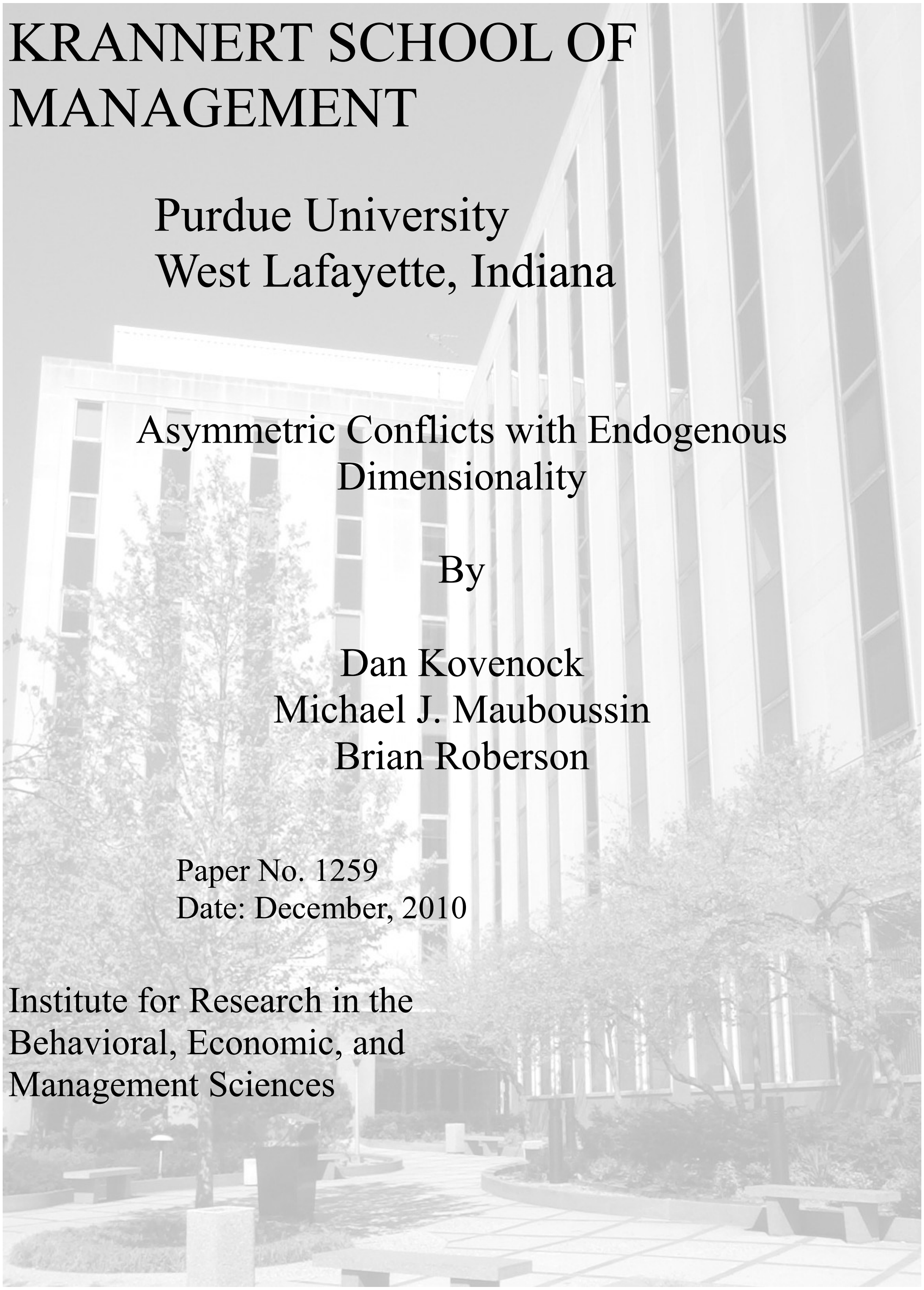

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Asymmetric Conflicts with Endogenous Dimensionality

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Abstract

This article examines a two-stage model of asymmetric conflict based on the classic Colonel Blotto game in which players have, in the first stage, the ability to increase the number of battlefields contested. It thereby endogenizes the "dimensionality" of conflict. In equilibrium, if the asymmetry in the players' resource endowments exceeds a threshold, the weak player chooses to add battlefields, while the strong player never does. Adding battlefields spreads the strong player's forces more thinly, increasing the incidence of favorable strategic mismatches for the weak player.

JEL Classification: C72, D74

Keywords: Asymmetric Conflict, Multi-battle Conflict, Colonel Blotto Game, Stochastic Guerilla Warfare, Endogenous Dimensionality

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1 Introduction

Conflict often takes place across multiple component contests or "battlefields." In this article we consider a two-player, two-stage game in which players start with a fixed endowment of a resource and a given number of battlefields. In the first stage, the players have the ability to simultaneously create additional battlefields. In the second stage they allocate their respective stocks of the resource across the updated set of battlefields in order to maximize the expected proportion of the individual component contests won. We are therefore concerned with the endogeneity of the number of battlefields, or the "dimensionality" of the conflict.

