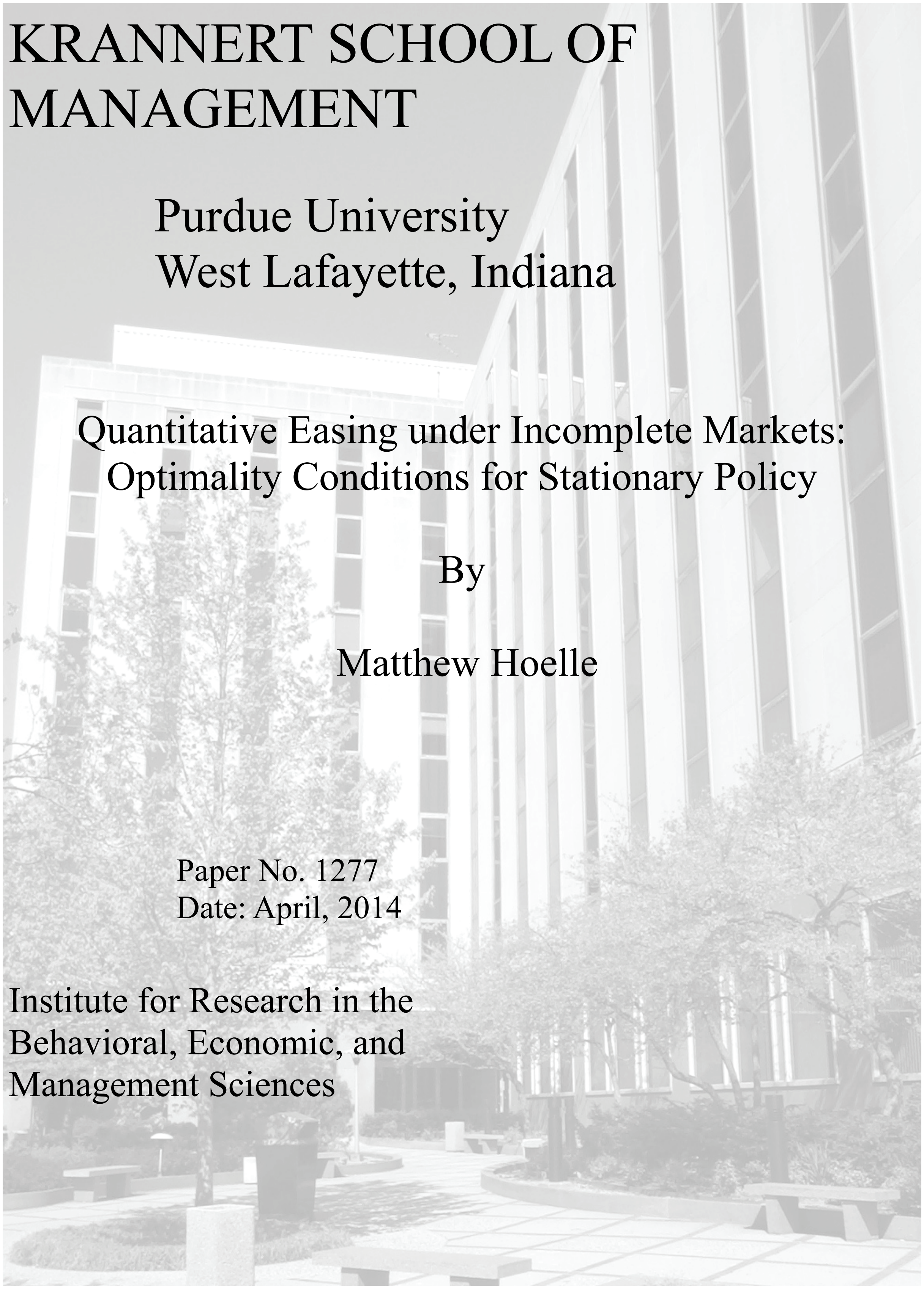

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Quantitative Easing under Incomplete Markets: Optimality Conditions for Stationary Policy

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Quantitative Easing under Incomplete Markets: Optimality Conditions for Stationary Policy

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Abstract

In a stochastic economy, the rebalancing of short and long term government debt positions can have real effects when markets are incomplete. Monetary policy aims to harnass these real effects to maximize social welfare. The policy rules can either be stationary, in which the current policy choice vector only depends on the current shock realization, or nonstationary, in which the policy choice vector is recursively updated. This paper characterizes the conditions under which a stationary policy rule is optimal.

Keywords monetary policy rules – asset span – term structure of interest rates – Markov equilibrium

JEL Classification **D52, E43, E44, E52**

1 Introduction

In a stochastic economy, the rebalancing of short and long term government debt positions can have real effects whenever markets are incomplete. Quantitative easing is a form of monetary policy that operates through the rebalancing of these short and long term debt positions. The monetary policy rules can either by stationary, meaning they are history-independent, or nonstationary, meaning they are determined by an autoregressive updating rule. This paper characterizes the conditions under which stationary policy suffices to maximize the objective function of the monetary authority.

The model to address quantitative easing is a stochastic nominal economy in which money has value via cash-in-advance constraints (as in Lucas and Stokey, 1983).[1] Heterogeneous households face endowment risk, and insure against this risk using a portfolio of nominal bonds. The bonds are nominally risk-free, but vary in terms of their maturity. The term structure of interest rates is determined by central bank policy, and except for a certain knife-edge policy, the bond yields are linearly independent across maturities meaning that an entire portfolio of assets affords better risk-sharing opportunities than a single short-term bond.

Hoelle (2014) analyzes the welfare implications under complete markets, namely when the number of assets equals the number of states of uncertainty each period. The main result of Hoelle (2014) is that all equilibrium allocations are Pareto efficient. The policy is only required to be feasible in the equilibrium definition, and is not chosen to maximize any central bank objective function. This result renders any policy exercises with complete markets both trivial and uninteresting.

This paper, therefore, derives its main result under incomplete markets. Policy in such a setting can have welfare effects, so I must endow the central bank with an objective function. The objective function will be a social welfare function equal to the weighted sum of household utilities for a given vector of Pareto weights. Under complete markets, the allocation that maximizes such an objective function is a Pareto efficient allocation, and I call this allocation the complete markets allocation.

The policy rule is the sequence of central bank variables that include the government debt levels (issued across the same set of maturities as the bonds), the interest rates, and the inflation rates. The policy rule can either be stationary or nonstationary. A stationary policy rule is such that the central bank variables are history-independent and only depend on the current shock realization. A nonstationary policy is such that the central bank variables follow an autoregressive updating rule.

