figure 2, and this randomization is discussed in greater detail directly following this outline.

3. There are ${}_nC_2$ ways of dividing the n all-pay auctions into disjoint subsets such that $n-2$ all-pay auctions receive bids of zero with probability 1 and 2 all-pay auctions involve randomizations of resources as in point 2. The n-variate distribution function formed by placing probability $[{}_nC_2]^{-1}$ on each of these n-variate distribution functions has univariate marginal distribution functions which each have a mass point of $(1 - (X_A/v))$ at 0 and randomize uniformly on $(0, X_A]$ with the remaining mass.

The pivotal step in this construction is point 2. Let x_i denote the allocation of resources to all-pay auction $i \in \{1, 2\}$. Consider the support of a bivariate distribution function, G_A, for x_1 and x_2 which uniformly places mass $1 - (nX_A/2v)$ on each of the two following line segments:

$$\{(x_1, x_2) |\ x_1 = 0 \text{ and } 2X_A - \tfrac{2v}{n} \leq x_2 \leq X_A\}$$

$$\{(x_1, x_2) |\ x_2 = 0 \text{ and } 2X_A - \tfrac{2v}{n} \leq x_1 \leq X_A\}.$$

and uniformly places the remaining mass, $(nX_A/v) - 1$, on the line segment

$$\{(x_1, x_2) \in \mathbb{R}_+^2 |\ x_1 + x_2 = 2X_A - \tfrac{2v}{n}\}.$$

This support is shown in panel (i) of Figure 2.

[Insert Figure 2 here]

In the expression for this bivariate distribution function we will use the following notation.

R1: $\left\{(x_1, x_2) \in [0, 2X_A - \frac{2v}{n}]^2\right\}$

R2: $\left\{(x_1, x_2) \in (2X_A - \frac{2v}{n}, X_A] \times [0, 2X_A - \frac{2v}{n}]\right\}$

R3: $\left\{(x_1, x_2) \in [0, 2X_A - \frac{2v}{n}] \times (2X_A - \frac{2v}{n}, X_A]\right\}$

R4: $\left\{(x_1, x_2) \in (2X_A - \frac{2v}{n}, X_A]^2\right\}$

The bivariate distribution function for x_1, x_2 is given by

$$G_A(x_1, x_2) = \begin{cases} \left(\frac{n}{2v}\right) \max\left\{x_1 + x_2 - 2X_A + \frac{2}{vn}, 0\right\} & \text{if} \quad (x_1, x_2) \in \text{R1} \\ \left(1 - \frac{nX_A}{v}\right) + \frac{nx_1}{2v} + \frac{nx_2}{2v} & \text{if} \quad (x_1, x_2) \in \text{R2} \cup \text{R3} \cup \text{R4} \end{cases}$$

The univariate marginal distributions are given by $G_A(x_1, X_A) = (1-(nX_A/2v)) + (nx_1/2v)$ and $G_A(X_A, x_2) = (1 - (nX_A/2v)) + (nx_2/2v)$. To see that G_A provides the necessary univariate marginal distributions, observe that given the randomization outlined above player A allocates zero resources to each all-pay auction j with probability $((n-2)/n) + (2/n)(1 - (nX_A/2v)) = (1 - (X_A/v))$ and randomizes uniformly over the interval $(0, X_A]$ with the remaining mass.

If $X_A < (v/n)$, then player A allocates zero resources to $n-1$ of the all-pay auctions and provides a random level of resources in the one remaining all-pay auction. In this one remaining all-pay auction player A has a mass point of $(1 - (nX_A/v))$ at 0 and randomizes uniformly over the interval $[0, X_A]$ with the remaining mass.

The proof for player B is similar. The construction of a sufficient n-variate distribution function for player B and $X_A \geq (v/n)$ is outlined as follows. In the Theorem 3 parameter region $X_B \geq \min\{nX_A, (n-2)X_A + (2v/n)\}$. If $X_A \geq (v/n)$ then $X_B \geq (n-2)X_A + (2v/n)$. The remaining case in which $X_A < (v/n)$ and $X_B \geq nX_A$ is addressed directly following this case.

1. Player B selects $n-2$ of the all-pay auctions, each all-pay auction chosen with equal probability, and bids X_A in each of those all-pay auctions.

2. On the remaining 2 all-pay auctions, player B randomizes uniformly on three line segments: (i) $\{(x_1, x_2) \in \mathbb{R}^2_+ |\ x_1 + x_2 = (2v/n)\}$, (ii) $\{(x_1, x_2)|\ x_1 = X_A$ and $0 \leq x_2 \leq (2v/n) - X_A\}$, and (iii) $\{(x_1, x_2)|\ x_2 = X_A$ and $0 \leq x_1 \leq (2v/n) - X_A\}$. This support is shown in Panel (ii) of Figure 2, and this randomization is discussed in greater detail directly following this outline.

3. There are ${}_nC_2$ ways of dividing the n all-pay auctions into disjoint subsets such that $n-2$ all-pay auctions receive X_A with probability 1 and 2 all-pay auctions involve randomizations of force as in point 2. The n-variate distribution function formed by placing probability $[{}_nC_2]^{-1}$ on each of these n-variate distribution functions has univariate marginal distribution functions which each have a mass point of $(1-(X_A/v))$ at X_A and randomize uniformly on $[0, X_A)$ with the remaining mass.

The pivotal step in this construction is again point 2. Let x_i denote the allocation to all-pay auction $i \in \{1, 2\}$. Consider the support of a bivariate distribution function, G_B, for x_1 and x_2 which uniformly places mass $1 - (nX_A/2v)$ on each of the two following line segments

$$\{(x_1, x_2)|\ x_1 = X_A \text{ and } 0 \leq x_2 \leq \tfrac{2v}{n} - X_A\}$$

$$\{(x_1, x_2)|\ x_2 = X_A \text{ and } 0 \leq x_1 \leq \tfrac{2v}{n} - X_A\}.$$

and uniformly places the remaining mass, $(nX_A/v) - 1$, on the line segment

$$\{(x_1, x_2)|\ x_1 + x_2 = \tfrac{2v}{n}\}$$

This support is shown in Panel (ii) of Figure 2.

The bivariate distribution function for x_1, x_2 is given by

$$G_B(x_1, x_2) = \begin{cases} \left(\frac{n}{2v}\right) \max\left\{x_1 + x_2 - \frac{2}{vn}, 0\right\} & \text{if } \ (x_1, x_2) \in [0, X_A)^2 \\ \frac{nx_1}{2v} & \text{if } \ x_2 = X_A,\ x_1 \in [0, X_A) \\ \frac{nx_2}{2v} & \text{if } \ x_1 = X_A,\ x_2 \in [0, X_A) \\ 1 & \text{if } \ x_1, x_2 \geq X_A \end{cases}$$

Following from the arguments given above for player A, it follows that G_B provides the necessary univariate marginal distributions for all-pay auctions 1 and 2.

If $X_A < (v/n)$ and $X_B \geq nX_A$, then player B allocates X_A to $n-1$ of the all-pay auctions and provides a random level of resources in the one remaining all-pay auction. In this one remaining all-pay auction player B has a mass point of $(1-(nX_A/v))$ at X_A and randomizes uniformly over the interval $[0, X_A)$ with the remaining mass.

This completes the proof of the existence of sufficient n-variate distributions for the Theorem 3 parameter range.

In the remaining region in which $\max\{(X_B - \frac{2v}{n})/(n-2), X_B/n\} < X_A < X_B/(n-1)$, as in the corresponding constant-sum parameter range, both players have atoms in the interior of the domains of their univariate marginal distribution functions. It should be noted that in this region the results are sensitive to the specification of the tie-breaking rule.

Let Δ denote the amount of resources available to player B if player B has bid X_A in $n-1$ of the auctions:

$$\Delta = X_B - (n-1)X_A.$$

Recalling that the floor function $\lfloor x \rfloor$ denotes the largest integer less than or equal to x, define k as

$$k = \left\lfloor \frac{X_A}{X_B - (n-1)X_A} \right\rfloor = \left\lfloor \frac{X_A}{\Delta} \right\rfloor.$$

In this region of the parameter space, $(n-1)X_A < X_B < nX_A$ and so $k \geq 1$. It will also be helpful to note that $X_A/(k+1) < \Delta \leq X_A/k$.

In this region of the parameter space the sets of equilibrium univariate marginal distributions are not unique.