The battlefields in this game may be taken literally to be points of combat or fronts of a military campaign, where opening up a new battlefield means expanding the geographical scope of the conflict. Or they may refer to collections of targets within a transportation or computer network in the context of counter-terrorism or information systems security efforts.

In sports contests, battlefields may be interpreted as the individual player contests within the overall match, such as a cornerback trying to "lock down" a receiver in American football, a defender marking a striker in football, or any of the five individual one-on-one matchups in basketball. Alternatively, they may be interpreted as the clash between distinct skill sets within the overall match, such as passing versus pass defense in American football or left-handed pitching versus left-handed hitting in baseball.

It is also possible to interpret the number of component contests or battlefields within a conflict more directly as the different dimensions of the conflict. For instance, when firms compete in a market for a bundled good that has multiple patentable components, one might interpret the race to patent a given component as a dimension of the conflict. Similarly, political campaigns involve multiple issues and multiple segments of voters. Winning the contest over each of these issues or segments may be viewed as succeeding in a given dimension of the contest.

To examine the issue of the endogeneity of the number of battlefields or dimensionality of a conflict, we augment the classic Colonel Blotto game of Borel (1921) to allow for a pre-conflict stage in which players have the ability to increase the dimensionality of the conflict. The Colonel Blotto game is a foundational model of conflict with multiple component contests and was one of the first problems examined in modern game theory.[1] In this game, two players simultaneously allocate their respective endowments of a resource across a finite number, n, of battlefields. In each battlefield, the player with the higher level of resources wins the battle, and each player maximizes the proportion of the battlefields that he wins.[2] Consider for example, a simple version of the game in which two symmetric players each have 1 unit of force to allocate across three battlefields. In this simple example, a pure strategy is a three-tuple of nonnegative numbers (x_1, x_2, x_3) such that $x_1 + x_2 + x_3 \leq 1$.

It is well-known that the Colonel Blotto game is a constant-sum game which, for initial resource endowments that are not too asymmetric, has no pure-strategy equilibria. Unless one player has a sufficient endowment of the resource to allocate an amount greater than or equal to the other player's endowment to each battlefield, equilibrium requires nondegenerate mixed strategies. These mixed strategies are n-variate distributions, one dimension for each battlefield.

When players have identical endowments, the fact that the Colonel Blotto game is constant-sum ensures that payoffs are identical, regardless of the number of battlefields. Our focus in this article is on conflicts in which players are asymmetrically endowed. There is a strong player, S, whose endowment of the resource is larger than that of the weak player,

[1]The Colonel Blotto game is an example of a multidimensional contest with linkages, where the linkages arise from the fact that if a player uses all of his endowment of the resource then an increase in his allocation of the resource to any one of the component contests requires a decrease in the resource allocated to one or more of the other contests. There are a number of related environments in which overall performance is a function of the outcomes in each of a set of component contests. For a survey of multidimensional contests with linkages see Kovenock and Roberson (2010).

[2]An alternative objective is for each player to maximize the probability that he wins a majority of the battlefields. For $n > 3$, the solution for the majority objective game is still an open question.

W.

In the context of the two-stage game described above, we demonstrate that given a sufficiently small initial number of battlefields and cost of adding battlefields, and given sufficiently asymmetric endowments of the resource (but not necessarily in the range of parameters leading to a pure-strategy equilibrium in the Colonel Blotto game), in any subgame perfect equilibrium the weak player W optimally chooses to add battlefields, whereas the strong player S abstains. Moreover, we characterize the optimal number of battlefields for the weak player to add.

The logic underlying our result is straightforward. Because the resources in the Colonel Blotto game are use-it-or-lose-it, conflicts with a small number of battlefields force the players to go toe-to-toe and directly compete strength-versus-strength. Because the strong player has the larger endowment, this means that when the conflict is focused on a small number of battlefields, the strong player is at a clear advantage. In the trivial case of one battlefield, or when the number of battlefields is sufficiently small that the strong player has a sufficient endowment of the resource to allocate an amount greater than or equal to the weak player's endowment to each battlefield, the strong player wins with certainty. As the number of battlefields increases and becomes sufficiently large that the strong player does not win every battlefield with certainty, this advantage persists even though equilibrium is in nondegenerate mixed strategies. For an initial range of parameters for which mixed strategies arise, the strong player employs a joint distribution with identical marginal distributions across the battlefields which on average allocate a larger amount to each battlefield than the weak player. Moreover, the strong player allocates an amount equal to the weak player's budget to a fraction of the battlefields. As the number of battlefields increases further, the strong player no longer has sufficient resources to find it optimal to allocate an amount equal to the weak player's budget to a subset of battlefields and employs an n-variate distribution which randomizes uniformly in each battlefield with a common support. The upper bound of this support, the highest single equilibrium allocation of either player to each battlefield, is decreasing in the number of battlefields. Over this range the weak player's equilibrium strategy takes a special form: the weak player stochastically neglects a proportion of the battlefields (in expectation equal to one minus the ratio of his endowment to the strong player's endowment), placing none of the resource in some battlefields with positive probability, while randomly allocating positive levels of the resource to a subset of battlefields in magnitudes that are comparable in expectation to the random allocations of his rival. Following Roberson (2006) we call this a "stochastic guerilla warfare" strategy.