I introduce recursive formulations for the equilibrium concept. The recursive formulations are termed Markov equilibria and allow me to characterize the entire set of competitive equilibria under both types of policy rules. For nonstationary policy in which the central bank variables can be changed every period, indeterminacy can become crippling. High nominal interest rates today are consistent with high inflation rates next period, which are

[1] Magill and Quinzii (1992 and 1996) use a form of a cash-in-advance constraint in order to model monetary economies using only 2 periods. The model in this paper is a full-fledged infinite time horizon model, with the asset structure of Magill and Quinzii (1992 and 1996) and the cash-in-advance constraints of Lucas and Stokey (1983).

consistent with low nominal interest rates next period, and the cycle continues. In response, I require that the central bank variables chosen in any period must be consistent with the equilibrium equations both on the equilibrium path and off the equilibrium path for the contingency whereby central bank variables remain constant for all future periods.

The recursive formulations are conducive to theory and also numerical computation. This paper only focuses on the former, while future work will use the recursive formulations to numerically solve for the optimal policies and resulting allocations of the model. Theory is very instructive for this process. Even under incomplete markets, the complete markets allocation can still be supported by optimal policy provided that (i) the number of households is no greater than the number of assets and (ii) the number of assets is one less than the number of states of uncertainty. Under these conditions, it is easy to analytically solve for the optimal policies and resulting allocations as all allocations are identical and Pareto efficient. Included in the set of optimal policies are stationary policies.

The sufficient conditions are also necessary (generically), meaning that in their absence, the optimal policies are both nonstationary and support a Pareto inefficient allocation. These policies can only be solved numerically using recursive computational methods.

This paper only models monetary policy. Any vector of interest rates, government debt levels, and inflation rates satisfying no arbitrage and the government budget constraints can be supported as equilibrium central bank variables. The optimal policy are those variables in this set that maximize the central bank objective function. The ability of a change in the central bank variables to effect a welfare change depends upon the degree to which the central bank variables can change the asset span that households face. This is determined by the state variables of the economy at the time when the change is made and the numbers of households, assets, and states of uncertainty.

Ricardian fiscal policy can be added to the model.[2] If the taxes are targeted, then government debt adjusts to satisfy budget balance. If the government debt levels are targeted, then taxes adjust. There is a natural indeterminacy, even when the monetary authority is adopting optimal monetary policy. Fiscal policy will introduce additional degrees of freedom, or additional policy tools, into the monetary-fiscal authority problem. These additional tools can be used such that: (i) joint monetary and fiscal policy is able to support more asset spans allowing for greater risk-sharing on the parts of households (and increased social welfare) or (ii) the central bank variables meet some stated targeting objective, such as interest rate

[2] Fiscal policy can take one of two forms: (i) non-Ricardian policy means that the future tax rates are fixed and government debt must adjust to balance the budget and (ii) Ricardian policy is such that both the government debt and taxes are indeterminate, where the only requirement is that they satisfy budget balance.

targeting or money supply targeting.[3]

The following section connects the results and methodology of the present paper with the diverse literature on monetary policy.

2 Literature Review

Two classes of models are used to determine optimal monetary policies. The first class of models is New Keynesian models and the second class of models is neoclassical models.

In the New Keynesian class of models, the frictions are nominal rigidities and monopolistic competition. The objective function of central banks is to return the economy to the steady state. The optimal monetary policy has zero or close to zero inflation in order to mitigate the effects of the nominal rigidities. The optimal policy is in terms of policy rules, which are committed to by the central bank, instead of discretionary policy changes. A representative sample of the important papers using this class of models in a closed economy include Galí (1992), Sims (1992), Bernanke and Mihov (1998), Christiano et al. (1999), Taylor (1999), Clarida et al. (2000), Woodford (2003), Schmitt-Grohé and Uribe (2004), Uhlig (2005), and the recent Gertler and Karadi (2011) that extend the business cycle models of Christiano et al. (2005) and Smets and Wouters (2007).

In the neoclassical class of models, policy rules are once again found to be optimal compared to discretionary policy (since the latter suffer from time inconsistency as shown in Kydland and Prescott (1977)). The optimal policy is able to achieve welfare gains through changes in the expected future inflation rates. For this reason, and in contrast to the New Keynesian class of models, optimal policy responds to the state realizations. Policy is considered along the equilibrium path, but with only a representative household, the objective function of the central bank includes a function that proxies for the economic costs of inflation. Examples of this class of models include Sargent and Wallace (1975), Lucas and Stokey (1983), Barro and Gordon (1983), Chari et al. (1991), and Calvo and Guidotti (1993).

Specifically on the topic of quantitative easing, both Krishnamurthy and Vissing-Jorgensen (2011) and Peiris and Polemarchakis (2013) analyze the theoretical implications of such policies. This current paper, unlike its predecessors, will characterize the optimal policies of quantitative easing. This is possible given the incomplete markets structure. Peiris and Polemarchakis (2013) consider a complete markets structure under which all feasible policies support a Pareto efficient allocation (see the proof in Hoelle, 2014).

The current paper belongs to the neoclassical class of models, with elements borrowed from general equilibrium (a stochastic environment with incomplete markets and heteroge-

[3] Such targets are chosen to satisfy the unmodeled goals of fixing expectations.

neous households) to allow for real effects of monetary policy.[4] The asset structure of the model, and resulting portfolio choice problem for households, is essential to understand the effects of quantitative easing, which was previously defined as the adjustment of short and long term government debt positions.

Contrary to New Keynesian models with a representative household and a single bond, the dynamics in my model are derived along the equilibrium path, which follows a Markov process. The dynamics in New Keynesian models are derived off the equilibrium path as log-linear deviations from the steady state. In both approaches, the real effects of inflation are endogenously determined, though the New Keynesian costs result from the nominal rigidity friction while the neoclassical costs result from the incomplete markets friction. In both cases, the objective of the central bank is to minimize the costs of inflation, though in this paper that objective is formalized as an allocational efficiency motive.

The remainder of the paper is organized as follows. Section 3 introduces the model. Section 4 introduces the recursive equilibrium concept of a Markov equilibrium and verifies several important properties. Section 5 characterizes the conditions under which stationary policy is optimal. Section 6 provides concluding remarks and the proofs of the main results are contained in Appendix A.

3 The Model

The model describes a closed economy with a single infinite-lived monetary authority. I prefer the term "monetary authority" to "central bank" as I want to focus exclusively on monetary policy and to abstract away from the other duties typically assigned to central banks.

Time is discrete and infinite with time periods $t \in \{0, 1, ...\}$. The filtration of uncertainty follows a one-period Markov process with finite state space $\mathbf{S} = \{1, ..., S\}$. The realized state of uncertainty in any period t, denoted s_t, is a function only of the realized state in the previous period $t-1$, denoted s_{t-1}. This random process is characterized by a transition matrix $\Pi \in \mathbb{R}^{S,S}$ whose elements are $\pi(s, s')$ for row s and column s'. I assume that no single state is an absorbing state, meaning that $\pi(s, \sigma) < 1$ for all pairs of consecutive states $(s, \sigma) \in \mathbf{S}^2$.