Theorem 5. *Let X_A, X_B, v, and $n \geq 3$ satisfy $\max\{(X_B - \frac{2v}{n})/(n-2), X_B/n\} < X_A < X_B/(n-1)$. The pair of n-variate distribution functions P_A^* and P_B^* constitute a Nash equilibrium of the game $NCB\{X_A, X_B, n, v\}$ if they satisfy the following two conditions: (1) For each player i, $\mathrm{Supp}(P_i^*) \subset \mathfrak{B}_i$ and (2) P_i^*, $i = A, B$, provides the corresponding set of univariate marginal distribution functions $\{F_{i,j}^*\}_{j=1}^n$ outlined below, $\forall\ j \in \{1, \ldots, n\}$*

$$F_{B,j}^*(x) = \begin{cases} \frac{x}{v} & \text{if } x \in \left[0, \frac{X_A}{k+1}\right) \\ \frac{\left(\frac{2}{n} - \frac{\Delta + X_A}{v}\right)}{k+1} + \frac{x}{v} & \text{if } x \in \left[\frac{X_A}{k+1}, \frac{2X_A}{k+1}\right) \\ \vdots & \vdots \\ \frac{i\left(\frac{2}{n} - \frac{\Delta + X_A}{v}\right)}{k+1} + \frac{x}{v} & \text{if } x \in \left[\frac{iX_A}{k+1}, \frac{(i+1)X_A}{k+1}\right) \\ \vdots & \vdots \\ \frac{k\left(\frac{2}{n} - \frac{\Delta + X_A}{v}\right)}{k+1} + \frac{x}{v} & \text{if } x \in \left[\frac{kX_A}{k+1}, X_A\right) \\ 1 & \text{if } x \geq X_A \end{cases}.$$

If $k \geq 2$, then $\forall\ j \in \{1,\ldots,n\}$

$$F^*_{A,j}(x) = \begin{cases} 1 - \frac{2}{n} + \frac{\left(\frac{2}{n} - \frac{X_A}{v}\right)}{k+1} + \frac{x}{v} & \text{if } \ x \in [0,\Delta) \\ 1 - \frac{2}{n} + \frac{2\left(\frac{2}{n} - \frac{X_A}{v}\right)}{k+1} + \frac{x}{v} & \text{if } \ x \in \left[\Delta, \Delta + \frac{X_A - \Delta}{k-1}\right) \\ \vdots & \vdots \\ 1 - \frac{2}{n} + \frac{(i+1)\left(\frac{2}{n} - \frac{X_A}{v}\right)}{k+1} + \frac{x}{v} & \text{if } \ x \in \left[\Delta + \left(i-1\right)\left(\frac{X_A-\Delta}{k-1}\right), \Delta + i\left(\frac{X_A-\Delta}{k-1}\right)\right) \\ \vdots & \vdots \\ 1 - \frac{2}{n} + \frac{k\left(\frac{2}{n} - \frac{X_A}{v}\right)}{k+1} + \frac{x}{v} & \text{if } \ x \in \left[\Delta + \left(k-2\right)\left(\frac{X_A-\Delta}{k-1}\right), X_A\right) \\ 1 & \text{if } \ x > X_A \end{cases}.$$

If $k = 1$, then $\forall\ j \in \{1,\ldots,n\}$

$$F^*_{A,j}(x) = \begin{cases} 1 - \frac{2}{n} + \frac{\left(\frac{2}{n} - \frac{X_A}{v}\right)}{2} + \frac{x}{v} & \text{if } \ x \in [0,\Delta) \\ 1 - \frac{2}{n} + \frac{1.5\left(\frac{2}{n} - \frac{X_A}{v}\right)}{2} + \frac{x}{v} & \text{if } \ x \in [\Delta, X_A) \\ 1 & \text{if } \ x > X_A \end{cases}.$$

In equilibria satisfying these conditions on P^*_A and P^*_B, the expected payoff for player A is $[(2vk/n) - k(\Delta + X_A)]/(k+1)$, the expected payoff for player B is $(n-1)(v - X_A) + v[1 - (2/n) + [(2/n) - (X_A/v)]/(k+1)]$, the total expected expenditure for player A is $X_A - (1 - nX_A/2v)(X_A - \Delta)/(k+1)$, and the total expected expenditure for player B is $nX_A(nX_B - (n-1)^2 X_A)/2v + (1 - n(\Delta + X_A)/2v)(k+2)X_A/(k+1)$.

We begin with a sketch of the proof that a pair of n-variate distribution functions which satisfy the conditions of Theorem 5 form an equilibrium, and then move on to the proof of existence of such a pair of n-variate distribution functions. We will focus primarily on the case that $k \geq 2$ and conclude with the case that $k = 1$.

Let P^*_B denote a feasible n-variate distribution function for player B with the univariate marginal distribution functions $\{F^*_{B,j}\}$ given in Theorem 5. We begin with case in which there are no ties, and then address the case of ties. If player B is using P^*_B, then player A's expected payoff π_A, when player A chooses any n-tuple of bids $\mathbf{x}_A \in \mathfrak{B}_A \bigcap [0, X_A]^n$ such that for all $j = 1, \ldots, n$ and $i = 1, \ldots, k+1$, $x_{A,j} \neq iX_A/(k+1)$, is

$$\pi_A\left(\mathbf{x}_A, P_B^*\right) = \sum_{j=1}^{n} \left[vF_{B,j}^*\left(x_{A,j}\right) - x_{A,j}\right]. \tag{17}$$

To simplify the following discussion, for each $j = 1, \ldots, n$ let player B's univariate marginal distributions be written as

$$F_{B,j}^*\left(x_{A,j}\right) = \gamma_B(x_{A,j}) + \frac{x_{A,j}}{v}, \tag{18}$$

where, because we are focusing on the case of no ties, the term $\gamma_B(x_{A,j})$ is the sum of the mass on all atoms that lie strictly below $x_{A,j}$ and is given by the expression for $F_{B,j}^*$ in the statement of Theorem 5. Note that for each of player B's univariate marginal distribution functions each atom that lies strictly in the interior of the domain has the same mass, $[(2/n) - ((\Delta + X_A)/v)]/(k+1)$. Thus, the term $\gamma_B(x_{A,j})$ is equal to the number of atoms that lie below $x_{A,j}$ multiplied by the mass on each atom. Inserting equation (18) into equation (17) and simplifying, player A's expected payoff is given by

$$\pi_A\left(\mathbf{x}_A, P_B^*\right) = v\sum_{j=1}^{n} \gamma_B(x_{A,j}), \tag{19}$$

which is equal to the value of the prize multiplied by both the number of player B's atoms that player A outbids and by the mass on each atom, $[(2/n) - ((\Delta + X_A)/v)]/(k+1)$.

Next note that in Theorem 5's set of univariate marginal distribution functions for player B, $\{F_{B,j}^*\}_{j=1}^n$, the step size between each atom is $X_A/(k+1)$, and the first atom occurs at $X_A/(k+1)$. There are a total of $k+1$ atoms in each of player B's univariate marginal distributions. Recall that the rule for breaking ties at a common bid of X_A in an auction is that player B wins the auction. In the event that player A bids X_A in auction j, then — because the $(k+1)$th atom is at X_A — player A outbids exactly k of player B's atoms. Suppose player A outbids $\theta \leq k$ of player B's atoms in auction j at a cost of at least $\theta X_A/(k+1)$. The maximal number of player B's atoms that player A can feasibly outbid with his remaining budget [i.e., $\sum_{j' \neq j} x_{A,j} < X_A(1 - (\theta/(k+1)))$] is $k - \theta$ of player B's atoms, for a total of k atoms across all auctions.

Because player B is following the equilibrium strategy, the maximum expected payoff to player A for an any n-tuple of bids $\mathbf{x}_A \in \mathfrak{B}_A \bigcap [0, X_A]^n$ such that for all $j = 1, \ldots, n$ and

$i = 1, \ldots, k+1$, $x_{A,j} \neq iX_A/(k+1)$ is

$$\pi_A\left(\mathbf{x}_A, P_B^*\right) = v\sum_{j=1}^{n}\gamma_B(x_{A,j}) \leq \frac{vk\left(\frac{2}{n} - \frac{\Delta + X_A}{v}\right)}{k+1}. \tag{20}$$

Recall that if a tie occurs and the common bid is neither X_A nor $X_B - (n-1)X_A$ then each player wins the auction with equal probability. It follows that when the sum of player A's bids is X_A and exactly two ties occur [such as $x_{A,j'} = X_A/(k+1)$ and $x_{A,j''} = (k-1)X_A/(k+1)$], player A's expected payoff is equal to his equilibrium expected payoff given in equation (20). However, if more than two ties occur [such as $x_{A,j'} = X_A/(k+1)$ and $x_{A,j''} = X_A/(k+1)$ and $x_{A,j'''} = (k-2)X_A/(k+1)$], then player A's expected payoff is strictly less than his equilibrium expected payoff.

