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Strategic Defense and Attack for Series and Parallel Reliability Systems: Comment

By

Dan Kovenock
Brian Roberson

Paper No. 1253
Date: September, 2010

Institute for Research in the Behavioral, Economic, and Management Sciences

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Dan Kovenock*and Brian Roberson†

September 5, 2010

Abstract

Hausken (2008a) formulates a contest-theoretic model of the attack and defense of a network of targets. This note identifies a technical error that invalidates Hausken's characterization of Nash equilibrium for a substantial portion of the parameter space that he examines and provides necessary conditions for his solution to form a pure-strategy Nash equilibrium. Many of the existing results in the contest-theoretic literature on the attack and defense of networks of targets rely upon Hausken's (2008a) characterization and require corresponding parameter restrictions. When these restrictions are not met, the analysis of Clark and Konrad (2007) and Kovenock and Roberson (2010a) provides a foundation for constructing mixed-strategy Nash equilibria.

JEL Classification: C72, D74, H56

Keywords: Game theory; Reliability theory; OR in military; Conflict; Contest; Network; Colonel Blotto game

*Dan Kovenock, University of Iowa, Department of Economics, Tippie College of Business, W284 PBB, 21 E. Market Street, Iowa City, IA 52242 USA t: 319-335-1810, E-mail: dan-kovenock@uiowa.edu

†Brian Roberson, Purdue University, Department of Economics, Krannert School of Management, 403 W. State Street, West Lafayette, IN 47907 USA t:765-494-4531, E-mail: brobers@purdue.edu (Correspondent)

1 Introduction

Clark and Konrad (2007) and Hausken (2008a) formulate models of the attack and defense of a network of targets in which the outcome of the conflict at each target is determined by a contest success function that maps the players' resource allocations into their probabilities of success.[1] If the defender (attacker) allocates a force of $d_i \geq 0$ ($a_i \geq 0$) to target, or component, i, then the probability that target i is successfully defended is given by the contest success function $p_{D,i}(d_i, a_i)$, which may take one of several forms including, but not limited to, the general ratio-form specification of the contest success function given by $p_{D,i}(d_i, a_i) = d_i^{m_i}/(a_i^{m_i} + d_i^{m_i})$ where $m_i \geq 0$ is an exogenous parameter that specifies the level of noise in contest i. In the trivial case where $m_i = 0$, each player wins contest i with probability (1/2) regardless of the players' force allocations. The interpretation of a strictly positive but low value for m_i is that contest i involves a substantial amount of noise conditional on the profile of allocations (see Konrad and Kovenock 2009 for a discussion of how much noise is implied). When m_i is high, with $m_i = \infty$ the limiting case, each contest has a low amount of, or no, noise.

The outcome of the overall conflict is determined by the outcomes at the individual targets. For example in infrastructure networks there often exist particular targets or combinations of targets which if destroyed would be sufficient to either disable or disrupt the entire network. In the context of an arbitrarily complex network of targets, Hausken (2008a) uses the players' sets of first-order conditions to identify a pure-strategy Nash equilibrium and then examines properties arising in that solution. In this comment we provide necessary conditions for the solution in Hausken (2008a) to form a pure-strategy Nash equilibrium point and show that these conditions are quite restrictive.

Section 2 reviews the characterization of equilibrium given by Hausken (2008a) and provides necessary conditions for the solution in Hausken (2008a) to form a pure-strategy Nash equilibrium point. For the portion of the parameter space in which the solution given in Hausken (2008a) does not form a pure-strategy Nash equilibrium point, there exist mixed-strategy Nash equilibria.[2] Section 3 summarizes the existing results on mixed-strategy equilibria.

Many of the existing results in the contest-theoretic literature on the attack and defense

[1] The contest success function may be interpreted as the specifying the probability that component i is completely destroyed or alternatively the degree to which it is damaged.

[2] For general Nash equilibrium existence proofs covering this special case see Montiero and Page (2007) and Tian (2009).

of networks of targets build upon the characterization in Hausken (2008a). Section 2 briefly discusses how the parameter restrictions on the first-order approach impact the related literature. Clark and Konrad (2007) and Kovenock and Roberson (2010a)[3] demonstrate how to construct mixed-strategy equilibria for benchmark special cases in this class of games. These two articles provide a foundation for examining the parameter ranges of those models in the related literature which have no pure-strategy Nash equilibria. We view the characterization of the mixed-strategy equilibria over these parameter ranges as an important area for future research.