The history of all realizations up to and including the current realization completely characterize the date-event and are required to uniquely identify the markets, household decisions, and policy choices. Define the history of realizations up to and include the real-

[4] The idea that monetary policy can have real effects under incomplete markets comes from the 2-period model of Magill and Quinzii (1992 and 1996).

ization s_t in period t as $s^t = (s_0, s_1, ..., s_t)$. For convenience, $\pi(s^t|s^\tau)$ for any $t > \tau$ refers to the probability that history s^t is realized conditional on the history s^τ. Additionally, let $s^{t+j} \succ s^t$ refer to the S^j histories $(s^t, \sigma_1, ..., \sigma_j)_{(\sigma_1,...,\sigma_j)\in\mathbf{S}^j}$ that are realized condition upon s^t in $j \geq 0$ periods. The trivial specification $s^{t+0} \succ s^t$ refers to the singleton $\{s^t\}$.

3.1 Households

In each date-event, a finite number of households $h \in \mathbf{H} = \{1, ..., H\}$ trade and consume a single physical commodity. Households receive endowments $e^h \in \ell^\infty_{++}$.[5] I assume that the endowments are stationary. Define the stationary endowment mapping as $\mathbf{e}^h : \mathbf{S} \to \mathbb{R}_{++}$ such that $e^h(s^t) = \mathbf{e}^h(s_t)$ for all date-events. Denote the aggregate endowment as $E : \mathbf{S} \to \mathbb{R}_{++}$ such that $E(s) = \sum_{h\in\mathbf{H}} \mathbf{e}^h(s)\ \forall s \in \mathbf{S}$. The model permits aggregate risk, i.e., $E(s) \neq E(\sigma)$ for some $s, \sigma \in \mathbf{S}$.

The consumption by household h in date-event s^t is denoted $c^h(s^t) \in \mathbb{R}_+$. The vector of consumption for household h is denoted $c^h = \left(c^h(s^t)\right)_{s^t\succ s_0;t\geq 0} \in \ell^\infty_+$.[6]

The household preferences are assumed to be identical and satisfy constant relative risk aversion:

$$\sum_{t=0}^{\infty} \beta^t \sum_{s^t\succ s_0} \pi\left(s^t|s_0\right) u\left(c^h\left(s^t\right)\right). \tag{1}$$

Assumption 1 The discount factor $\beta \in (0, 1)$ and $u(c) = \frac{c^{1-\rho}}{1-\rho}$ for $\rho \neq 1$ and $u(c) = \ln(c)$ for $\rho = 1$.

The exact form in Assumption 2 is chosen for convenience, as the theoretical results only require identical and homothetic preferences (of which CRRA is the leading example).

In each date-event s^t, the money supply is $M(s^t) > 0$ and the nominal price level is $p(s^t) > 0$.

In each date-event, a finite number of nominal assets J are traded. The assets are indexed by $j \in \mathbf{J} = \{1, ..., J\}$, where $J \leq S$. Complete markets occurs when $J = S$ and incomplete markets occurs when $J < S$. Quantitative easing requires $J > 1$.

Assumption 2 $J > 1$.

Asset 1 is a 1-period nominally risk-free bond. The nominal payout of a 1-period bond purchased in date-event s^t equals 1 for all date-events $s^{t+1} \succ s^t$ and 0 otherwise. Similarly,

[5] ℓ^∞_{++} is the space of bounded and strictly positive infinite sequences under the sup norm.
[6] ℓ^∞_+ is the space of bounded and nonnegative infinite sequences under the sup norm.

asset $j \in \{2, ..., J\}$ is a $j-$period nominally risk-free bond. The nominal payouts of a $j-$period bond purchased in date-event s^t equals 1 for all date-events $s^{t+j} \succ s^t$ and 0 otherwise. The $j-$period nominally risk-free bond can be freely traded in all interim periods up to the maturation date. The nominal asset price for a $j-$period bond issued in date-event s^t is denoted $q_j(s^t)$.

Each date-event is divided into two subperiods. In the initial subperiod, the money markets and bond markets open. Denote $\hat{m}^h(s^t)$ as the holding of money by household h by the close of the money market in date-event s^t. Denote $b_j^h(s^t) \in \mathbb{R}$ as the holding of a $j-$period nominal bond by household h by the close of the bond markets in date-event s^t. Each bond can either be held long or short by the household. Denote the entire portfolio as the $J-$ dimensional column vector $b^h(s^t) = \left(b_j^h(s^t)\right)_{j \in \mathbf{J}} \in \mathbb{R}^J$.

Denote $\omega^h(s^t) \in \mathbb{R}$ as the nominal wealth held by household h for use in the date-event s^t. The initial period value $\omega^h(s_0)$ is a parameter of the model. The budget constraint, at the close of the money markets and bond markets in date-event s^t is given by:

$$\hat{m}^h\left(s^t\right) + \sum_{j \in \mathbf{J}} q_j\left(s^t\right) b_j^h\left(s^t\right) \leq \omega^h\left(s^t\right). \tag{2}$$

The budget constraint is specified in nominal terms.

In the second subperiod of each date-event, the commodity markets open. The purchase of the commodity is subject to the cash-in-advance constraint:

$$p\left(s^t\right) c^h\left(s^t\right) \leq \hat{m}^h\left(s^t\right). \tag{3}$$

At the same time that consumption is being purchased on the commodity markets, the households receive income from selling their endowments. Denote $m^h(s^t)$ as the money holding of household h by the close of the commodity markets in date-event s^t :

$$m^h\left(s^t\right) = \hat{m}^h\left(s^t\right) + p\left(s^t\right) \mathbf{e}^h\left(s_t\right) - p\left(s^t\right) c^h\left(s^t\right). \tag{4}$$

Given the money definition (4), the cash-in-advance constraint (3) can be rewritten as:

$$m^h\left(s^t\right) \geq p\left(s^t\right) \mathbf{e}^h\left(s_t\right). \tag{5}$$

Entering into the date-events $s^{t+1} \succ s^t$, the wealth available to household h is equal to the money holding plus the portfolio payout:

$$\omega^h\left(s^{t+1}\right) = m^h\left(s^t\right) + b_1^h\left(s^t\right) + \sum_{j\in\mathbf{J}\setminus\{1\}} q_{j-1}\left(s^{t+1}\right) b_j^h\left(s^t\right).$$

For simplicity, I define $q_0\left(s^t\right) = 1$ for all date-events. The wealth can then be expressed as:

$$\omega^h\left(s^{t+1}\right) = m^h\left(s^t\right) + \sum_{j\in\mathbf{J}} q_{j-1}\left(s^{t+1}\right) b_j^h\left(s^t\right). \tag{6}$$

Households are permitted to short-sell the nominal bonds, so I require the following implicit debt constraint for all bonds (in real terms):

$$\inf_{t,s^t}\left(\frac{q_j\left(s^t\right) b_j^h\left(s^t\right)}{p_j\left(s^t\right)}\right) > -\infty. \tag{7}$$

The debt constraint states that the real debt position must be bounded for all random state realizations.