Using a similar argument for player B, it can be shown that the maximal number of player A's atoms that player B can outbid is $(n-1)(k+1)+1$. One difference in this case is that player A's atom at zero has more mass than the mass on each of the other atoms, but each other atom has the same mass. Observe that in the Theorem 5 equilibrium univariate marginal distributions for player B, player B's bid is almost surely strictly positive. Therefore, player B outbids player A's atom at zero in each of the auctions.

As before, the case in which there are no ties, and then address the case of ties. If player A is using P_A^* and player B chooses any n-tuple of bids $\mathbf{x}_B \in \mathfrak{B}_B \bigcap (0, X_A]^n$ such that for each auction j and $i = 1, \ldots, k-1$, $x_{B,j} \neq \Delta + i[(X_A - \Delta)/(k-1)]$, then player B's expected payoff may be written as

$$\pi_B\left(\mathbf{x}_B, P_A^*\right) = v\sum_{j=1}^{n}\gamma_A\left(x_{B,j}\right), \tag{21}$$

where $\gamma_A(x_{B,j})$ is the sum of the mass on all atoms that lie strictly below $x_{B,j}$.

Recall that $\Delta = X_B - (n-1)X_A$, and that $X_A/(k+1) < \Delta \leq X_A/k$. If player B bids X_A in $(n-1)$ of the auctions and in the remaining auction j bids $x_{B,j} \in (0, \Delta]$, then, because player A has $k+1$ atoms in each univariate marginal, player B outbids $(n-1)(k+1)+1$ of A's atoms and the expected payoff for player B is[11]

$$\pi_B\left(\mathbf{x}_B, P_A^*\right) = \Big(n-1\Big)\Big(v - X_A\Big) + v\Big(1 - \frac{2}{n} + \frac{\frac{2}{n} - \frac{X_A}{v}}{k+1}\Big). \tag{22}$$

[11]Observe that when player B bids Δ in auction j, the tie-breaking rule applies and player A wins the auction. Therefore, equation (22) provides player B's expected payoff at this point and there is no jump in the expected payoff.

If player B chooses any n-tuple of bids $\mathbf{x}_B \in \mathfrak{B}_B \bigcap (0, X_A]^n$ such that a bid of X_A is made in all but two auctions, denoted j' and j'', then player B's budget constraint implies that $x_{B,j'} + x_{B,j''} \leq \Delta + X_A$. In this case, player B's expected payoff is

$$\pi_B\left(\mathbf{x}_B, P_A^*\right) = \Big(n-2\Big)\Big(v - X_A\Big) + v\gamma_A(x_{B,j'}) + v\gamma_A(x_{B,j''}), \tag{23}$$

and for any feasible pair of bids $x_{B,j'}$ and $x_{B,j''}$ in $(0, X_A]^2$ such that $x_{B,j'} + x_{B,j''} = \Delta + X_A$ player B outbids $k+2$ of A's atoms in auctions j' and j'', which results in the expected payoff in equation (22). Lastly, it is important to note that, because player A has an atom at X_A in each of his univariate marginal distributions, player B does not have incentive to lower the bids in the $(n-2)$ auctions which receive a bid of X_A. This follows from two facts. First, for each of player A's univariate marginal distribution functions each atom that lies strictly in the interior of the domain has the same mass, $[(2/n) - (X_A/v)]/(k+1)$. Second, whereas player A has atoms at 0 and at Δ, the step size between the remaining atoms is $(X_A - \Delta)/(k-1) > \Delta$. In regard to ties, the comments given in the case for player A apply directly, with the caveat of ties at X_A and Δ.

For the proof of existence of a pair of n-variate distribution functions which satisfy the conditions of Theorem 5, consider the following constructions which are shown in Figure 3. The equilibrium construction is briefly described as follows. The support of each player's n-variate joint distribution function consists of an absolutely continuous distribution over a set of line segments in $\mathbb{R}_n^+$ combined with a set of atoms on n-tuples. Mass is distributed among the atoms and line segments in such a way that the opponent is indifferent among all feasible pure strategies and the mass sums to one. To avoid confusion between the atoms in the constructions outlined below and the atoms in the resulting univariate marginal distributions, 2-tuples which receive positive mass will be referred to as bivariate atoms. Similarly, in the resulting univariate marginal distribution functions, we will refer to a point with positive mass as a univariate atom.

[Insert Figure 3 here]

Player A randomly allocates 0 resources to $n-2$ of the all-pay auctions, each all-pay auction chosen with equal probability, $(n-2)/n$. On the remaining 2 all-pay auctions player A utilizes a bivariate distribution function with $k+1$ bivariate atoms,[12] each bivariate atom

[12]Observe that at each of the "bivariate atoms" described here player A allocates 0 resources to the other $n-2$ auctions. Thus, each of these bivariate atoms is actually an atom on an n-tuple.

receiving the same weight, $(1-(nX_A)/(2v))/(k+1)$. Player A's bivariate atoms on these two remaining all-pay auctions are located at the points $(0, X_A)$, $(X_A, 0)$, and

$$\left(\Delta + \left(k-1-i\right)\left(\frac{X_A-\Delta}{k-1}\right), \Delta + \left(i-1\right)\left(\frac{X_A-\Delta}{k-1}\right)\right),\ i = 1, \ldots, k-1. \tag{24}$$

Player A uniformly distributes the remaining $(nX_A)/(2v)$ of the mass along the line segment $\{(x_1, x_2) \in \mathbb{R}^2_+ |\ x_1+x_2 = X_A\}$. To see that this construction provides the necessary univariate marginal distributions, observe that in the randomization outlined above player A allocates zero resources to each all-pay auction j with probability $(n-2)/n+(2/n)[1-(nX_A/2v)]/(k+1) = 1-(2/n)+[2/(n(k+1))]-[X_A/(v(k+1))]$, randomizes uniformly over the interval $(0, X_A]$ with probability $(2/n)(nX_A)/(2v) = X_A/v$, and has the specified univariate atoms with the remaining probability.

Player B randomly allocates X_A forces to $n-2$ all-pay auctions, each all-pay auction chosen with equal probability, $(n-2)/n$. On the remaining 2 all-pay auctions player B utilizes a bivariate distribution function with $k+1$ bivariate atoms, each bivariate atom receiving the same weight, $[1-(n/2v)(\Delta+X_A)]/(k+1)$. Player B's bivariate atoms on the 2 remaining all-pay auctions are located at

$$\left(\frac{(k+1-i)X_A}{(k+1)}, \frac{(1+i)X_A}{(k+1)}\right),\ i = 0, \ldots, k. \tag{25}$$

Player B uniformly distributes the remaining $n(\Delta + X_A)/(2v)$ of the mass along the three line segments $\{(x_1, x_2) \in \mathbb{R}^2_+ |\ x_1 + x_2 = X_B - (n-2)X_A\}$, $\{(x_1, x_2) \in \mathbb{R}^2_+ |\ x_1 = X_A \text{ and } 0 \leq x_2 \leq \Delta\}$, and $\{(x_1, x_2) \in \mathbb{R}^2_+ |\ x_2 = X_A \text{ and } 0 \leq x_1 \leq \Delta\}$. To see that this construction provides the necessary univariate marginal distributions, observe that in the randomization outlined above player B allocates X_A resources to each all-pay auction j with probability $((n-2)/n)+(\Delta/v)+[(2/n)-((\Delta+X_A)/v))]/(k+1)$, randomizes uniformly over the interval $[0, X_A)$ with probability $(2/n)(nX_A)/(2v) = X_A/v$, and has the specified univariate atoms with the remaining probability.

It is important to note that in the construction of the bivariate distributions given above none of player B's bivariate atoms exhaust his budget, and only two of player A's bivariate atoms exhaust his budget. However, as shown below, each of the bivariate atoms yields the equilibrium expected payoff for the corresponding player.