2 Model and Main Result

The basic model of attack and defense of networks of targets is formally described as follows. Two players, an attacker, A, and a defender, D, simultaneously allocate their forces across a finite number, $n \geq 2$, of targets indexed by $i \in \{1, \ldots, n\}$. The overall outcome of the conflict across the network of targets is determined by the outcomes at the individual targets.

Beginning with the outcomes at the individual targets, let a_i (d_i) denote the level of force allocated by the attacker (defender) to target i. The probability that target i is successfully defended is given by

$$p_{D,i}(d_i, a_i) = \begin{cases} \frac{1}{2} & \text{if } \ a_i = d_i = 0 \\ \frac{d_i^{m_i}}{a_i^{m_i} + d_i^{m_i}} & \text{otherwise} \end{cases} . \tag{1}$$

The probability that target i is successfully attacked is given by $p_{A,i}(a_i, d_i) = 1 - p_{D,i}(d_i, a_i)$. Let $p_D(p_{D,1}, p_{D,2}, \ldots, p_{D,n})$ denote the probability that the network of n targets is successfully defended.

As in Hausken (2008a), we will focus on the simple *series network*, or what Clark and Konrad (2007) refer to as a *weakest-link network*.[4] The second simple type of network, referred to as a *parallel network* by Hausken (2008a) or a *best-shot network* by Clark and Konrad (2007), is successfully defended if the defender successfully defends at least one target within the network. As noted by both Clark and Konrad (2007) and Hausken (2008a), a best-shot network is the same as a weakest-link network with the labeling of the attacker and defender reversed.

[3] An earlier version of Kovenock and Roberson (2010a) appeared under the title, "Terrorism and the Optimal Defense of Networks of Targets," (Kovenock and Roberson, 2006).

[4] It is straightforward to extend the arguments given below to cover what Hausken (2008a) refers to as an "arbitrarily complex system."

A weakest-link (series) network is successfully defended if the defender wins all of the targets within the network. Conversely, an attack on a weakest-link network is successful if the attacker wins at least one target in the network. In this case,

$$p_D(p_{D,1}, p_{D,2}, \ldots, p_{D,n}) = \prod_{i=1}^{n} p_{D,i}(d_i, a_i). \tag{2}$$

Assume that players are risk neutral and have asymmetric objectives. The attacker's objective is to successfully attack at least one target, and the attacker's payoff for the successful attack of at least one target is $v_A > 0$. Let $\mathbf{a}$ ($\mathbf{d}$) denote the attacker's (defender's) n-tuple of force allocations. The attacker's expected payoff function is given by

$$\pi_A(\mathbf{a}, \mathbf{d}) = v_A \left(1 - \prod_{i=1}^{n} p_{D,i}(d_i, a_i)\right) - \sum_{i=1}^{n} c_{A,i} a_i \tag{3}$$

where $c_{A,i}$ is the attacker's constant unit cost of force expenditure to target i. The defender's objective is to preserve the entire network, and the defender's payoff for successfully defending the network is $v_D > 0$. The defender's expected payoff function is given by

$$\pi_D(\mathbf{a}, \mathbf{d}) = v_D \left(\prod_{i=1}^{n} p_{D,i}(d_i, a_i)\right) - \sum_{i=1}^{n} c_{D,i} d_i \tag{4}$$

where $c_{D,i}$ is the defenders constant unit cost of force expenditure to target i. The force allocated to each target must be nonnegative.