The household optimization problem is given by:

$$\begin{array}{ll} \underset{\left(c^h,b^h,m^h\right)\in(\ell^\infty)^{J+2}}{max} & \sum_{t=0}^{\infty}\beta^t\pi\left(s^t|s_0\right)u^h\left(c^h\left(s^t\right)\right) \\ subj.\ to & \text{budget constraint (2) with (4) and (6) } \forall t, s^t \\ & \text{cash-in-advance constraint (5) } \forall t, s^t \\ & \text{debt constraint (7) } \forall j \end{array}. \tag{8}$$

3.2 Monetary authority

The monetary authority chooses the debt positions $B\left(s^t\right) = \left(B_j\left(s^t\right)\right)_{j\in\mathbf{J}} \in \mathbb{R}_+^J$ in each date-event s^t. The choice $B_j\left(s^t\right) \in \mathbb{R}_+$ refers to the amount of debt issued by the government in terms of the $j-$period bond. Governments are only permitted to issue debt, so the debt positions are nonnegative.

The monetary authority issues the money supply $M\left(s^t\right)$ in the date-event s^t. In the initial period s_0, the monetary authority has the nominal obligation $W\left(s_0\right)$.

The monetary authority has the following budget constraints, where the liabilities of the monetary authority are on the left-hand side of the equations and the assets of the monetary authority are on the right-hand side of the equations:

$$\begin{aligned} W(s_0) &= M(s_0) + \sum_{j\in\mathbf{J}} q_j(s_0)\, B_j(s_0). \qquad (9) \\ M\left(s^{t-1}\right) + \sum_{j\in\mathbf{J}} q_{j-1}\left(s^t\right) B_j\left(s^{t-1}\right) &= M\left(s^t\right) + \sum_{j\in\mathbf{J}} q_j\left(s^t\right) B_j\left(s^t\right). \end{aligned}$$

Similar to the households, the monetary authorities face the implicit debt constraint:

$$\sup_{t,s^t}\left(\frac{q_j\left(s^t\right) B_j\left(s^t\right)}{p_j\left(s^t\right)}\right) < \infty. \qquad (10)$$

3.3 Sequential competitive equilibrium

The following equilibrium concept is the most general. It only requires that the monetary authority variables are feasible. It does not require that the monetary authority variables are optimal.

Definition 1 *A sequential competitive equilibrium (SCE) is the household variables $\left(c^h, b^h, m^h\right)_{h\in\mathbf{H}}$, the monetary authority variables $\{B(s^t), M(s^t)\}$, and the price variables $\left\{p(s^t), \left(q_j(s^t)\right)_{j\in\mathbf{J}}\right\}$ such that:*

1. *Given $\left\{p(s^t), \left(q_j(s^t)\right)_{j\in\mathbf{J}}\right\}$ and $\omega^h(s_0)$, each household chooses $\left(c^h, b^h, m^h\right)$ to solve the household problem (8).*

2. *Given $W(s_0)$, the monetary authority variables $\{B(s^t), M(s^t)\}$ satisfy (9) and (10).*

3. *Markets clear:*

   (a) $\sum_{h\in\mathbf{H}} c^h\left(s^t\right) = \sum_{h\in\mathbf{H}} \mathbf{e}^h(s_t)$ *for every* t, s^t.

   (b) $\sum_{h\in\mathbf{H}} \omega^h(s_0) = W(s_0)$.

   (c) $\sum_{h\in\mathbf{H}} m^h\left(s^t\right) = M\left(s^t\right)$ *for every* t, s^t.

   (d) $\sum_{h\in\mathbf{H}} b_j^h\left(s^t\right) = B_j\left(s^t\right)\ \forall j \in \mathbf{J}$ *and for every* t, s^t.

3.4 Equilibrium conditions

A necessary condition for equilibrium is no arbitrage in all date-events.

Definition 2 *No arbitrage is satisfied in date-event s^t if $\exists \mu^T \in \mathbb{R}^S_{++}$ such that*

$$\left(q_1\left(s^t\right),...,q_J\left(s^t\right)\right) = \mu \begin{bmatrix} 1 & q_1\left(s^t,1\right) & .. & q_{J-1}\left(s^t,1\right) \\ : & : & : & : \\ 1 & q_1\left(s^t,S\right) & .. & q_{J-1}\left(s^t,S\right) \end{bmatrix}.$$

The equilibrium asset price $q_1\left(s^t\right) \leq 1$. Otherwise, the market clearing condition on the bond markets is not satisfied as households prefer to save using money holdings and not bond holdings. If $q_1\left(s^t\right) < 1$, the cash-in-advance constraints (5) will bind for all households. With binding cash-in-advance constraints (5), the market clearing condition for the money markets implies that the Quantity Theory of Money holds:

$$M\left(s^t\right) = p\left(s^t\right) \sum_{h \in \mathbf{H}} \mathbf{e}^h\left(s_t\right) = p\left(s^t\right) E\left(s_t\right). \tag{11}$$

3.5 Discussion of the Friedman rule

The Friedman rule in date-event s^t is such that $q_1\left(s^t\right) = 1$. Under the Friedman rule, money and the 1-period bond are perfect substitutes. Market clearing for both implies that the sum of the two is pinned down for all households and the monetary authority, but not the composition. The cash-in-advance constraints (5) need not bind under the Friedman rule.

Suppose the Friedman rule is imposed in all date-events. The Friedman rule is a special case of interest rate targeting, where the nominal interest rate is set equal to 0. The asset prices would then be $q_j\left(s^t\right) = 1$ for all assets and all date-events. Such a targeting policy leads to a payout matrix in which all assets are linearly dependent. The objective function of the monetary authority is a social welfare function. Section 5 shows that the policy that maximizes any social welfare function must be such that the payout matrix has full rank. Therefore, the interest rate targeting policy of the Friedman rule cannot be optimal.[7]

It is innocuous (i.e., without real effects) under the Friedman rule to set the household money holdings such that the cash-in-advance constraints (5) bind. This would allow the Quantity Theory of Money (11) to hold.

[7] If the objective of the monetary authority includes both the maximizatin of a social welfare function and the minimization of the social cost of holding money (minimized at the Friedman rule), then the optimal policy will involve lower interest rates than predicted in this paper. However, so long as the monetary authority places strictly positive weight on maximizing the social welfare function, the Friedman rule will continue to be a suboptimal policy rule.

3.6 Discussion of the fiscal theory of the price level

The initial price level $p(s_0)$ is not pinned down in equilibrium.[8] There do not exist any real effects from changing this variable. Thus, it is immaterial (in terms of the real allocation) if $p(s_0)$ is included in the policy vector chosen by the monetary authority, or is left as an undetermined value. For simplicity, I consider $p(s_0)$ as a policy choice of the monetary authority.