Recall that each player i's expected payoff [see equations (19) and (21)] is proportional to the number of player $-i$'s univariate atoms [which lie strictly in the interior of the domains of player $-i$'s univariate marginal distributions] which are outbid. As can be seen in Figure 3, for each player i each of his bivariate atoms outbids the same number of player $-i$'s univariate atoms as in his equilibrium expected payoff [k atoms for player A and $(n-2)(k+1)+k+2$ for player B].[13] It is straightforward, albeit tedious, to show this algebraically. The key step in this is given by the following inequality

$$\frac{(k-i)X_A}{k+1} < \Delta + \Big(k-1-i\Big)\Big(\frac{X_A-\Delta}{k-1}\Big) < \frac{(k+1-i)X_A}{k+1}, \tag{26}$$

which holds for all $i = 1, \ldots, k-1$. This inequality follows directly from the relationship between Δ, k, and X_A. In particular, $X_A/(k+1) < \Delta \leq X_A/k$.

The inequality in equation (26) shows that when player A bids $\Delta + (k-1-i)((X_A - \Delta)/(k-1))$ in an auction she outbids $k-i$ of player B's univariate atoms in that auction. Conversely, as equation (26) holds for all $i = 1, \ldots, k-1$, it also shows that when player B bids $(k+1-i)X_A/(k+1)$ in an auction she outbids $k+1-i$ of player A's univariate atoms in that auction. From the locations of each player's bivariate atoms given in equations (24) and (25), it follows that for each player i each of his bivariate atoms outbids the same number of player $-i$'s univariate atoms as in his equilibrium expected payoff [k atoms for player A and $(n-2)(k+1)+k+2$ for player B]. This completes the proof of Theorem 5 for $k \geq 2$.

We now address the case of $k = 1$. Just as with $k \geq 2$, the sets of equilibrium univariate marginal distributions are not unique, but, as shown in Lemma 3 of the Appendix, the equilibrium expected payoffs are unique. For $k = 1$ the construction specified above for player A fails to be feasible given his budget constraint. In this case, player A's univariate marginals are modified, but for player B the construction specified above, but with $k = 1$, still applies.

The sketch of the proof that a pair of n-variate distribution functions, which satisfy the conditions of Theorem 5 with $k = 1$, form an equilibrium follows along the same lines as for $k \geq 2$. For the proof of existence of such an n-variate distribution function for player A, consider the following construction.

Player A randomly allocates 0 resources to $n-2$ of the all-pay auctions, each all-pay auction chosen with equal probability, $(n-2)/n$. On the remaining 2 all-pay auc-

[13]Consider for example, player A's bivariate atom at the point $(0, X_A)$. This is the left most atom in panel (i) of Figure 3. When player A chooses this bivariate atom, he outbids (due to the tie-breaking rule) all but one ($k = 3$) of player B's (univariate) atoms in the x_2 direction and none of player B's univariate atoms in the x_1 direction. Similarly, at each of his bivariate atoms player A outbids a total of k of player A's univariate atoms. The case for player B follows directly.

tions player A utilizes a bivariate distribution function with 4 bivariate atoms, each bivariate atom receiving the same weight, $(1 - (nX_A)/(2v))/4$. Player A's bivariate atoms on these two remaining all-pay auctions are located at the points $(0, X_A)$, $(X_A, 0)$, $(0, \Delta)$, and $(\Delta, 0)$. Player A uniformly distributes the remaining $(nX_A)/(2v)$ of the mass along the line segment $\{(x_1, x_2) \in \mathbb{R}^2_+ | \ x_1 + x_2 = X_A\}$. To see that this construction provides the necessary univariate marginal distributions, observe that in the randomization outlined above player A allocates zero resources to each all-pay auction j with probability $(n-2)/n + (2/n)[1 - (nX_A/2v)]/2 = 1 - (2/n) + (1/n) - [X_A/(2v)]$, randomizes uniformly over the interval $(0, X_A]$ with probability $(2/n)(nX_A)/(2v) = X_A/v$, and has the specified univariate atoms with the remaining probability.

As before, two of player A's atoms do not exhaust player A's budget. However, each of these bivariate atoms clearly outbids one of player B's univariate atoms and results in the unique equilibrium expected payoff for player A.

Two Auctions

Before outlining the case of two auctions, it is important to note that for $n = 2$ the sets of equilibrium univariate marginal distributions are non-unique for all parameter regions.[14] However, as is shown in the Appendix, in the Theorem 1 and 2 parameter ranges with $X_A \neq (2v/n)$ the equilibrium payoffs and total expenditures are unique.

Recall that in both panels of Figure 1, the parameter space is partitioned by the four rays: (a) $X_A = X_B/n$, (b) $X_A = X_B/(n-1)$, (c) $X_A = 2X_B/n$, and (d) $X_A = X_B$. In the case that $n = 2$, the last three of these collapse to the single ray $X_A = X_B$, and the first of these becomes $X_A = X_B/2$. The following partition of the parameter space, for $n = 2$, provides the portions of the parameter space in which the theorems in the preceding subsection provide sufficient, but not necessary, conditions for equilibrium.

T1*: $\left\{(X_A, X_B) \in \mathbb{R}^2_+ \middle| v < X_A \leq X_B\right\}$

T2*: $\left\{(X_A, X_B) \in \mathbb{R}^2_+ \middle| X_B = X_A \leq v \ \text{ or } \ X_A = v \ \text{ and } \ X_B > v\right\}$

T3a*: $\left\{(X_A, X_B) \in \mathbb{R}^2_+ \middle| X_B \geq v \ \text{ and } \ \frac{X_B}{2} < X_A < v\right\}$

T3b*: $\left\{(X_A, X_B) \in \mathbb{R}^2_+ \middle| X_A < v \ \text{ and } \ X_A \leq \frac{X_B}{2}\right\}$

[14]With $n = 2$ each player's pair of univariate marginals need not be independent of the identity of the auction. For example, the location of and/or the mass placed on atoms need not be symmetric across auctions. For further information see Macdonell and Mastronardi (2010).

T5*: $\left\{(X_A, X_B) \in \mathbb{R}^2_+ \middle| X_B < v \text{ and } \frac{X_B}{2} < X_A < v\right\}$

These regions and the resulting modified budgets are illustrated in Figure 4 below.

[Insert Figure 4 here]

Recall that in the constructions provided for the Theorem 3 and 5 parameter regions, each player allocated a specified bid to $(n-2)$ of the all-pay auctions [for player A this was a bid of 0, and for player B this was a bid of X_A]. When $n = 2$, $(n-2)/n = 0$ and the constructions for both of those regions simply become the bivariate distributions that were specified for the remaining two auctions. It is straightforward to show that in the Theorem 1 and 2 regions the Fréchet-Hoeffding lower bound 2-copula combined with the univariate marginals specified in Theorems 1 and 2, which for player $i = A, B$ are given by

$$P_i^*(b_{i,1}, b_{i,2}) = \max\left\{F_{i,1}^*(b_{i,1}) + F_{i,2}^*(b_{i,2}) - 1, 0\right\},$$

results in a pair of bivariate distribution functions for which $\text{Supp}(P_i^*) \subset \mathfrak{B}_i$ and that provide an equilibrium pair of univariate marginal distribution functions.

5 Conclusion

Kvasov (2007) introduces a non-constant-sum version of the Colonel Blotto game which relaxes the "use it or lose it" feature of the traditional constant-sum formulation of the game. In the case of symmetric budgets, that article establishes that there exists a one-to-one mapping from the set of unique univariate marginal distribution functions in the constant-sum game to those in the non-constant-sum game. As the analysis of the non-constant-sum version of the Colonel Blotto game is extended to allow for asymmetric budget constraints, we find that — as long as the level of asymmetry between the players' budgets is below a threshold — there still exists a one-to-one mapping from the unique set of equilibrium univariate marginal distribution functions in the constant-sum game to those in the non-constant-sum game. The classic Colonel Blotto game provides an important benchmark in the study of the logic of strategic multi-dimensional conflict, and, as our results show, the nature of the incentives in such conflicts remain largely unchanged when the use it or lose it feature of the constant-sum game is relaxed.

Appendix

For the Theorem 1 parameter range with $n \geq 3$ (denoted as T1), this Appendix characterizes each player's unique: set of equilibrium univariate marginal distribution functions, equilibrium payoffs, and equilibrium total expected expenditures. We also show that the uniqueness of the equilibrium payoffs and equilibrium total expected expenditures extends to the case of $n = 2$. The proof for the Theorem 2 parameter range with $X_A \neq (2v/n)$, follows along similar lines, and we conclude with a sketch of that proof.