As this description implies, by adding battlefields, the weak player forces the strong player to allocate resources more thinly, thereby weakening the strong player's relative advantage. In other words, underdogs who increase the number of battlefields have a better chance of an upset.

In asymmetric conflicts, by forcing the strong player to spread his resources more thinly, the weak player increases the incidence of favorable mismatches, in which the weak player, despite his overall disadvantage, is in an advantageous position within a given battlefield. Note that, in equilibrium, each player attempts to strategically allocate resources to maximize the proportion of battlefields won, creating strategic matches or mismatches of resources as conforms to this objective. With a large number of battlefields the net outcome of this behavior is that the weak player on average stochastically allocates resources to a proportion of the set of battlefields equal to the ratio of the weak and strong endowments and neglects the remaining battlefields. In those battlefields in which the weak player is active, he on average allocates the same level of the resource as the stronger player. The strong player, in turn, randomizes by allocating positive amounts to all battlefields, treating each symmetrically but satisfying his resource constraint with probability one.

Other well-known constant-sum games in which players attempt to achieve favorable mismatches include Matching Pennies and Rock-Paper-Scissors. Like the Colonel Blotto game, equilibrium in these games requires nondegenerate mixed strategies. However, the mismatch incentives in the Colonel Blotto game are quite different from those arising in Matching Pennies or Rock-Paper-Scissors and are somewhat more complicated. In the Colonel Blotto game, because of the feasibility constraint on resource allocations imposed by the endowment, in the range in which equilibrium is in mixed strategies each player would like to win each battlefield that he wins by barely winning and lose each battlefield that he loses by a large margin (thereby assuring that he has more or his rival has less to allocate elsewhere). Indeed, these offsetting incentives are the source of the instability that leads to the requirement of mixed equilibrium strategies. In asymmetric conflicts, these incentives are also the source of the nature of the weak player's equilibrium strategies, when they are uniquely determined. Because the weak player is at a disadvantage, competing toe-to-toe in every battlefield requires stochastically allocating a lower level of the resource to each battlefield. This however, means a high probability of losing any given contested battlefield, an outcome that the weak player would rather suffer while allocating zero to that battlefield. As a consequence, the weak player chooses instead to bid aggressively on a randomly chosen subset of battlefields, competing on those battlefields on equal footing with the stronger player, while expending zero in the remaining battlefields, in effect conceding them. In turn, these offsetting incentives cause the strong player to randomize uniformly across a randomly chosen subset of battlefields and to lock down the remaining battlefields by allocating an amount equal to the weak player's endowment to those battlefields.

One article related to our contribution is Arbatskaya and Mialon (2010a).[3] This article examines a single winner-take-all contest for a prize of common and known value in which each player's level of resource employed is a one-dimensional output derived from multiple inputs or activities entered into a Cobb-Douglas production function. The players' respective

[3] See also Arbatskaya and Mialon (2010b) which examines a dynamic version of Arbatskaya and Mialon (2010a).

outputs are then inserted into a lottery contest success function, so that the probability that each player wins the prize is the ratio of his output to the sum of the two players' outputs (or one-half in the event that both outputs are zero). Players incur constant unit costs of employing the inputs, which may be asymmetric across both inputs and players. Inputs may also enter the production function with different exponents.

One of the issues examined by Arbatskaya and Mialon is the effect of increasing the number of inputs. Because the multiple inputs determine an aggregate output variable, rather than giving rise to an extra component contest, the effects of increasing the number of inputs or activities in their model is quite different from increasing the number of battlefields in our model. Specifically, adding an input in the Arbatskaya-Mialon model leads to two distinct effects. First, it tends to increase the discriminatory power of the contest, leading to more intense competition and more dissipation of the value of the prize. Second, depending on the initial strength of each player in the contest and the players' per unit costs of the input, adding an input may alter the relative strengths of the players in the overall contest. Because symmetry in player strengths tends to increase competition, this effect could go either way, intensifying competition if the addition makes players more symmetric than they were previously or softening competition if it makes players more asymmetric. None of these effects involve the basic forces at work in the game we examine where, by forcing the strong player to spread his resources more thinly across component contests, the weak player increases the incidence of favorable strategic mismatches.

Also related are issues such as sabotage which can be seen as an additional input in a one-dimensional aggregate output variable for a single winner-take-all contest. In contests with sabotage (c.f. Lazear 1989, Konrad 2000, Chen 2003, and Münster 2007) each player's aggregate output is a function of his own productive effort and the effort that each of his rivals expends on sabotaging, or hindering, his output. As in Arbatskaya and Mialon, the outcome of the contest is determined by the players' aggregate outputs. In general, stronger players, as measured by the cost of effort, are more heavily sabotaged and thus the relative positions of weaker players improve. The intuition for this result is that at the margin hindering a stronger opponent increases a player's odds of winning more than hindering a weaker opponent. Also related is the issue doping or performance enhancement in contests (cf. Berentsen 2002, Konrad 2005, and Kräkel 2007). In these studies, each player's aggregate output is a function of his productive effort and his investment in doping. Again, in each of these examples the basic forces at work differ from those forces that we describe above.