4 Markov Equilibrium

Define the real debt positions for the monetary authority and the real bond positions for the households as:

$$\begin{aligned} \hat{B}_j\left(s^t\right) &= \frac{B_j\left(s^t\right)}{p\left(s^t\right)} \quad \forall j, t, s^t. \\ \hat{b}_j^h\left(s^t\right) &= \frac{b_j^h\left(s^t\right)}{p\left(s^t\right)} \quad \forall h, j, t, s^t. \end{aligned}$$

The portfolios are denoted $\hat{B}\left(s^t\right) = \left(\hat{B}_j\left(s^t\right)\right)_{j\in\mathbf{J}}$ and $\hat{b}^h\left(s^t\right) = \left(\hat{b}_j^h\left(s^t\right)\right)_{j\in\mathbf{J}}$, respectively. Market clearing in terms of nominal bond positions occurs if and only if market clearing in the real bond positions occurs.

Additionally, define the stochastic price ratio

$$\nu\left(s^t\right) = \frac{p\left(s^{t-1}\right)}{p\left(s^t\right)} \quad \forall t, s^t.$$

4.1 Monetary authority constraints

The monetary authority constraints (9) in real terms, after using the Quantity Theory of Money (11), are given by:

$$\nu\left(s^t\right)\left(E\left(s_{t-1}\right) + \sum_{j\in\mathbf{J}} q_{j-1}\left(s^t\right)\hat{B}_j\left(s^{t-1}\right)\right) = E\left(s_t\right) + \sum_{j\in\mathbf{J}} q_j\left(s^t\right)\hat{B}_j\left(s^t\right). \tag{12}$$

[8] Notable works on the fiscal theory of the price level include Leeper (1991), Sims (1994), and Woodford (1994, 1995).

4.2 Household problem

The household problem will be recursive in terms of wealth. If the cash-in-advance constraint (5) is binding, the household budget constraint under a SCE is given by:

$$c^h\left(s^t\right)+\sum_{j\in\mathbf{J}} q_j\left(s^t\right)\hat{b}_j^h\left(s^t\right)\leq\nu\left(s^t\right)\left(\mathbf{e}^h\left(s_{t-1}\right)+\sum_{j\in\mathbf{J}} q_{j-1}\left(s^t\right)\hat{b}_j^h\left(s^{t-1}\right)\right). \tag{13}$$

Define the scaled wealth for household h entering date-event s^t as

$$\hat{\omega}^h(s^t)=\frac{\omega^h(s^t)}{p\left(s^t\right)}=\nu\left(s^t\right)\left(\mathbf{e}^h\left(s_{t-1}\right)+\sum_{j\in\mathbf{J}} q_{j-1}\left(s^t\right)\hat{b}_j^h\left(s^{t-1}\right)\right).$$

The first order conditions with respect to bonds $\hat{b}_j^h\left(s^t\right)$ of any maturities $j\in\mathbf{J}$ are given by:

$$q_j\left(s^t\right)=\beta\sum_{\sigma\in\mathbf{S}}\pi\left(s_t,\sigma\right)\left(\frac{c^h(s^t,\sigma)}{c^h(s^t)}\right)^{-\rho}\nu\left(s^t,\sigma\right)q_{j-1}\left(s^t,\sigma\right). \tag{14}$$

4.3 Discussion of the yield curve

The yield in state σ for asset j traded in date-event s^t is defined by:

$$y_j\left(s^t,\sigma\right)=\frac{q_{j-1}\left(s^t,\sigma\right)}{q_j\left(s^t\right)}.$$

For the case of a 1-period bond, this yields the standard expression $y_1\left(s^t,\sigma\right)=\frac{1}{q_1(s^t)}$. The yield is equal to 1 plus the nominal interest rate. Empirical measures of the yield curve, which is the relation between the maturity j and the yield, show that the yield is strictly increasing in j. This property is one component of the so-called term structure of interest rates.[9] Consider the expected yields for maturities j and 1 :

$$\begin{aligned}\sum_{\sigma\in\mathbf{S}}\pi\left(s_t,\sigma\right)y_j\left(s^t,\sigma\right) &= \frac{1}{q_j\left(s^t\right)}\sum_{\sigma\in\mathbf{S}} q_{j-1}\left(s^t,\sigma\right).\\ \sum_{\sigma\in\mathbf{S}}\pi\left(s_t,\sigma\right)y_1\left(s^t,\sigma\right) &= \frac{1}{q_1\left(s^t\right)}.\end{aligned}$$

The expected yield is higher for maturity j provided that:

$$q_1\left(s^t\right)\sum_{\sigma\in\mathbf{S}} q_{j-1}\left(s^t,\sigma\right)\geq q_j\left(s^t\right). \tag{15}$$

[9] Another property is that the yield is a strictly concave function of j.

The first order conditions from (14) for maturities j and 1 imply that the inequality (15) is equivalent to:

$$\begin{aligned} & \beta \sum_{\sigma \in \mathbf{S}} \pi\left(s_t, \sigma\right)\left(\frac{c^h(s^t, \sigma)}{c^h(s^t)}\right)^{-\rho} \nu\left(s^t, \sigma\right)\left(\sum_{\sigma \in \mathbf{S}} q_{j-1}\left(s^t, \sigma\right)\right) \\ \geq\ & \beta \sum_{\sigma \in \mathbf{S}} \pi\left(s_t, \sigma\right)\left(\frac{c^h(s^t, \sigma)}{c^h(s^t)}\right)^{-\rho} \nu\left(s^t, \sigma\right) q_{j-1}\left(s^t, \sigma\right). \end{aligned}$$

The yield curve relation (15) then holds provided that $q_{j-1}\left(s^t, \sigma\right)$ is inversely related to $\nu\left(s^t, \sigma\right)$. I conjecture that this inverse relation holds as high values for $q_{j-1}\left(s^t, \sigma\right)$ (low interest rates) in a deterministic model imply low values of $\nu\left(s^t, \sigma\right)$ (high inflation rates). The stochastic model in this paper can be used to verify this conjecture.

4.4 Stationary policy and Markov equilibrium

The initial conditions are s_0, $\left(\omega^h(s_0)\right)_{h \in \mathbf{H}}$, $W\left(s_0\right)$, and $p\left(s_0\right)$. The price $p\left(s_0\right)$ is chosen by the monetary authority. A Markov equilibrium is defined in terms of a policy correspondence (a multi-valued mapping) and a transition function. These mappings, together with the initial conditions, determine the entire sequence of SCE variables.

This subsection introduces stationary policy, in which the monetary authority chooses the policy vector at the intial date-event and then maintains this policy vector in all future date-events. The following subsection introduces dynamic policy, in which the monetary authority can change policy over time. I use the adjective 'dynamic' to refer to the general class of policies and the adjective 'nonstationary' to refer to the complement of stationary. I will soon show that stationary policy is a special case of dynamic policy (since the Markov equilibrium concept does not require optimality on the part of the monetary authority).

In this paper, the policy vector will be the vector of asset prices. Equivalently, I could specify the model such that the policy vector is the vector of debt positions. The vector of asset prices and the vector of debt positions both belong to the space $\mathbb{R}_+^{SJ}$.