For $(X_A, X_B) \in T1$, the proof of the uniqueness of the set of univariate marginal distributions involves formally showing that, as the Euler-Lagrange equations given in equation (5) of Section 3 suggest, there exists a one-to-one correspondence between the equilibrium univariate marginal distributions in the Non-Constant-Sum Colonel Blotto game and the equilibrium distributions of bids from a unique set of two-bidder independent and identical simultaneous all-pay auctions. The uniqueness of the equilibrium univariate marginal distributions follows from the characterization of the all-pay auction by Hillman and Riley (1989) and Baye, Kovenock and de Vries (1996).

In the case of the standard constant-sum formulation of the Colonel Blotto game, the proof of the uniqueness of the equilibrium marginal distributions (Roberson 2006) utilizes the fact that in a two-player constant-sum game with multiple equilibria all equilibria are interchangeable. In Lemmas 1-3 we show that for the Theorem 1 parameter range this interchangeability of equilibria property also applies to the Non-Constant-Sum Colonel Blotto game. Given this result on the interchangeability of equilibria, the rest of the proof follows along lines similar to Roberson (2006).

In the discussion that follows we will utilize the following notational conventions. Given an n-variate distribution function P_i with $\text{Supp}(P_i) \subset \mathcal{B}_i$ and the set of univariate marginal distribution functions $\{F_{i,j}\}_{j=1}^{n}$, let M_{X_i} denote the total expected expenditure across the entire set of auctions, that is $M_{X_i} \equiv \sum_{j=1}^{n} E_{F_{i,j}}(x_{i,j})$. Also, let $\bar{s}_{i,j}$ and $\underline{s}_{i,j}$ denote the upper and lower bounds of player i's distribution of resources for all-pay auction j.

We begin the proof of the interchangeability of equilibria in the Non-Constant-Sum Colonel Blotto game by showing that if the pair of the players' resources $(X_A, X_B) \in T1$ [i.e., $(2/n)\min\{v, X_B\} < X_A \leq X_B$], then in any equilibrium the pair of total expected expenditures (M_{X_A}, M_{X_B}) are uniquely determined by (X_A, X_B) and equal to those give in Theorem 1. The proof of this result in done two steps. First, Lemma 1 shows that if $(X_A, X_B) \in T1$, then in any equilibrium $\{P_A, P_B\}$ the pair of equilibrium total expected expenditures (M_{X_A}, M_{X_B}) must lie in the set of equilibrium total expected expenditures for

Theorem 1 as illustrated by the shaded region 1c in panel (ii) of Figure 1 and delineated by the conditions: (i) $(2/n)M_{X_B} \leq M_{X_A} \leq M_{X_B}$, (ii) $M_{X_i} \leq (nv/2)$ for $i = A, B$, and (iii) if $M_{X_A} > (2v/n)$ then $M_{X_B} \leq (nvM_{X_A}/2)^{1/2}$. Then, Lemma 2 shows that in the Theorem 1 parameter region the equilibrium total expected expenditures are uniquely determined by the pair of the players' resources (X_A, X_B).

Lemma 1. *If $(X_A, X_B) \in \mathrm{T1}$, then in any equilibrium $\{P_A, P_B\}$ the pair of total expected expenditures (M_{X_A}, M_{X_B}) are contained in the region delineated by: (i) $(2/n)M_{X_B} \leq M_{X_A} \leq M_{X_B}$, (ii) $M_{X_i} \leq (nv/2)$ for $i = A, B$, and (iii) if $M_{X_A} > (2v/n)$ then $M_{X_B} \leq (nvM_{X_A}/2)^{1/2}$.*

Proof. First, note that the total value at stake in the auctions is nv. Let α_i denote the fraction of the total value of the auctions that player i expects to win in this equilibrium,

$$\alpha_i = \frac{1}{n} E_{P_i}\left[\sum_{j=1}^{n} F_{-i,j}(x_{i,j})\right] = 1 - \frac{1}{n} E_{P_{-i}}\left[\sum_{j=1}^{n} F_{i,j}(x_{-i,j})\right] \tag{27}$$

where the first [second] expectation is taken with respect to player i's joint distribution P_i [player $-i$'s joint distribution P_{-i}] and the second equality follows from $\alpha_A + \alpha_B = 1$. It it instructive to note that the α_i term is precisely player i's expected payoff in the corresponding constant-sum Colonel Blotto game with budget constraints given by the expected expenditures (M_{X_A}, M_{X_B}). Player i's expected payoff may be written as:

$$\pi_i(P_i, P_{-i}) = nv\alpha_i - M_{X_i}. \tag{28}$$

First, we show that there exist no equilibria in which $M_{X_A} + M_{X_B} > nv$. This proof is by contradiction. Suppose that $(X_A, X_B) \in T1$, and that there exists an equilibrium $\{P_A, P_B\}$ in which $M_{X_A} + M_{X_B} > nv$. From equation (28), it follows that the sum of the players' expected payoffs is

$$\pi_A(P_A, P_B) + \pi_B(P_B, P_A) = nv - M_{X_A} - M_{X_B}. \tag{29}$$

Because in any equilibrium each player must have a nonnegative expected payoff, it follows that the sum of the players' expected payoffs must also be nonnegative. Thus, from equation (29) there exist no equilibria in which $M_{X_A} + M_{X_B} > nv$, a contradiction to the assumption that there exists such an equilibrium..

Focusing now on equilibria in which $M_{X_A} + M_{X_B} \leq nv$, for the T1 region there are two remaining cases to consider:[15] (i) $M_{X_A} > M_{X_B}$, and (ii) $M_{X_A} \leq M_{X_B}$, $M_{X_A} > (2v/n)$, and $M_{X_B} > (nvM_{X_A}/2)^{1/2}$.

We begin with case (i). By way of contradiction, suppose that there exists an equilibrium $\{P_A, P_B\}$ in which $M_{X_A} > M_{X_B}$. Because $X_B \geq X_A$, player B can always duplicate player A's strategy and earn an expected payoff of at least $(nv/2) - M_{X_A}$. That is

$$\pi_B(P_B, P_A) \geq \frac{nv}{2} - M_{X_A} \tag{30}$$

From equations (28) and (30) it follows that $\alpha_B \geq (1/2) - (M_{X_A} - M_{X_B})/nv$. Because $\alpha_A + \alpha_B = 1$ it follows that

$$\pi_A(P_A, P_B) \leq \frac{nv}{2} - M_{X_B} \tag{31}$$

We will now use the upper bound on player A's expected payoff from the strategy profile $\{P_A, P_B\}$, given in equation (31), to show that there exists a profitable deviation, $\widetilde{P}_A$, for player A. From Roberson (2006) [see the comments following Theorem 1 in this article] we know that there exists a joint distribution function $\widetilde{P}_A$ which satisfies the three following properties: $\mathrm{Supp}(\widetilde{P}_A) \subset \mathfrak{B}_A$, the total equilibrium expected expenditures are given by $\widetilde{M}_{X_A} = \min\{X_A, (nvM_{X_B}/2)^{1/2}\}$, and the set of univariate marginal distributions are given by

$$\forall\ j \in \{1, \ldots, n\} \qquad \widetilde{F}^*_{A,j}\left(x\right) = \tfrac{x}{(2/n)\widetilde{M}_{X_A}} \qquad \text{for } \ x \in \left[0, \tfrac{2}{n}\widetilde{M}_{X_A}\right].$$

Player A's expected payoff from the feasible deviation $\widetilde{P}_A$ is

$$\pi_A(\widetilde{P}_A, P_B) = nv\left(1 - E_{P_B}\left(\sum_{j=1}^{n} \widetilde{F}^*_{A,j}\left(x_{B,j}\right)\right)\right) - \widetilde{M}_{X_A} \geq nv\left(1 - \frac{M_{X_B}}{2\widetilde{M}_{X_A}}\right) - \widetilde{M}_{X_A}. \tag{32}$$

If $\mathrm{Supp}(P_B) \subset [0, \frac{2}{n}\widetilde{M}_{X_A}]^n$, then equation (32) holds with equality.

Recall that in the equilibrium $\{P_A, P_B\}$ equation (31) provides an upper bound on player A's expected payoff. However, $\widetilde{P}_A$ is a feasible payoff increasing deviation from P_A. That is, because $M_{X_A} + M_{X_B} \leq nv$ and by assumption $M_{X_A} > M_{X_B}$, it follows that $M_{X_B} < (nv/2)$. Thus, $M_{X_B} < \widetilde{M}_{X_A} < (nv/2)$, and it follows from equations (31) and (32) that $\pi_A(\widetilde{P}_A, P_B) > \pi_A(P_A, P_B)$. A contradiction to the assumption that there exists an equilibrium $\{P_A, P_B\}$ in which $M_{X_A} > M_{X_B}$.