The remainder of this paper is organized as follows. In the next section, we describe the two-stage game including the pre-conflict and conflict stages. In section 3 we determine the subgame perfect equilibrium strategies and payoffs and discuss the implications of the results. Section 4 concludes.

2 The Model

To allow for an endogenous number of battlefields, we examine a two-player, two-stage game that augments the classic Colonel Blotto game by allowing for a pre-conflict battlefield creation stage. Initially, there are a finite number $n_0 \geq 3$, of independent battlefields, and each of the two players has a fixed endowment of a resource (henceforth, available forces or budget). Let S denote the strong player with budget B_S, and W denote the weak player with budget B_W, where $B_S \geq B_W$.

Let the two-stage game with resource endowments (B_W, B_S) and the initial number of battlefields n_0 be denoted by $\Gamma(B_W, B_S, n_0)$. We start the description of the model in the first stage.

Pre-Conflict Stage

In the first or pre-conflict stage both of the players have the opportunity to invest in the creation of additional battlefields. If player S chooses to create n_s new battlefields and player W chooses to create n_w new battlefields, then the number of battlefields changes from the initial number n_0 to the updated number n as follows

$$n = n_0 + n_s + n_w. \tag{1}$$

The costs of creating additional battlefields are symmetric and given by the cost function $c : \mathbb{R}_+ \to \mathbb{R}_+$, which is assumed to be strictly increasing, strictly convex, and continuously differentiable on $\mathbb{R}_+$. Although each player is restricted to choosing a nonnegative integer number of battlefields to create, assuming that the cost function is defined over the set of nonnegative real numbers simplifies the analysis. To ensure an interior equilibrium, we will also assume that $c'(0) = 0$. Each of the players is risk neutral and maximizes the expected proportion of battles won minus the opportunity cost of the funds invested in the creation of new battlefields.

Conflict Stage

In the final or conflict stage, the two players play a Colonel Blotto game with the updated number of battlefields. The final stage's Colonel Blotto game is described as follows. The players simultaneously allocate the resource (henceforth, forces) across the updated (finite) number, $n \geq 3$, of independent battlefields. Each player has a fixed level of available forces. In order to keep the total value of the final stage's Colonel Blotto game constant with respect to changes in the number of battlefields, each player is assumed to maximize the expected proportion of battlefields won. Thus, the total value of the final stage's Colonel Blotto game is one, or, equivalently, each battlefield has a common value of $(1/n)$ for each player.

Let $\mathbf{x}_i$ denote player i's n-tuple of force allocations $(x_{i,1}, \ldots, x_{i,j}, \ldots, x_{i,n})$, which specifies the allocation of force to each battlefield j. In the case that the players allocate the same level of force to a particular battlefield, it is assumed that the strong player (S) wins the battlefield. This specification of the tie-breaking rule avoids the need to have the strong player (S) provide an allocation of force arbitrarily close to, but above, the weak player's (W's) maximal allocation of force. A range of tie-breaking rules yields similar results.

In each battlefield j, let $\pi_{i,j}$ denote the payoff to player i when he allocates force $x_{i,j}$ and his rival allocates $x_{-i,j}$:

$$\pi_{i,j}\left(x_{i,j}, x_{-i,j}\right) = \begin{cases} \frac{1}{n} & \text{if } \; x_{i,j} > x_{-i,j} \\ 0 & \text{if } \; x_{i,j} < x_{-i,j} \end{cases}$$

where ties are handled as described above. Given the updated number of battlefields n and the force allocation profile $(\mathbf{x}_i, \mathbf{x}_{-i})$, each player i's total payoff, denoted by π_i, for the two-stage game $\Gamma(B_W, B_S, n_0)$ is equal to the proportion of battlefields to which player i allocates a higher level of force minus the opportunity cost of the funds invested in the creation of new battlefields

$$\pi_i\left(\{\mathbf{x}_i, n_i\}, \{\mathbf{x}_{-i}, n_{-i}\}\right) = \sum_{j=1}^{n} \pi_{i,j} - cn_i.$$

In each battlefield the allocation of force must be nonnegative. For player i, the set of feasible force allocations across the n battlefields is denoted by

$$\mathfrak{B}_i = \left\{ \mathbf{x} \in \mathbb{R}_+^n \middle| \sum_{j=1}^{n} x_{i,j} \leq B_i \right\}.$$

If the strong player (S) has enough resources (B_S) to outbid the weaker player's (W's) maximal force allocation B_W on all n battlefields (i.e., if $B_S \geq nB_W$) then there, trivially, exists a pure-strategy equilibrium, and the strong player (S) wins all of the battlefields. It is well known that for the remaining parameter configurations, $(1/n)B_S < B_W \leq B_S$, there is no pure-strategy equilibrium for this class of games. A mixed strategy, which we term a *distribution of force*, for player i is an n-variate distribution function $P_i : \mathbb{R}^n_+ \to [0,1]$ with support (denoted $\mathrm{Supp}(P_i)$) contained in player i's set of feasible force allocations $\mathfrak{B}_i$ and with the set of one-dimensional marginal distribution functions $\{F_{i,j}\}_{j=1}^n$, one univariate marginal distribution function for each battlefield j. The n-tuple of player i's allocation of force to each of the n battlefields is a random n-tuple drawn from the n-variate distribution function P_i. When players employ mixed local strategies in the second stage, each player's objective in the two-stage game is to maximize the expected proportion of battlefields won minus the opportunity cost of creating new battlefields.