Under stationary policy, a stationary policy vector $\mathbf{q} = \left(\mathbf{q}_j\left(s\right)\right)_{(j,s) \in \mathbf{J} \times \mathbf{S}} \in \mathbb{R}_+^{SJ}$ is chosen by the monetary authority in the initial date-event and is such that $q_j\left(s^t\right) = \mathbf{q}_j\left(s_t\right)$ for all assets and all date-events. This vector need not be a state variable, as it remains unchanged throughout the entire length of the model.

The wealth distribution is $\hat{\omega}(s^t) = \left(\hat{\omega}^h(s^t)\right)_{h \in \mathbf{H}} \in \mathbb{R}^H$. Notice that (i) the possibility of borrowing in date-event s^t allows for negative wealth values and (ii) the wealth distribution is $H-$dimensional as the total household wealth depends upon the choices of the monetary authority and does not equal a fixed parameter.

4.4.1 State space

With stationary policy, the Markov equilibrium concept is referred to as a $(\mathbf{q})$ −Markov equilibrium in which $\mathbf{q}$ refers to the stationary policy vector $\mathbf{q} = (\mathbf{q}_j(s))_{(j,s)\in\mathbf{J}\times\mathbf{S}} \in \mathbb{R}_+^{SJ}$. The state space includes the aggregate shock realization in the current period, the wealth distribution, and the household bond holdings for assets $j > 1$. Define $\hat{b}_{\backslash 1}^h(s^t) = \left(\hat{b}_j^h(s^t)\right)_{j\in\mathbf{J}\backslash\{1\}}$ and $\hat{b}_{\backslash 1}(s^t) = \left(\hat{b}_{\backslash 1}^h(s^t)\right)_{h\in\mathbf{H}}$. The state space is then $\Omega_{\mathbf{q}} = \mathbf{S}\times\mathbb{R}^H\times\mathbb{R}^{H(J-1)}$ with typical element $\left(s, \hat{\omega}(s^t), \hat{b}_{\backslash 1}(s^t)\right)$.

4.4.2 Expectations correspondence

Define $\hat{\mathbf{Z}}_{\mathbf{q}} = \mathbb{R}_+^H\times\mathbb{R}^H\times\mathbb{R}_+\times\mathbb{R}_+^J\times\mathbb{R}_+\times\mathbb{R}_+$ as the set of current period variables, with typical element

$$\hat{z}(s^t) = \left(\left(c^h(s^t), \hat{b}_1^h(s^t)\right)_{h\in\mathbf{H}}, \nu(s^t), \hat{B}(s^t), M(s^t), p(s^t)\right).$$

Define $\mathbf{Z}_{\mathbf{q}} = \mathbb{R}^H\times\mathbb{R}^{H(J-1)}\times\hat{\mathbf{Z}}_{\mathbf{q}}$ as the set containing the state variables $\left(\hat{\omega}(s^t), \hat{b}_{\backslash 1}(s^t)\right)$ and the set of current period variables $\hat{z}(s^t)$. For simplicity, define

$$z(s^t) = \left(\hat{\omega}(s^t), \hat{b}_{\backslash 1}(s^t), \hat{z}(s^t)\right).$$

The key mapping for existence is the expectations correspondence

$$g_{\mathbf{q}} : \Omega_{\mathbf{q}}\times\hat{\mathbf{Z}}_{\mathbf{q}} \rightrightarrows (\mathbf{Z}_{\mathbf{q}})^S$$

that describes all next period variables that are consistent with the budget constraints, household optimization, and market clearing. The expectations correspondence is defined such that for

$$\begin{aligned} z &= \left(\hat{\omega}, \left(c^h, \hat{b}^h\right)_{h\in\mathbf{H}}, \nu, \hat{B}, M, p\right) \text{ and} \\ z'(\sigma) &= \left(\hat{\omega}'(\sigma), \left(c'^h(\sigma), \hat{b}'^h(\sigma)\right)_{h\in\mathbf{H}}, \nu'(\sigma), \hat{B}'(\sigma), M'(\sigma), p'(\sigma)\right) \quad \forall\sigma\in\mathbf{S}, \end{aligned}$$

the vector of variables $(z'(1), ..., z'(S)) \in g_{\mathbf{q}}(s, z)$ if the following conditions hold.

1. For all $\sigma\in\mathbf{S}$, the nominal price levels

$$p'(\sigma) = \frac{p(s)}{\nu'(\sigma)}. \tag{16}$$

2. For all $\sigma \in \mathbf{S}$, the money supplies

$$M'(\sigma) = p'(\sigma) E(\sigma). \tag{17}$$

3. For all $h \in \mathbf{H}$ and all $\sigma \in \mathbf{S}$, the household wealth

$$\hat{\omega}'^h(\sigma) = \nu'(\sigma) \left(\mathbf{e}^h(s) + \sum_{j \in \mathbf{J}} \mathbf{q}_{j-1}(\sigma) \hat{b}_j^h \right). \tag{18}$$

4. For all $\sigma \in \mathbf{S}$, the monetary authority constraint (12):

$$\nu'(\sigma) \left(E(s) + \sum_{j \in \mathbf{J}} \mathbf{q}_{j-1}(\sigma) \hat{B}_j \right) = E(\sigma) + \sum_{j \in \mathbf{J}} \mathbf{q}_j(\sigma) \hat{B}'_j(\sigma). \tag{19}$$

5. For all $h \in \mathbf{H}$ and all $\sigma \in \mathbf{S}$, the household consumptions satisfy the budget constraint:

$$c'^h(\sigma) + \sum_{j \in \mathbf{J}} \mathbf{q}_j(\sigma) \hat{b}'^h_j(\sigma) = \hat{\omega}'^h(\sigma). \tag{20}$$

6. For all $j \in \mathbf{J}$ and all $h \in \mathbf{H}$, the Euler equation (14):

$$\mathbf{q}_j(s) = \beta \sum_{\sigma \in \mathbf{S}} \pi(s, \sigma) \left(\frac{c'^h(\sigma)}{c^h} \right)^{-\rho} \nu'(\sigma) \mathbf{q}_{j-1}(\sigma). \tag{21}$$

7. For all $j \in \mathbf{J}$ and all $\sigma \in \mathbf{S}$, markets clear:

$$\hat{B}'_j(\sigma) = \sum_{h \in \mathbf{H}} \hat{b}'^h_j(\sigma). \tag{22}$$

By definition, the graph of $g_{\mathbf{q}}$ is a closed subset of $\Omega_{\mathbf{q}} \times \hat{\mathbf{Z}}_{\mathbf{q}} \times (\mathbf{Z}_{\mathbf{q}})^S$.