The proof of case (ii) follows along a similar line as the proof for case (i). By way of

[15] Observe that cases (i) and (ii) together with $M_{X_A} + M_{X_B} > nv$ correspond to the non-shaded portions of the T1 region in panel (ii) of Figure 1.

contradiction, suppose that there exists an equilibrium $\{P_A, P_B\}$ in which $M_{X_A} \leq M_{X_B}$, $M_{X_A} > (2v/n)$, and $M_{X_B} > (nvM_{X_A}/2)^{1/2}$. Parallel to the lower bound of player B's expected payoff in case (i) given in equation (30), in case (ii) player A can establish a lower bound on his expected payoff. As with the upper bound of player A's expected payoff in case (i) given in equation (31), in case (ii) the upper bound on player B's expected payoff is given by $nv(1-\alpha_A) - M_{X_B}$. It can then be shown that there exists a profitable deviation for player B, a contradiction to the assumption that such an equilibrium exists. This completes the proof of Lemma 1. □

Lemma 2. *If $(X_A, X_B) \in \mathrm{T1}$, then in any equilibrium $\{P_A, P_B\}$ the pair of total expected expenditures (M_{X_A}, M_{X_B}) is equal to the pair of equilibrium total expected expenditures uniquely determined by (X_A, X_B) in Theorem 1. Furthermore, the equilibrium expected payoffs are also uniquely determined by (X_A, X_B).*

Proof. By way of contradiction suppose that for some $(X_A, X_B) \in \mathrm{T1}$ there exists an equilibrium $\{P_A, P_B\}$ with a pair of total expected expenditures (M_{X_A}, M_{X_B}) that satisfies Lemma 1 [i.e., (M_{X_A}, M_{X_B}) is contained in the set of equilibrium total expected expenditures for Theorem 1] but in which the pair (M_{X_A}, M_{X_B}) differs from the pair of total expected expenditures uniquely determined by (X_A, X_B) in Theorem 1.

The outline of the proof is as follows. First, we show how feasible and total-expected-expenditure invariant deviations from $\{P_A, P_B\}$ may be used to determine the payoffs in the original equilibrium $\{P_A, P_B\}$. Then we show that because the pair (M_{X_A}, M_{X_B}) differs from the pair of total expected expenditures uniquely determined by (X_A, X_B) in Theorem 1 at least one player i has a strictly payoff increasing deviation — in which player i's total-expected-expenditure differs from M_{X_i} — from the assumed equilibrium $\{P_A, P_B\}$.

Beginning with the first step, because (M_{X_A}, M_{X_B}) satisfies Lemma 1, we know from Roberson (2006) that there exists a joint distribution function P_A^* which satisfies the two following properties: $\mathrm{Supp}(P_A^*) \subset \overline{\mathfrak{B}}_A$ and the set of univariate marginal distributions are given by

$$\forall\, j \in \{1, \dots, n\} \qquad F_{A,j}^*(x) = \left(1 - \frac{M_{X_A}}{M_{X_B}}\right) + \frac{x}{(2/n)M_{X_B}} \left(\frac{M_{X_A}}{M_{X_B}}\right) \qquad \text{for } \; x \in \left[0, \tfrac{2}{n} M_{X_B}\right]$$

Observe that the feasible deviation P_A^* has a total expected expenditure of M_{X_A}. Such a feasible deviation ensures that

$$\alpha_A \geq \frac{M_{X_A}}{2M_{X_B}} \;\; \text{and} \;\; \alpha_B \leq 1 - \frac{M_{X_A}}{2M_{X_B}}. \tag{33}$$

Similarly, there exists a feasible deviation P_B^* with $\mathrm{Supp}(P_B^*) \subset \overline{\mathfrak{B}}_B$ and the set of univariate marginal distributions:

$$\forall\ j \in \{1, \ldots, n\} \qquad F_{B,j}^*(x) = \tfrac{x}{(2/n)M_{X_B}} \qquad \text{for}\ \ x \in \left[0, \tfrac{2}{n}M_{X_B}\right]$$

Note that P_B^* is a feasible deviation which is invariant with respect to the total expected expenditure M_{X_B}. Such a strategy ensures that

$$\alpha_A \leq \frac{M_{X_A}}{2M_{X_B}} \ \text{ and } \ \alpha_B \geq 1 - \frac{M_{X_A}}{2M_{X_B}}. \tag{34}$$

From equations (33) and (34), it follows that the original equilibrium strategy profile $\{P_A, P_B\}$ yields the respective total expected fractions of contests won

$$\alpha_A = \frac{M_{X_A}}{2M_{X_B}} \ \text{ and } \ \alpha_B = 1 - \frac{M_{X_A}}{2M_{X_B}}. \tag{35}$$

Inserting equation (35) back into equation (28), the players' expected payoffs from the original equilibrium strategy profile $\{P_A, P_B\}$ are

$$\pi_A(P_A, P_B) = \frac{nvM_{X_A}}{2M_{X_B}} - M_{X_A} \ \text{ and } \ \pi_B(P_B, P_A) = nv\big(1 - \frac{M_{X_A}}{2M_{X_B}}\big) - M_{X_B}. \tag{36}$$

We now show that because the pair of total expected expenditures (M_{X_A}, M_{X_B}) in the the original equilibrium strategy profile $\{P_A, P_B\}$ differ from the pair of total expected expenditures uniquely determined by (X_A, X_B) in Theorem 1 (denoted $M_{X_i}^*$ for $i = A, B$) at least one player has a strictly payoff increasing deviation from the assumed equilibrium $\{P_A, P_B\}$.

By assumption $\{P_A, P_B\}$ is an equilibrium, and thus neither player i can increase his expected payoff by deviating to a feasible strategy with a different total expected expenditure M_{X_i}. Recall that in Theorem 1 player A's equilibrium total expected expenditure is $M_{X_A}^* = \min\{X_A, (nv/2)\}$. By way of contradiction assume that $M_{X_A} \neq M_{X_A}^*$. If $M_{X_B} = (nv/2)$, then because (M_{X_A}, M_{X_B}) satisfies Lemma 1 it must be the case that $M_{X_A} = (nv/2)$. A contradiction to the assumption that $M_{X_A} \neq M_{X_A}^*$. We now examine the remaining case that $M_{X_B} < (nv/2)$. Because (M_{X_A}, M_{X_B}) satisfies Lemma 1 and $M_{X_A} \neq M_{X_A}^*$, it follows that either: (a) $X_A \geq (nv/2)$ and $M_{X_A} < (nv/2)$ or (b) $X_A < (nv/2)$ and $M_{X_A} < X_A$. Following along similar lines to the feasible deviations outlined above, from Roberson (2006) there exists a joint distribution function $\widetilde{P}_A$ which satisfies the property that $\mathrm{Supp}(\widetilde{P}_A) \subset \mathcal{B}_A$, has a total expected expenditure $\widetilde{M}_{X_A}$ such that $M_{X_A} < \widetilde{M}_{X_A} \leq M_{X_A}^*$, and ensures that

$\widetilde{\alpha}_A \geq (\widetilde{M}_{X_A}/2M_{X_B})$. Thus, it follows from equation (36) that in both cases player A has a strictly payoff increasing deviation. A contradiction to the assumption that $\{P_A, P_B\}$ is an equilibrium.

A similar argument shows that if $M_{X_B} \neq M^*_{X_B}$, then at least one player has a feasible strictly payoff increasing deviation. To summarize, we have shown that if (X_A, X_B) lies in the T1 parameter range and $\{P_A, P_B\}$ is an equilibrium with the pair of total expected expenditures (M_{X_A}, M_{X_B}), then $M_{X_A} = \min\{X_A, (nv/2)\}$ and $M_{X_B} = \min\{X_B, (nv/2), (nvX_A/2)^{1/2}\}$.

Given the uniqueness of the equilibrium total expected expenditures, the uniqueness of the equilibrium payoffs follows directly. □

Lemma 3. *If $(X_A, X_B) \in$ T1, then any equilibrium $\{P_A, P_B\}$ is interchangeable with any equilibrium $\{P^*_A, P^*_B\}$ which satisfies the conditions of Theorem 1.*

Proof. Suppose that — in addition to an equilibrium $\{P^*_A, P^*_B\}$ which satisfies the conditions in Theorem 1 — there exists an equilibrium $\{P_A, P_B\}$ that violates condition (2) of Theorem 1 [i.e., the condition on the sets of univariate marginal distributions]. From Lemma 2 all equilibria have the same expected expenditures (M_{X_A}, M_{X_B}) and the same expected payoffs. From equation (28) it follows that there is a unique equilibrium pair (α^*_A, α^*_B).