3 Subgame Perfect Equilibrium

Conflict Stage

We begin in the final conflict stage and move back through the game tree to the pre-conflict stage. Theorem 1 summarizes Roberson's (2006) characterization of the unique equilibrium payoffs in the Colonel Blotto game. Recall that the floor function $\lfloor x \rfloor$ provides the largest integer less than or equal to x, and that the ceiling function $\lceil x \rceil$ provides the smallest integer greater than or equal to x.

Theorem 1 (Roberson (2006))**.** *Let n denote the updated number of battlefields in a subgame beginning at the final conflict stage of the game. The unique Nash equilibrium payoffs of the final stage Colonel Blotto game are given as follows:*

1. If B_W/B_S satisfies $(2/n) < (B_W/B_S) \leq 1$, then the expected proportion of battlefields won by the weak player (W) is $(B_W/2B_S)$ and the expected proportion of battlefields won by the strong player (S) is $1 - (B_W/2B_S)$.

2. If B_W/B_S satisfies $(1/(n-1)) \leq (B_W/B_S) \leq (2/n)$, then the expected proportion of battlefields won by the weak player (W) is $(2/n)-(2B_S/(n^2B_W))$ and the expected proportion of battlefields won by the strong player (S) is $1-(2/n)+(2B_S/(n^2B_W))$.

3. If B_W/B_S satisfies $(1/n) < (B_W/B_S) < (1/(n-1))$, then define $m = \lceil(B_W/(B_S - B_W(n-1)))\rceil$, and note that $2 \leq m < \infty$. The expected proportion of battlefields won by the weak player (W) is $(2m-2)/(mn^2)$, and the expected proportion of battlefields won by the strong player (S) is $1-(2m-2)/(mn^2)$.

Theorem 1 follows immediately from Theorems 2, 3, and 5 of Roberson (2006), which demonstrate the existence of equilibrium, properties of equilibrium univariate marginal distributions, and the unique equilibrium payoffs in the two-payer Colonel Blotto game with asymmetric budgets. Note that uniqueness of the equilibrium expected proportions of battlefields won follows immediately from the fact that the Colonel Blotto game is constant sum.

Before moving on to the pre-conflict stage, it is helpful to briefly summarize the nature of the optimal strategies in the conflict stage.[4] Beginning with the most asymmetric parameter range and moving towards symmetric endowments, if $B_S \geq nB_W$ then the strong player can trivially win each of the n battlefields by allocating B_W to each battlefield. If $(1/n) < (B_W/B_S) < (1/(n-1))$, then the weak player utilizes a stochastic guerilla warfare strategy consisting of a joint distribution function that randomly allocates zero resources to all but two of the battlefields, and the strong player utilizes a stochastic complete coverage strategy consisting of a joint distribution function that randomly allocates B_W to all but two of the battlefields. In the two battlefields in which the weaker player competes, he randomizes over a set of bivariate mass points. Similarly, in the two battlefields which the stronger player does not lock down by allocating B_W, he also randomizes over a set of bivariate mass

[4]For more information on the equilibrium joint distribution functions see Theorems 2, 3, and 5 of Roberson (2006).

points. If $(1/(n-1)) \leq (B_W/B_S) \leq (2/n)$, then the weak player continues to randomly allocate zero resources to all but two battlefields, but on the remaining two battlefields he now randomizes continuously. The strong player continues to lock down a random subset of the battlefields with an allocation of B_W, but the expected proportion of battlefields that are locked down is decreasing in the ratio of the weak to strong resource endowments, (B_W/B_S). If $(2/n) < (B_W/B_S) \leq 1$, then the weak player continues to randomly allocate zero resources to a subset of the battlefields, but now the proportion of battlefields that are conceded is decreasing in the ratio of the weak to strong resource endowments. Over this last range, the strong player optimally chooses not to lock down any of the battlefields.

Pre-Conflict Stage

We now examine the pre-conflict stage. We will focus on cases 1 and 2 of Theorem 1. The analysis for case 3 differs in that both players' expected proportions of battlefields won have points of discontinuity, which makes the analysis considerably more complicated. Although we will not treat case 3 formally in this article, the basic intuition underlying the analysis in this parameter range is similar to that of case 2.