4.4.3 Markov equilibrium definition

Claim 1 *For all* $\left(\sigma, \hat{\omega}'(\sigma), \hat{b}'_{\setminus 1}(\sigma)\right) \in \Omega_{\mathbf{q}}$, *there exists a unique vector*

$$\hat{z}'(\sigma) = \left(\left(c'^h(\sigma), \hat{b}'^h_1(\sigma) \right)_{h \in \mathbf{H}}, \nu'(\sigma), \hat{B}'(\sigma), M'(\sigma), p'(\sigma) \right)$$

satisfying (16), (17), (19), (20), (21), and (22).

Proof. See Section A.1. ■

Define the function $\phi_{\mathbf{q}} : \Omega_{\mathbf{q}} \rightarrow \hat{\mathbf{Z}}_{\mathbf{q}}$ such that

$$\phi_{\mathbf{q}}\left(\sigma, \hat{\omega}'(\sigma), \hat{b}'_{\backslash 1}(\sigma)\right) = \left\{\hat{z}'(\sigma) : (16), (17), (19), (20), (21), \text{ and } (22) \text{ satisfied}\right\}.$$

From claim 1, $\phi_{\mathbf{q}}$ is a well-defined function (whose image is single-valued).

A $(\mathbf{q})$ $-$Markov equilibrium is defined by a policy correspondence $\mathbf{V}_{\mathbf{q}} : \mathbf{S} \times \mathbb{R}^{H} \rightrightarrows \mathbb{R}^{H(J-1)}$ and a transition function $\mathbf{F}_{\mathbf{q},\sigma} : graph(\mathbf{V}_{\mathbf{q}}) \rightarrow \mathbb{R}^{H} \times \mathbb{R}^{H(J-1)}$ for all $\sigma \in \mathbf{S}$ satisfying the following two properties:

1. For all $\left(s, \hat{\omega}, \hat{b}_{\backslash 1}\right) \in graph(\mathbf{V}_{\mathbf{q}})$ and all $\sigma \in \mathbf{S}$,

$$\left(\mathbf{F}_{\mathbf{q},\sigma}\left(s, \hat{\omega}, \hat{b}_{\backslash 1}\right), \phi_{\mathbf{q}}\left(\sigma, \mathbf{F}_{\mathbf{q},\sigma}\left(s, \hat{\omega}, \hat{b}_{\backslash 1}\right)\right)\right) \in g_{\mathbf{q},\sigma}\left(s, \hat{\omega}, \hat{b}_{\backslash 1}, \phi_{\mathbf{q}}\left(s, \hat{\omega}, \hat{b}_{\backslash 1}\right)\right).$$

2. For all $\left(s, \hat{\omega}, \hat{b}_{\backslash 1}\right) \in graph(\mathbf{V}_{\mathbf{q}})$ and all $\sigma \in \mathbf{S}$,

$$\left(\sigma, \mathbf{F}_{\mathbf{q},\sigma}\left(s, \hat{\omega}, \hat{b}_{\backslash 1}\right)\right) \in graph(\mathbf{V}_{\mathbf{q}}).$$

4.4.4 Full rank

Given $\left(s, \hat{\omega}, \hat{b}_{\backslash 1}\right) \in \Omega_{\mathbf{q}}$, define the real payout matrix $R(s) \in \mathbb{R}^{S,J}$ as the real payouts for all J assets in all S states $\sigma \in \mathbf{S}$:

$$R(s) = \begin{bmatrix} \nu'(1) & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & \nu'(S) \end{bmatrix} \begin{bmatrix} 1 & \mathbf{q}_1(1) & .. & \mathbf{q}_{J-1}(1) \\ : & : & : & : \\ 1 & \mathbf{q}_1(S) & .. & \mathbf{q}_{J-1}(S) \end{bmatrix}.$$

Since the diagonal matrix $[v'] = \begin{bmatrix} \nu'(1) & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & \nu'(S) \end{bmatrix}$ has full rank, $R(s)$ has full column rank J iff $\mathbf{Q}_0^{J-1} = \begin{bmatrix} 1 & \mathbf{q}_1(1) & .. & \mathbf{q}_{J-1}(1) \\ : & : & : & : \\ 1 & \mathbf{q}_1(S) & .. & \mathbf{q}_{J-1}(S) \end{bmatrix}$ has full column rank J. This matrix contains the asset prices for assets numbered $j = 0$ through $j = J - 1$.

Standard existence results require that the payout matrix has full column rank. At this point in the proceedings, I simple assume that the policy is chosen from the subset for which the payout matrix has full column rank. Later, upon introducing the objective function for the monetary authority, I prove that the optimal policy can only belong to this subset for which the payout matrix has full column rank.

Claim 2 *Given the initial conditions $p(s_0)$ and $\left(\omega^h(s_0)\right)_{h\in\mathbf{H}}$, for any $(\mathbf{q})$ −Markov equilibrium in which $\mathbf{Q}_0^{J-1}$ has full column rank J, the sequence of variables generated by the policy correspondence $\mathbf{V_q} : \mathbf{S}\times\mathbb{R}^H \rightrightarrows \mathbb{R}^{H(J-1)}$ and the transition function $\mathbf{F}_{\mathbf{q},\sigma} : graph(\mathbf{V_q}) \rightarrow \mathbb{R}^H\times\mathbb{R}^{H(J-1)}$ is a SCE.*

Proof. To show that $(\mathbf{q})$ −Markov equilibria satisfy the SCE definition, the Euler equation (21) must be necessary and sufficient for household optimality. Necessary is immediate. Sufficiency follows as in Proposition 3.2 from Duffie et al. (1994). The assets in the present model are uniformly bounded under the debt contraint (7) and the assumption that $\mathbf{Q}_0^{J-1}$ has full column rank J. ■

4.4.5 Existence

Definition 3 *The stationary policy vector $\mathbf{q}=(\mathbf{q}_j(s))_{(j,s)\in\mathbf{J}\times\mathbf{S}}$ satisfies no arbitrage if the sequence $\left\{(q_j(s^t))_{j\in\mathbf{J}}\right\}$ defined by $q_j(s^t)=\mathbf{q}_j(s_t)$ for all assets and all date-events satisfies no arbitrage.*

Claim 3 *If $\mathbf{q}$ satisfies no arbitrage, then a SCE exists in which $q_j(s^t)=\mathbf{q}_j(s_t)$ for all assets and all date-events.*