The set of first-order conditions of the players' optimization problems is given by equation (14) of Hausken (2008a). In each contest $i = 1, \ldots, n$,

$$\frac{\partial \pi_A}{\partial a_i} = v_A \frac{m_i a_i^{m_i - 1} d_i^{m_i}}{(a_i^{m_i} + d_i^{m_i})^2} \prod_{j \neq i} \frac{d_j^{m_j}}{a_j^{m_j} + d_j^{m_j}} - c_{A,i} = 0 \tag{5}$$

and

$$\frac{\partial \pi_D}{\partial d_i} = v_D \frac{m_i d_i^{m_i - 1} a_i^{m_i}}{(a_i^{m_i} + d_i^{m_i})^2} \prod_{j \neq i} \frac{d_j^{m_j}}{a_j^{m_j} + d_j^{m_j}} - c_{D,i} = 0. \tag{6}$$

Let $\mathbf{a}^*$ and $\mathbf{d}^*$ denote the solution to the first-order conditions, for the attacker and defender

respectively, where as given by equation (15) of Hausken (2008a),

$$a_i^* = \frac{1}{(c_{A,i}/v_A)} \frac{m_i(c_{D,i}/v_D)^{m_i}}{(c_{A,i}/v_A)^{m_i} + (c_{D,i}/v_D)^{m_i}} \prod_{j=1}^{n} \frac{(c_{A,j}/v_A)^{m_i}}{(c_{A,j}/v_A)^{m_i} + (c_{D,j}/v_D)^{m_i}} \tag{7}$$

and

$$d_i^* = \frac{1}{(c_{D,i}/v_D)} \frac{m_i(c_{D,i}/v_D)^{m_i}}{(c_{A,i}/v_A)^{m_i} + (c_{D,i}/v_D)^{m_i}} \prod_{j=1}^{n} \frac{(c_{A,j}/v_A)^{m_i}}{(c_{A,j}/v_A)^{m_i} + (c_{D,j}/v_D)^{m_i}} \tag{8}$$

for each $i = 1, \ldots, n$.

The players' expected payoffs from the pair of n-tuples $(\mathbf{a}^*, \mathbf{d}^*)$ are given by equation (17) of Hausken (2008a),

$$\pi_A(\mathbf{a}^*, \mathbf{d}^*) = v_A \left(1 - \left(1 + \sum_{i=1}^{n} \frac{m_i(c_{D,i}/v_D)^{m_i}}{(c_{A,i}/v_A)^{m_i} + (c_{D,i}/v_D)^{m_i}} \right) \prod_{i=1}^{n} \frac{(c_{A,i}/v_A)^{m_i}}{(c_{A,i}/v_A)^{m_i} + (c_{D,i}/v_D)^{m_i}} \right) \tag{9}$$

and

$$\pi_D(\mathbf{a}^*, \mathbf{d}^*) = v_D \left(1 - \sum_{i=1}^{n} \frac{m_i(c_{D,i}/v_D)^{m_i}}{(c_{A,i}/v_A)^{m_i} + (c_{D,i}/v_D)^{m_i}} \right) \prod_{i=1}^{n} \frac{(c_{A,i}/v_A)^{m_i}}{(c_{A,i}/v_A)^{m_i} + (c_{D,i}/v_D)^{m_i}} \tag{10}$$

Because each player can ensure a payoff of 0 by doing nothing, in any equilibrium each player must have a nonnegative payoff. When the solution of the first-order conditions, the strategy profile $(\mathbf{a}^*, \mathbf{d}^*)$, results in negative expected payoffs for either or both players, the first-order conditions do not yield a pure-strategy Nash equilibrium point. Furthermore, it is incorrect to assert, as does Hausken (2008a), that if a player's expected payoff given in (9) or (10) is negative, then there exists an equilibrium in which that player allocates zero forces to each target (yielding a payoff of zero) and the other player earns arbitrarily close to his valuation for the system by allocating an arbitrarily small amount of force to one or more of the targets.[5] To see that such a pair of n-tuples does not form an equilibrium, assume that the defender allocates zero forces to each target and that the attacker allocates an arbitrarily small amount of force to one or more targets. In such a case, the defender's best response is to outbid the attacker, by an arbitrarily small amount, at each of the attacked targets. Such a strategy increases the defender's payoff from zero to arbitrarily close to his valuation

[5]This argument is given in the discussion following equation (17) on p.864, the discussion following example 1 on p.865, and the discussion of example 2 on p.866 of Hausken (2008a).

for the entire system. A similar argument rules out the possibility that allocating zeros everywhere could be an equilibrium strategy for the attacker.

Proposition 1 below provides necessary conditions for the strategy-profile $(\mathbf{a}^*, \mathbf{d}^*)$ to form a pure-strategy Nash equilibrium point.