In the pre-conflict stage, each player maximizes his total expected payoff for the game by choosing a number of new battlefields to create. For each player $i = W, S$ let $E(\sum_{i=1}^{n} \pi_{i,j})$ denote the proportion of battlefields that player i expects to win given that there are n battlefields. For player i, this optimization problem may be written as

$$\max_{n_i \in \mathbb{Z}_+} E\Big(\pi_i \Big| n_i, n_{-i}\Big) = E\Big(\sum_{i=1}^{n_0+n_i+n_{-i}} \pi_{i,j}\Big) - c\Big(n_i\Big) \tag{2}$$

Before stating the unique subgame-perfect equilibrium strategies in stage one, it is helpful to illustrate how the players' total expected payoffs vary as n increases. The solid curve in Figure 1 provides the weaker player's expected proportion of battlefields won as a function of B_W/B_S. In case 3 of Theorem 1 both players' expected proportions of battlefields won as a function of B_W/B_S are step functions, and in Figure 1 the solid curve in the case 3 range runs through the left endpoints of the individual steps. The dashed line in Figure 1 shows how the weak player's expected proportion of battlefields won increases as the number of battlefields increases. Because this is a constant-sum game, the strong player's expected proportion of battlefields won necessarily decreases as the number of battlefields increases. Finally, the dotted line in Figure 1 shows the weak player's maximal expected proportion of battlefields won as the number of battlefields becomes arbitrarily large.

[Insert Figure 1 here]

Observe that in the case 1 parameter range $((2/n) < (B_W/B_S) \leq 1)$ the player's expected payoffs are linear with respect to (B_W/B_S). As shown in Figure 1, within the case 1 range the weak player's proportion of expected battlefields won is at its maximal level and is invariant with respect to increases in the number of battlefields. Moving into the case 2 parameter range $((1/(n-1)) \leq (B_W/B_S) \leq (2/n))$ both players' expected proportions of battlefields won are continuous functions of (B_W/B_S). As illustrated in Figure 1, as the number of battlefields increases the boundary between case 1 and case 2 shifts left. As a result the weak player's expected proportion of battlefields won increases by $\Delta E\left(\sum_{i=1}^{n} \pi_{W,j}\right)$, and depending on the cost of creating additional battlefields $c(\cdot)$, the weak player has a potential gain in his total payoff from increasing the number of battlefields.

Theorem 2. *In the pre-conflict stage of the game with n_0 initial battlefields and resource endowments B_W and B_S that satisfy $(1/(n_0-1)) \leq (B_W/B_S) \leq 1$ (i.e., the case 1 and 2 parameter configurations), the subgame perfect equilibrium stage-game strategies are described as follows:*

1) If B_W/B_S satisfies $(2/n_0) \leq (B_W/B_S) \leq 1$, then $n_s^* = 0$ and $n_w^* = 0$.

2) If B_W/B_S satisfies $(1/(n_0-1)) \leq (B_W/B_S) < (2/n_0)$, then $n_s^* = 0$, and if $n_{w,R} \in (0,(2B_S/B_W)-n_0)$ denotes the real number that implicitly solves

$$-\frac{2}{(n_0+n_{w,R})^2}+\frac{4B_S}{B_W(n_0+n_{w,R})^3}-c'(n_{w,R})=0,$$

then

a. $n_w^* = \lfloor n_{w,R} \rfloor$ *if* $E\Big(\pi_w \Big| \lfloor n_{w,R} \rfloor, 0\Big) > E\Big(\pi_w \Big| \lceil n_{w,R} \rceil, 0\Big)$.

b. $n_w^* = \lceil n_{w,R} \rceil$ *if* $E\Big(\pi_w \Big| \lfloor n_{w,R} \rfloor, 0\Big) < E\Big(\pi_w \Big| \lceil n_{w,R} \rceil, 0\Big)$.

c. $n_w^* = \lfloor n_{w,R} \rfloor$ *or* $\lceil n_{w,R} \rceil$ *if* $E\Big(\pi_w \Big| \lfloor n_{w,R} \rfloor, 0\Big) = E\Big(\pi_w \Big| \lceil n_{w,R} \rceil, 0\Big)$.[5]

Proof. We begin with the strong player. As is clear from the expressions for the stage payoffs in parts 1 and 2 of Theorem 1 and as is illustrated in Figure 1, for all n_0, B_W, and B_S that satisfy $(1/(n_0-1)) \leq (B_W/B_S) \leq 1$ the strong player's expected proportion of battlefields won weakly decreases in n, and therefore the strong player optimally chooses $n_s^* = 0$ regardless of the weak player's choice of n_w.

Moving on to the weak player, if n_0, B_W, and B_S satisfy $(2/n_0) \leq (B_W/B_S) \leq 1$ as in part 1 of Theorem 2, then from part 1 of Theorem 1 and as shown in Figure 1, the expected proportion of battlefields won is invariant to the number of battlefields for the weak player, and the weak player optimally chooses $n_w^* = 0$.