If $\{P_A, P_B\}$, with (M_{X_A}, M_{X_B}), is an equilibrium, then it must be the case that neither player has a feasible payoff increasing deviation. Without loss of generality, suppose that player A deviates to the strategy P^*_A which satisfies the conditions in Theorem 1. Because this is a feasible deviation which is invariant to the expected expenditure M_{X_A} and player A's expected payoff π_A does not increase, it follows that α_A does not increase. As $\alpha_A + \alpha_B = 1$, this implies directly that α_B does not decrease.

Conversely, because $\{P^*_A, P^*_B\}$ is an equilibrium neither player has a feasible payoff increasing deviation. Thus, if player B deviates from P^*_B to P_B, player B's expected payoff π_B does not increase. Then, because the deviation P_B is invariant to the expected expenditure M_{X_B}, it follows that α_B must not increase under this deviation. Because $\alpha_A + \alpha_B = 1$ and α_B does not increase, it must be the case that α_A does not decrease.

Because, when player B chooses P_B and player A choose P^*_A, both α_A and α_B neither increase nor decrease we can conclude that they stay at the unique values (α^*_A, α^*_B), and that the players' expected payoffs remain at the unique levels specified by Lemma 2. Furthermore, neither player has a feasible payoff increasing deviation. We have thus shown that if player B chooses P_B and player A choose P^*_A, then $\{P^*_A, P_B\}$ forms an equilibrium which satisfies Lemmas 1 and 2. By a symmetric argument it follows that $\{P_A, P^*_B\}$ also forms an equilibrium which satisfies Lemmas 1 and 2. Thus, any equilibrium $\{P_A, P_B\}$ is interchangeable with any equilibrium $\{P_A^*, P_B^*\}$ which satisfies the conditions in Theorem 1. $\square$

Because of Lemma 3's result on the interchangeability of equilibria, arguments along the lines of the proofs in Baye et al. (1996) can be used to establish the next three lemmas.

Lemma 4. *If* $(X_A, X_B) \in \text{T1}$*, then in any equilibrium* $\{P_A, P_B\}$*,* $\bar{s}_{i,j} = \bar{s} = (2/n)M_{X_B}$ *and* $\underline{s}_{i,j} = \underline{s} = 0$ *for each* $i \in \{A, B\}$ *and* $j \in \{1, \ldots, n\}$*.*

Lemma 5. *If* $(X_A, X_B) \in \text{T1}$*, then in any equilibrium* $\{P_A, P_B\}$ *no* $F_{i,j}$ *can place an atom in the half-open interval* $(0, (2/n)M_{X_B}]$

Lemma 6. *If* $(X_A, X_B) \in \text{T1}$*, then in any equilibrium* $\{P_A, P_B\}$ *there exists, for* $i = A, B$*, a* $\lambda_i \geq 0$ *such that* $\forall\ j = 1, \ldots, n$*,* $vF_{-i,j}(x) - (1 + \lambda_i)x$ *is constant* $\forall\ x \in (0, (2/n)M_{X_B}]$*.*

Note that the conditions stated in Lemma 6 are equivalent to the Euler-Lagrange equations given in equation (5) of Section 3. We now complete the proof of the uniqueness of the univariate marginals.

Lemma 7. *If* $(X_A, X_B) \in \text{T1}$*, then in any equilibrium* $\{P_A, P_B\}$*,* $\lambda_A = -1 + ((nv)/(2M_{X_B}))$ *and* $\lambda_B = -1 + ((nvM_{X_A})/(2M_{X_B}^2))$*. Therefore, each set of univariate marginal distribution functions* $\{F_{i,j}\}_{j=1}^n$*,* $i = A, B$ *satisfies the conditions in Theorem 1.*

Proof. By definition $M_{X_i} = \sum_{j=1}^n \int_0^{\bar{s}} x dF_{i,j}(x)$. From Lemma 4, $\bar{s} = (2/n)M_{X_B}$ and the lower bounds for each univariate marginal distribution is 0. From Lemma 6, $dF_{i,j}(x) = ((1 + \lambda_{-i})/v)dx$.

For player B,

$$M_{X_B} = \frac{(1 + \lambda_A)}{v} \sum_{j=1}^{n} \int_0^{(2/n)M_{X_B}} x dx. \tag{37}$$

Solving equation (37) for λ_A, uniquely yields $\lambda_A = -1+((nv)/(2M_{X_B}))$. From Lemmas 5 and 6, it follows that for each auction j, $F_{B,j}(x) = F_{B,j}(0) + x(n/2M_{X_B})$ for $x \in [0, (2/n)M_{X_B}]$. Then, because $F_{B,j}((2/n)M_{X_B}) = 1$ it follows that $F_{B,j}(0) = 0$.

For player A,

$$M_{X_A} = \frac{(1 + \lambda_B)}{v} \sum_{j=1}^{n} \int_0^{(2/n)M_{X_B}} x dx. \tag{38}$$

Solving equation (38) for λ_B, uniquely yields $\lambda_B = -1+((nvM_{X_A})/(2M_{X_B}^2))$. From Lemmas 5 and 6, it follows that for each auction j, $F_{A,j}(x) = F_{A,j}(0) + x((nM_{X_A})/(2M_{X_B}^2))$ for $x \in [0, (2/n)M_{X_B}]$. Then, because $F_{A,j}((2/n)M_{X_B}) = 1$ it follows that $F_{A,j}(0) = 1 - M_{X_A}/M_{X_B}$.

In the Theorem 1 range there were three cases: (a) neither player uses all of his resources, (b) only the weaker player (A) uses all of his resources, and (c) both players use all of their resources. In case (a), it follows that $\lambda_A = \lambda_B = 0$. Otherwise, from equation (40), at least one player i would have an incentive to increase his expenditure up towards X_i. Similarly, in case (b) it follows that $\lambda_B = 0$, and $\lambda_A \geq 0$, and in case (c) $\lambda_B \geq 0$, and $\lambda_A \geq 0$. Returning to the definition of M_{X_i} and the expressions for each $F_{i,j}$ given above, it follows that in the Theorem 1 parameter range, $M_{X_A} = \min\{X_A, (nv/2)\}$ and $M_{X_B} = \min\{X_B, (nvM_{X_A}/2)^{1/2}\}$. □

We conclude the Appendix with a brief discussion of how these results extend to the case of $n = 2$ and the Theorem 2 parameter range with $X_A \neq (2v/n)$ and $n \geq 3$. Note that Lemma 1 holds for all $n \geq 2$ and can be extended to cover all parameter configurations. Similarly, Lemma 2 holds for all $n \geq 2$, but the lemma can only be extended to the case that the player's resources (X_A, X_B) lie in the Theorem 2 parameter range with $X_A \neq (2v/n)$. If $M_{X_A} = (2v/n)$ then M_{X_B} can take any value in the interval $[v(2-(2/n)), X_B]$. That is any feasible pair of strategies $\{P_A, P_B\}$ with $M_{X_A} = (2v/n)$ and $M_{X_B} \in [v(2-(2/n)), X_B]$ and which provide the corresponding sets of univariate marginal distributions stated in Theorem 2 is an equilibrium. Similar issues regarding the nonuniqueness of the players' equilibrium total expected expenditures arise in the Theorem 3 and 5 parameter ranges.

In the Theorem 2 parameter range with $X_A \neq (2v/n)$ Lemma 4 applies to both players but $\overline{s} = X_A$. For this parameter range, Lemma 5 only applies to player A. The issue is that when (X_A, X_B) is in the Theorem 2 parameter range, interchangeability of equilibrium strategies is no longer sufficient to rule out mass points in player B's univariate marginal distributions. In particular, because $\overline{s} = X_A$ each $F_{B,j}$ can now place an atom at $\bar{s} = X_A$. Furthermore, mass points may exist in the interior of the domain of the player B's univariate marginals. Consider an equilibrium $\{P_A, P_B\}$, with $M_{X_A} = X_A$ and $M_{X_B} = X_B = (n/2)X_A$, in which player A uses a strategy consistent with Theorem 2 and player B uses the strategy formed by player B bidding $(X_A/2)$ in each auction with probability $(1-(2/n))$ and with probability $(2/n)$ player B utilizing a strategy consistent with Theorem 2. This is a feasible strategy for player B [which satisfies Lemma 2], and in this strategy player B's univariate marginals are given by:

$$\forall\ j \in \{1, \ldots, n\} \qquad F_{B,j}\left(x\right) = \begin{cases} \frac{2x}{nX_A} & \text{for} \quad x \in \left[0, \frac{X_A}{2}\right) \\ 1 - \frac{2}{n} + \frac{2x}{nX_A} & \text{for} \quad x \in \left[\frac{X_A}{2}, X_A\right] \end{cases}.$$

As long as player A uses all of his available resources X_A and bids above $(X_A/2)$ in a single auction — as is the case if player A is using a strategy consistent with Theorem 2 — this yields the unique equilibrium expected payoff $[v - X_A]$ for player A.[16] Furthermore, it is straightforward to show that there are no profitable deviations for player A, and thus, such a pair of joint distributions forms an equilibrium. The issue here is that at those points in the support of player A's equilibrium strategy where ties occur with positive probability: (i) player A is at his budget constraint and (ii) ties occur in at most two auctions. In order for player B to create such a situation, it must be the case that $X_B \geq (n/2)X_A$, and so this issue does not arise in the Theorem 1 range.