Proposition 1. *Two necessary conditions for the strategy-profile* $(\mathbf{a}^*, \mathbf{d}^*)$ *to form a pure-strategy Nash equilibrium in the model of attack and defense of networks of targets are,*[6]

$$\begin{aligned} &\sum_{i=1}^{n} \frac{m_i (c_{D,i}/v_D)^{m_i}}{(c_{A,i}/v_A)^{m_i} + (c_{D,i}/v_D)^{m_i}} \leq 1 \\ \text{and} \quad &\prod_{i=1}^{n} \frac{(c_{A,i}/v_A)^{m_i}}{(c_{A,i}/v_A)^{m_i} + (c_{D,i}/v_D)^{m_i}} \leq \frac{1}{1 + \sum_{i=1}^{n} \frac{m_i (c_{D,i}/v_D)^{m_i}}{(c_{A,i}/v_A)^{m_i} + (c_{D,i}/v_D)^{m_i}}}. \end{aligned} \tag{11}$$

First, observe that if either or both of the conditions in Proposition 1 fail to hold, then one or both of the players' expected payoffs, given in equations (9) and (10), are negative and the strategy profile $(\mathbf{a}^*, \mathbf{d}^*)$ is not a pure-strategy Nash equilibrium. Because the players' objective functions fail to be quasi-concave with respect to the relevant choice variables (each player's n-tuple of force) for a large portion of the parameter space, standard arguments on the existence of pure-strategy equilibrium do not apply. Therefore solving the system of first-order conditions, as does Hausken (2008a), does not guarantee equilibrium.

Whether the conditions stated in equation (11) are, in general, sufficient for $(\mathbf{a}^*, \mathbf{d}^*)$ to form a pure-strategy equilibrium is still an open problem. However, we know that under certain parameter restrictions, necessity and sufficiency are guaranteed. For example, an extension of the arguments given in Clark and Konrad (2007) implies that if for all targets $i = 1, \ldots, n$ (i) $c_{A,i} = c_A$, (ii) $c_{D,i} = c_D$, and (iii) $m_i = m \leq 1$, then the conditions in equation (11) are indeed necessary and sufficient for $(\mathbf{a}^*, \mathbf{d}^*)$ to form a pure-strategy equilibrium.

To see just how restrictive the conditions in Proposition 1 are, assume as in Clark and Konrad (2007) that: (1) at each target the per unit cost of allocating force is symmetric across players (i.e., $c_{A,i} = c_{D,i}$ for all i) and (2) $m_i = 1$ for all i. As Clark and Konrad (2007) show, both of the conditions in Proposition 1 hold (i.e., the first-order conditions identify a pure-strategy Nash equilibrium point) if and only if $v_D \geq (n-1)v_A$. That is, pure-strategy equilibria may fail to exist even in the popular case of $m_i = 1$ for all i. Note also that if $(c_{D,i}/v_D) = (c_{A,i}/v_A)$ for all $i = 1, \ldots, n$, then the conditions in (11) hold only if

[6] Observe that these conditions can be written more succinctly as $\sum_{i=1}^{n} \frac{m_i (c_{D,i}/v_D)^{m_i}}{(c_{A,i}/v_A)^{m_i} + (c_{D,i}/v_D)^{m_i}} \leq \min\{1, \prod_{i=1}^{n} [1 + ((c_{D,i} v_A)/(c_{A,i} v_D))^{m_i}] - 1\}$.

$\sum_{i=1}^{n} m_i \leq 2$. Clearly, this inequality places severe restrictions on $\{m_i\}_{i=1}^{n}$.

Because the solution in Hausken (2008a) forms a pure-strategy Nash equilibrium point only for a restrictive set of parameters, statements that are based on this solution — such as all of the propositions in Hausken (2008a) — fail to hold for all combinations of m_i, v_A, v_D, $c_{A,i}$ and $c_{D,i}$ that violate either of the two conditions in Proposition 1. Although we do not know at this time whether these conditions are sufficient, as a general proposition the nature of equilibrium strategies is quite nuanced and varies over the parameter space, as we detail below in our discussion of mixed-strategy equilibria.