For part 2 of Theorem 2, $((1/(n_0-1)) \leq (B_W/B_S) < (2/n_0))$, it will be helpful to note the following. Recalling equation (2) and given that $n_s^* = 0$, the weak player's optimization problem may be written as

$$\max_{n_w \in \mathbb{Z}_+} E\Big(\pi_w \Big| n_w\Big) = E\Big(\sum_{i=1}^{n_0+n_w} \pi_{w,j}\Big) - c\Big(n_w\Big). \tag{3}$$

Given that initially case 2 of Theorem 1 applies, the weak player's expected proportion of

[5]In case 2 c, the weak player may randomize between $\lfloor n_{w,R} \rfloor$ and $\lceil n_{w,R} \rceil$.

battlefields won is given by

$$E\Big(\sum_{i=1}^{n_0+n_w}\pi_{w,j}\Big)=\begin{cases}\frac{2}{n_0+n_w}-\frac{2B_S}{B_W(n_0+n_w)^2} & \text{if } 0\leq n_w\leq \frac{2B_S}{B_W}-n_0\\ \frac{B_W}{2B_S} & \text{if } n_w>\frac{2B_S}{B_W}-n_0\end{cases} \tag{4}$$

which, with respect to n_w, is continuously differentiable on $\mathbb{R}_+$ and concave. By assumption, the cost function $c(\cdot)$ is continuously differentiable on $\mathbb{R}_+$ and strictly convex.

We will now show that (i) the weak player's objective function is strictly concave, (ii) there exists a unique real number $n_{w,R}\in(0,(2B_S/B_W)-n_0)$ that maximizes the weak player's objective function, and (iii) given $n_{w,R}$ the weak player's optimal (discrete) number of battlefields to create is either $\lceil n_{w,R}\rceil$, or $\lfloor n_{w,R}\rfloor$, or both.

We begin with point (i), the strict concavity of the weak player's objective function. In equation (3), the weak player's objective function is equal to a concave function minus a strictly convex function. Thus, the strict concavity of the weak player's objective function follows immediately.

For point (ii), note that by assumption $c'(0)=0$ and, thus, it follows that

$$\frac{\partial E\Big(\pi_w\big|n_{w,R}\Big)}{\partial n_{w,R}}\bigg|_{n_{w,R}=0}=-\frac{2}{(n_0)^2}+\frac{4B_S}{B_W(n_0)^3}>0, \tag{5}$$

where the strict inequality holds because in part 2 of Theorem 2 the initial parameters satisfy $(B_W/B_S)<(2/n_0)$. Similarly, note that from the expression for the expected number of battlefields won, given in equation (4), it follows that

$$\frac{\partial E\Big(\pi_w\big|n_{w,R}\Big)}{\partial n_{w,R}}\bigg|_{n_{w,R}\geq\frac{2B_S}{B_W}-n_0}=-c'(n_{w,R})<0. \tag{6}$$

Combining equations (5) and (6) with the strict concavity of the weak player's objective function, we have shown that there exists a unique real number $n_{w,R} \in (0, (2B_S/B_W) - n_0)$ that maximizes the weak player's objective function.

Point (iii), that the weak player's optimal (discrete) number of battlefields to create, n_w^*, is either or both $\lceil n_{w,R} \rceil$ and $\lfloor n_{w,R} \rfloor$, follows from the strict concavity of the objective function over $\mathbb{R}_+$. This completes the proof of Theorem 2. $\square$

When the asymmetry in the players' initial resource endowments exceeds a threshold $((B_W/B_S) < (2/n_0))$, the weaker player chooses to add additional battlefields if the cost is sufficiently small. By adding additional battlefields the weaker player forces the stronger player to spread his resources over a larger number of battlefields. Outside of the pure strategy range where $((B_W/B_S) \leq (1/n_0))$ the weaker player employs a guerilla warfare strategy that stochastically allocates zero forces to a subset of the battlefields. This remains true as the number of battlefields increases, but the stronger player's position is weakened by the thinning out of his forces.

To obtain some intuition as to the magnitude of this effect, suppose the ratio of the weak player's endowment to the strong player's endowment is 0.12. Then when the initial number of battlefields n_0 satisfies $n_0 \leq 8$, the weak player has no chance of winning any battlefield. The strong player can simply allocate an amount greater than the weak player's endowment to every battlefield and win with certainty. If the weak player increases the number of battlefields to $n = 9$, from Theorem 1, part 3, the expected proportion of battlefields won by the weak player is .016. When $n = 10$, from Theorem 1, part 2, this proportion increases to .033. For $n \geq 17$, Theorem 1, part 1, gives us .06.

Suppose instead that the ratio of the weak player's endowment to the strong player's endowment is 0.22. When the initial number of battlefields n_0 satisfies $n_0 \leq 4$, the weak player again has no chance of winning any battlefield. If the weak player increases the number of battlefields to $n = 5$, from Theorem 1, part 3, the expected proportion of battlefields won by the weak player is .04. When $n = 9$, from Theorem 1, part 2, this proportion increases to .11, and from Theorem 1, part 1, it remains at that level for $n \geq 10$.

In ending, it is important to note that these effects arise from the combination of sufficiently asymmetric endowments and the cost effectiveness of expanding the set of battlefields. The Colonel Blotto game does not as a general rule confer a decisive advantage to the player with the larger endowment. For instance, when there are three battlefields ($n_0 = 3$) a player with a 25% higher endowment than his rival wins on average a proportion .6 of the battlefields. The weaker rival wins on average a proportion .4 of the battlefields. That is, despite the asymmetry in this case, the weak player is able on average to secure slightly more than one battlefield victory. Moreover, over this range of parameters, since the results in Theorem 2, part 1 apply, there is no incentive for the weak player to expand the number of battlefields in the contest. Asymmetry by itself is not sufficient to justify an endogenous increase in dimensionality.