Because the extension of Lemma 5 to the Theorem 2 parameter range applies to only player A's set of univariate marginal distributions, it clearly follows that the extensions of Lemmas 6 and 7 also only apply to player A's set of univariate marginal distributions.

[16] Note that if player A bids $(X_A/2)$ in two auctions, then the tie-breaking rule applies and player A's expected payoff is equal to the unique equilibrium payoff.

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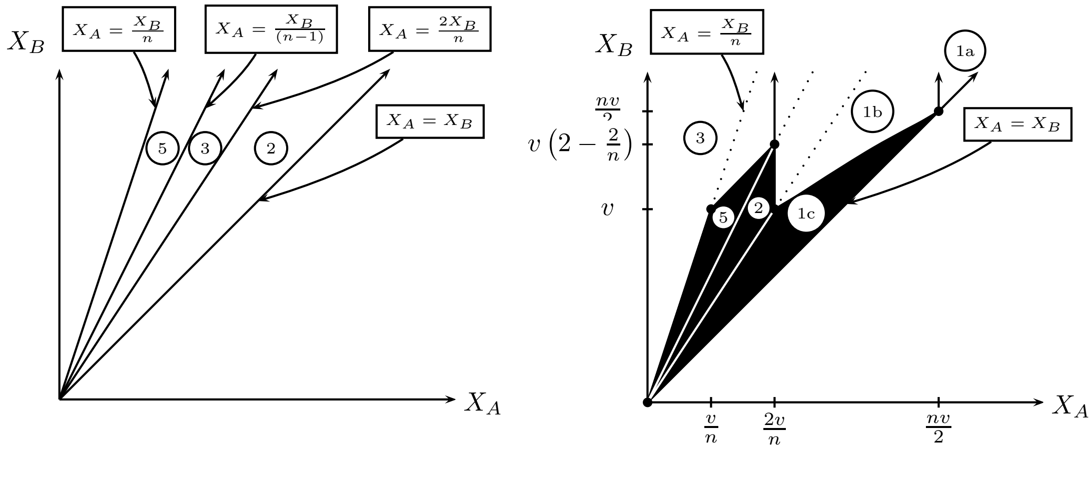
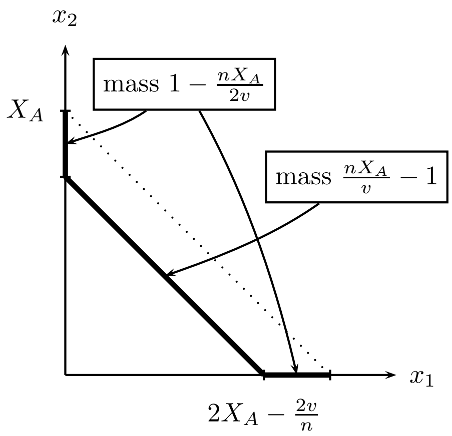


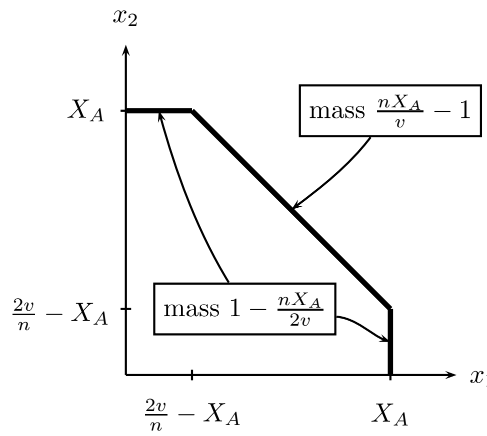
(i) Constant-Sum

(ii) Non-Constant-Sum

Figure 1: Parameter Space $n \geq 3$

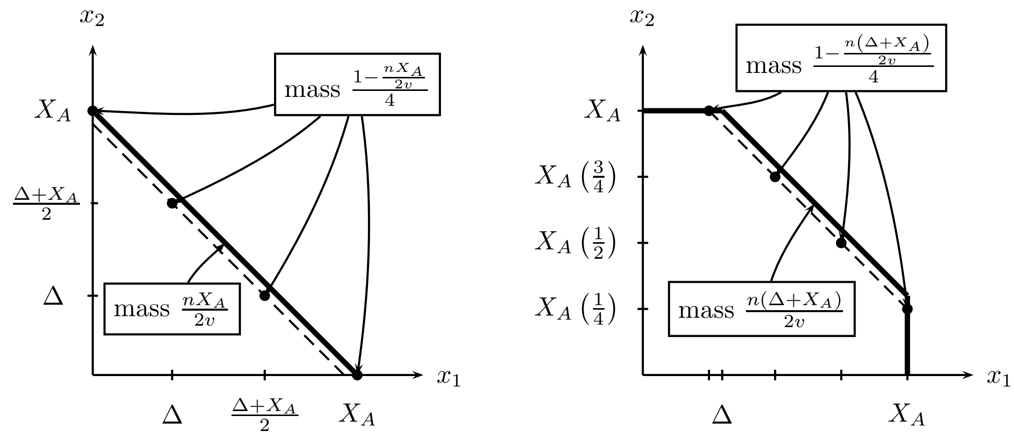


(i) Player A



(ii) Player B

Figure 2: Supports of players' bivariate distributions in Theorem 3 parameter range



(i) Player A

(ii) Player B

Figure 3: Supports of players' bivariate distributions in Theorem 5 parameter range ($k = 3$)

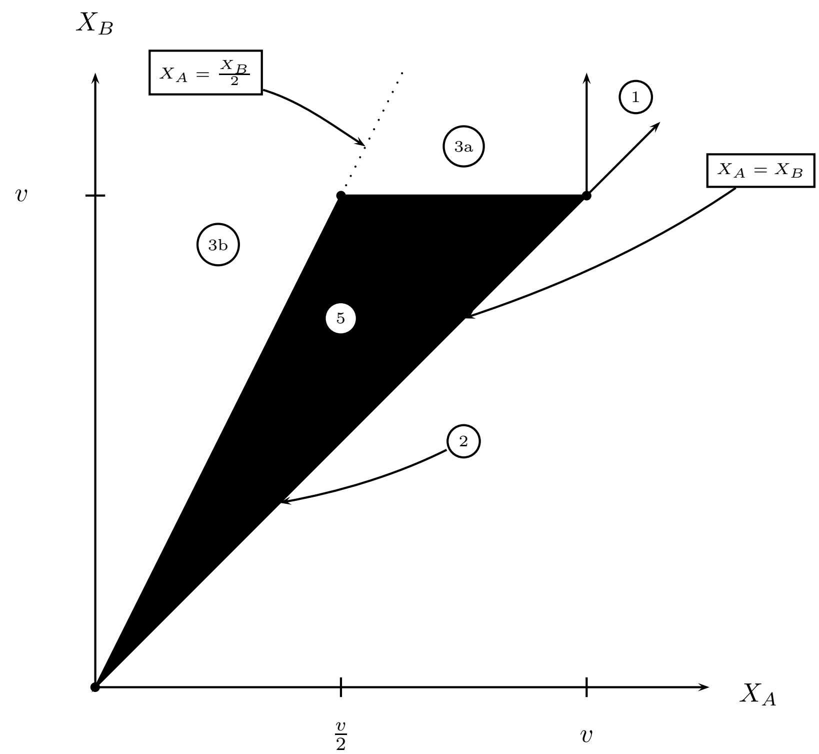


Figure 4: Parameter Space $n = 2$