There are a number of articles that analyze variations of the Hausken (2008a) framework that are also erroneous without the imposition of more restrictive parameter assumptions. These variations fall into two groups. First, both Hausken (2008b) and Hausken (2010) contain closely related technical errors that invalidate those characterizations of Nash equilibrium for a substantial portion of the parameter space. In particular, in both of those games each of the players has a secure utility that can be assured regardless of the action of the other player. But in the solutions that are given Hausken (2008b) and Hausken (2010) there exist large portions of the parameter space in which one or both of the players obtain a level of utility that is below their secure utility level, and hence these solutions do not form pure-strategy equilibria.

There is also a growing literature that relies upon the equilibrium characterization in Hausken (2008a) in order to analyze more complex environments. In the context of an "arbitrarily complex system," Proposition 1 of Hausken (2008a) states that,

> The investment expenditures relative to the system value for the defender and the attacker are equal for each component ... regardless of the parameter m_i. (Hausken 2008a, p.862)

If correct, this result would make it very easy to embed a multidimensional resource allocation game into a more complex game and would facilitate straightforward comparative statics statements regarding changes in the levels of exogenous noise $\{m_i\}_{i=1}^{n}$. However, as our Proposition 1 shows, Hausken's characterization of equilibrium fails to hold for most values of m_i. Unfortunately, there are a number of papers that, relying on Hausken (2008a), incorrectly assume that when $c_{A,i} = c_A$, $c_{D,i} = c_D$, and $m_i = m$ for all components $i = 1, \ldots, n$, it is optimal for each player to allocate forces evenly across components. Examples include, but are not limited to, treatments of the basic networks environment (Hausken and Levitin 2009a; Levitin and Hausken 2009f, 2010a, b), the partitioning of networks by the defender, the creation of false targets, and the creation of network redundancy (Hausken and

Levitin 2008, 2009c; Levitin and Hausken 2008, 2009a, b, c, d, 2010d, f; Peng, Levitin, Xie, and Ng 2010a, b), and multiple time periods and sequential attack (Hausken and Levitin 2009b, 2010; Levitin and Hausken 2009e, 2010c, e). It is beyond the scope of this comment to provide necessary and sufficient conditions for the existence of a pure-strategy equilibrium with an even allocation of forces that is assumed in all of these articles. However, it is straightforward to modify the necessary conditions given in Proposition 1 above for each of those particular models. Furthermore, as discussed in Section 3 below, Clark and Konrad (2007) and Kovenock and Roberson (2010a) demonstrate how to correctly construct the mixed-strategy equilibria for benchmark cases for which there exist no pure-strategy Nash equilibria. We conjecture that employing their methods, the set of mixed-strategy Nash equilibria can be derived for the relevant parameter ranges of those models. We view the completion of this equilibrium characterization as an important area for future research.[7]

3 Mixed-Strategy Equilibria

A mixed strategy, which we term a *distribution of force*, for player i is an n-variate distribution function $P_i : \mathbb{R}^n_+ \to [0, 1]$. The n-tuple of player i's allocation of force across the n targets is a random n-tuple drawn from the n-variate distribution function P_i. It is beyond the scope of this comment to provide a complete characterization for this class of games. We will instead highlight the existing results on mixed strategy equilibria for this game (Clark and Konrad 2007 and Kovenock and Roberson 2010a).[8]

In dealing with multivariate mixed strategies it is helpful to reduce the number of exogenous parameters in the model. Towards this end, assume that for each player the unit costs are symmetric across targets and players and are normalized to one (i.e., $c_{A,i} = c_{D,i} = 1$ for all i) and that the level of noise at each target is the same (i.e., $m_i = m$ for all i).

The case of $m = 1$ is analyzed by Clark and Konrad (2007). If $v_D \geq (n-1)v_A$, then this game has a pure-strategy Nash equilibrium that coincides with the characterization given above for $\mathbf{a}^*$ and $\mathbf{d}^*$ with $m = 1$ and $c_{A,i} = c_{D,i} = 1$ for all i. If $v_D < (n-1)v_A$, then Clark and Konrad (2007) show that the following pair of strategies constitutes an equilibrium.[9]

[7] See for example Arce, Kovenock and Roberson (2010) who examine the mixed strategy equilibria in a model with a network of targets and a terrorist organization with multiple attack technologies.