4 Conclusion

This article examines a two-stage model of conflict based on the classic Colonel Blotto game with n battlefields. In the first stage, each of two resource-constrained players has the opportunity to incur a cost to increase the number of battlefields that are contested. In the second stage the players play the resulting Colonel Blotto game with their respective endowments of the resource taken parametrically and the number of battlefields determined by the endogenous choices in the first stage. Players are assumed to maximize the expected proportion of the battlefields won, net of their costs of battlefield expansion.

Our focus is on the case where players are asymmetrically endowed: that is, asymmetric warfare. In the context of our two-stage game, we demonstrate that — given a sufficiently small cost of adding battlefields — for either a sufficiently small initial number of battlefields or sufficiently asymmetric endowments of the resource, the weak player chooses to add battlefields in any subgame perfect equilibrium. In contrast, the strong player, never adds battlefields. We therefore provide a model of the endogeneity of the number of battlefields, or the dimensionality of the conflict.

The basic force at work in our model is one that, to our knowledge, has not been modeled elsewhere: In asymmetric conflicts, by forcing the strong player to spread his resources more thinly, the weak player increases the incidence of favorable mismatches, in which the weak player, despite his overall disadvantage, obtains a battlefield-specific advantage.

The topic of asymmetric conflict has attracted considerable interest since the biblical story of David and Goliath. Our analysis demonstrates that the possibility of increasing the number of battlefields through an endogenous process can have a meaningful impact on outcomes, to the benefit of weaker players. We also believe that this observation has real-world application in many strategic interactions with multiple components such as war, sports matches, and business.

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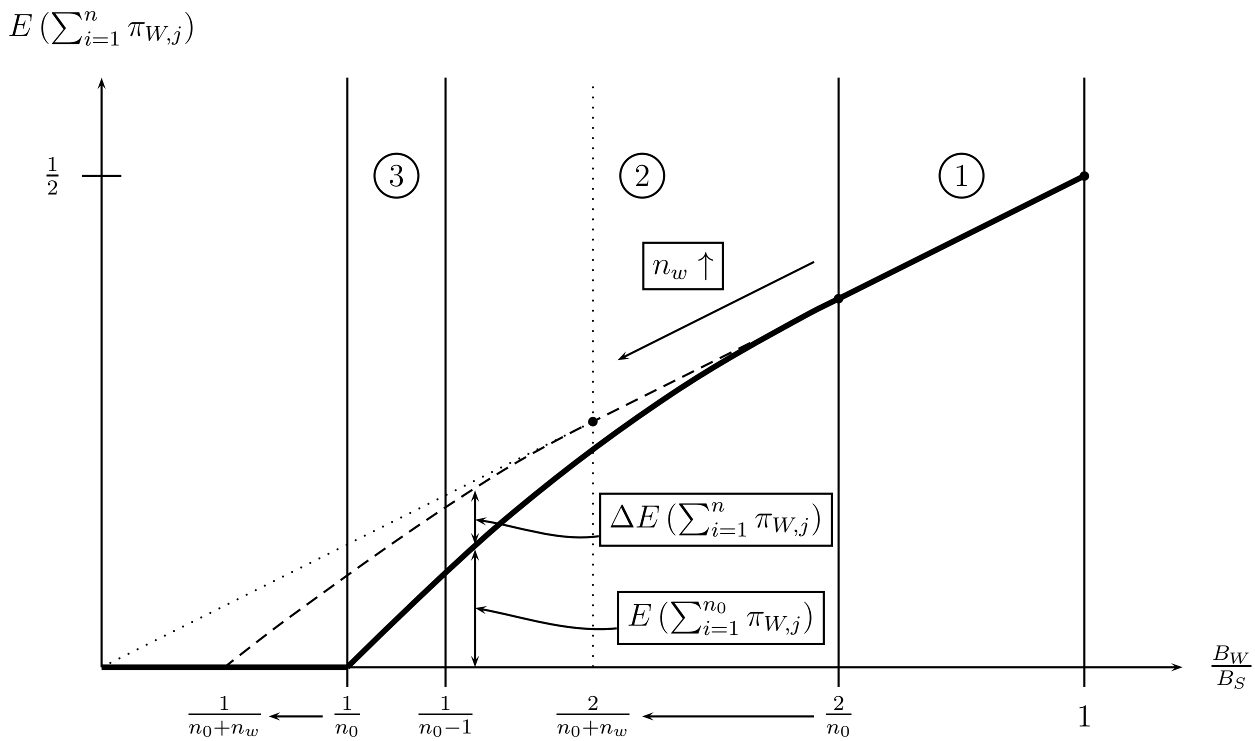

Figure 1: Player W's stage 2 unique subgame perfect equilibrium expected proportion of battlefields won