[8] This multi-dimensional contest is an example of a "Blotto-type" game. For more on the Colonel Blotto game see Roberson (2006). For a survey of multi-dimensional contests see Kovenock and Roberson (2010b) or Roberson (2010).

[9] It is straightforward to extend this analysis to cover cases in which $m \leq 1$.

Proposition 2 (Clark and Konrad (2007)). *If $m = 1$ and $v_D < (n-1)v_A$, then the following pair of strategies forms an equilibrium. Player A chooses the n-tuple $\mathbf{a} = (a, a, \ldots, a)$ where*

$$a = \frac{(n-1)^{n-1}}{n^{n+1}} v_D.$$

With probability $q = \frac{v_D}{v_A} \frac{1}{(n-1)}$ player D chooses the n-tuple $\mathbf{d} = (d, d, \ldots, d)$ where

$$d = \frac{(n-1)^{n}}{n^{n+1}} v_D.$$

With the remaining probability player D chooses the n-tuple $\mathbf{d} = (0, 0, \ldots, 0)$.

Player A's expected payoff is $v_A - 2v_D((n-1)^{n-1}/n^n)$, and player D's expected payoff is 0.

The case of $m = \infty$ is analyzed by Kovenock and Roberson (2010a). In this deterministic case we assume that in the event that the players allocate the same level of resources to a target, the defender wins the target.[10] There exist multiple mixed-strategy Nash equilibria in this case. Kovenock and Roberson (2010a) provide one equilibrium and characterize properties that hold in all equilibria. Proposition 3 provides one property that demonstrates how the nature of equilibrium drastically changes.

Proposition 3 (Kovenock and Roberson (2010a)). *If $m = \infty$, then in any equilibrium $\{P_A, P_D\}$ the attacker allocates a strictly positive level of force to at most one target in the weakest-link network.*

In the case of $m = \infty$, equilibrium requires that the attacker (who may disable the network by winning any single target) randomly allocate his forces to at most one target, rather than to multiple targets. In fact, recent experimental work provides some support for this behavior. Kovenock, Roberson and Sheremeta (2010) find that in experiments undertaken with $m = \infty$ and $n = 4$, over 80% of attackers utilize a *stochastic guerilla warfare strategy* that entails launching an attack on only a single target. (Contrary to the equilibrium prediction, Kovenock et al. (2010) also find that with $m = 1$ attackers launch a single attack almost 45% of the time.)

The case of $m = \infty$ is an extreme benchmark case, playing a role for contests with sunk expenditures similar to classical Bertrand competition in price-setting games with perfect

[10] Often, in games with discontinuous payoffs, such as winner-take-all contests with $m = \infty$, the modeler must employ a judicious choice of a tie-breaking rule in order to avoid having to revert to the use of ε-equilibrium concepts.

substitutes. However, there is good reason to believe that cases involving finite but high m are qualitatively closer to this benchmark than to the pure-strategy profiles examined in Hausken (2008a). A complete treatment of simultaneous move games of attack and defense in which the conditions in equation (11) are violated is still an open question. However, we do know that for a single contest with linear costs, (the famous Tullock rent-seeking model), a pure-strategy equilibrium exists only for m less than or equal to 2. For m greater than 2, as in the $m = \infty$ case, no pure-strategy equilibria exist. Although there has not been a complete characterization of the equilibrium set for $m > 2$, except for Baye, Kovenock and de Vries (1996) characterization for $m = \infty$, we do know that when $m > 2$ there exist Nash equilibria that yield payoffs identical to the unique equilibrium payoffs when $m = \infty$ (Alcalde and Dahm 2010).

Unfortunately, the nonexistence of pure-strategy equilibrium and characterization of mixed-strategy equilibria has been almost entirely overlooked in the contest-theoretic literature on attack and defense of networks of targets. In fact, as shown in Proposition 1, the pure-strategy equilibria applied extensively in the literature exist only for a very restrictive set of parameters. Clark and Konrad (2007) and Kovenock and Roberson (2010a) provide a foundation for constructing the mixed-strategy Nash equilibria in the parameter ranges of those models for which no pure-strategy equilibria exist. This characterization remains an important area for future research.

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