Proof. See Section A.2. ■

Select any vector $\mathbf{q}$ that satisfies no arbitrage and for which $\mathbf{Q}_0^{J-1}$ has full column rank J. The existence of a SCE suffices to ensure that $\hat{b}_{\backslash 1}(s^t)$ lies in a compact set for all date-events. Denote this compact set $\Delta\subseteq\mathbb{R}^{H(J-1)}$. The next steps provide an iterative algorithm to determine the policy correspondence $\mathbf{V_q}$. Define the initial correspondence $\mathbf{V}_\mathbf{q}^0 : \mathbf{S}\times\mathbb{R}^H \rightrightarrows \mathbb{R}^{H(J-1)}$ such that $\mathbf{V}_\mathbf{q}^0(s,\hat{\omega})=\Delta$ for all $(s,\hat{\omega})\in\mathbf{S}\times\mathbb{R}^H$. Define the operator G_Δ that maps from the correspondence $\mathbf{V}_\mathbf{q}^n : \mathbf{S}\times\mathbb{R}^H \rightrightarrows \mathbb{R}^{H(J-1)}$ to a new correspondence $\mathbf{V}_\mathbf{q}^{n+1} : \mathbf{S}\times\mathbb{R}^H \rightrightarrows \mathbb{R}^{H(J-1)}$ as follows:

$$\mathbf{V}_\mathbf{q}^{n+1}(s,\hat{\omega})=\left\{\hat{b}_{\backslash 1}\in\Delta : \begin{array}{c}\exists\,(z'(1),...,z'(S))\in g\left(s,\hat{\omega},\hat{b}_{\backslash 1},\phi_\mathbf{q}\left(s,\hat{\omega},\hat{b}_{\backslash 1}\right)\right)\\ \text{such that } \hat{b}'_{\backslash 1}(\sigma)\in\mathbf{V}_\mathbf{q}^n(\sigma,\hat{\omega}'(\sigma))\quad\forall\sigma\in\mathbf{S}\end{array}\right\}.$$

Define $\mathbf{V}_\mathbf{q}^* : \mathbf{S}\times\mathbb{R}^H \rightrightarrows \mathbb{R}^{H(J-1)}$ such that

$$\mathbf{V}_\mathbf{q}^*(s,\hat{\omega})=\bigcap_{n=0}^{\infty}\mathbf{V}_\mathbf{q}^n(s,\hat{\omega})\quad\text{for all }\ (s,\hat{\omega})\in\mathbf{S}\times\mathbb{R}^H.$$

Claim 4 *If* $\mathbf{q}$ *satisfies no arbitrage and* $\mathbf{Q}_0^{J-1}$ *has full column rank* J*, then the correspondence* $\mathbf{V}_{\mathbf{q}}^{*}:\mathbf{S}\times\mathbb{R}^{H}\rightrightarrows\mathbb{R}^{H(J-1)}$ *is nonempty valued and there exists a* $(\mathbf{q})-$*Markov equilibrium with policy correspondence* $\mathbf{V}_{\mathbf{q}}^{*}$.

Proof. See the proof of Theorem 1 in Kubler and Schmedders (2003). ■

4.5 Dynamic policy and Markov equilibrium

4.5.1 State space

The state space includes the aggregate shock realization in the current period, the wealth distribution, the household bond holdings for assets $j > 1$, and the vector of asset prices. The state space is $\Omega = \mathbf{S}\times\mathbb{R}^{H}\times\mathbb{R}^{H(J-1)}\times\mathbb{R}_{+}^{J}$ with typical element $\left(s,\hat{\omega}(s^t),\hat{b}_{\setminus 1}\left(s^t\right),q\left(s^t\right)\right)$.

4.5.2 Expectations correspondence

Define $\hat{\mathbf{Z}} = \mathbb{R}_{+}^{H}\times\mathbb{R}^{H}\times\mathbb{R}_{+}\times\mathbb{R}_{+}^{J}\times\mathbb{R}_{+}\times\mathbb{R}_{+}$ as the set of current period variables, with typical element

$$\hat{z}\left(s^t\right) = \left(\left(c^h(s^t),\hat{b}_1^h\left(s^t\right)\right)_{h\in\mathbf{H}},\nu\left(s^t\right),\hat{B}\left(s^t\right),M\left(s^t\right),p\left(s^t\right)\right).$$

Define $\mathbf{Z} = \mathbb{R}^{H}\times\mathbb{R}^{H(J-1)}\times\mathbb{R}_{+}^{J}\times\hat{\mathbf{Z}}_{\mathbf{q}}$ as the set containing the state variables $\left(\hat{\omega}(s^t),\hat{b}_{\setminus 1}\left(s^t\right),q\left(s^t\right)\right)$ and the set of current period variables $\hat{z}\left(s^t\right)$. For simplicity, define

$$z\left(s^t\right) = \left(\hat{\omega}(s^t),\hat{b}_{\setminus 1}\left(s^t\right),q\left(s^t\right),\hat{z}\left(s^t\right)\right).$$

The expectations correspondence $g:\Omega\times\hat{\mathbf{Z}}\rightrightarrows(\mathbf{Z})^{S}$ is defined such that for

$$\begin{aligned} z &= \left(\hat{\omega},q,\left(c^h,\hat{b}^h\right)_{h\in\mathbf{H}},\nu,\hat{B},M,p\right) \text{ and} \\ z'(\sigma) &= \left(\hat{\omega}'(\sigma),q'(\sigma),\left(c'^h(\sigma),\hat{b}'^h(\sigma)\right)_{h\in\mathbf{H}},\nu'(\sigma),\hat{B}'(\sigma),M'(\sigma),p'(\sigma)\right) \quad \forall\sigma\in\mathbf{S}, \end{aligned}$$

the vector of variables $(z'(1),...,z'(S))\in g(s,z)$ if the following conditions hold:

1. Equations (16), (17), and (22).
2. For all $\sigma\in\mathbf{S}$, equations (18), (19), and (20) with $\left(q_j'(\sigma)\right)_{j\in\mathbf{J}}$ instead of $(\mathbf{q}_j(\sigma))_{j\in\mathbf{J}}$.
3. For all $j\in\mathbf{J}$ and all $h\in\mathbf{H}$, the Euler equation (14):

$$q_j(s) = \beta\sum_{\sigma\in\mathbf{S}}\pi(s,\sigma)\left(\frac{c'^h(\sigma)}{c^h}\right)^{-\rho}\nu'(\sigma)\,q_{j-1}'(\sigma). \tag{23}$$

4. For all $\sigma \in \mathbf{S}$,

$$\hat{b}'_{\backslash 1}(\sigma) \in \mathbf{V}_{(q'(\sigma))_{\sigma \in \mathbf{S}}}(\sigma, \hat{\omega}'(\sigma)). \tag{24}$$

The condition (24) states that the household portfolios must be consistent with the policy correspondence $\mathbf{V}_{(q'(\sigma))_{\sigma \in \mathbf{S}}} : \mathbf{S} \times \mathbb{R}^H \rightrightarrows \mathbb{R}^{H(J-1)}$ associated with a $\left((q'(\sigma))_{\sigma \in \mathbf{S}}\right)$ −Markov equilibrium.

4.5.3 Markov equilibrium

Define the function $\phi : \Omega \to \hat{\mathbf{Z}}$ such that

$$\phi\left(\sigma, \hat{\omega}'(\sigma), \hat{b}'_{\backslash 1}(\sigma), q'(\sigma)\right) = \{\hat{z}'(\sigma) : (16), (17), (19), (20), (23), \text{and } (22) \text{ satisfied}\}.$$

From Claim 1, ϕ is single-valued.

A Markov equilibrium is defined by a policy correspondence $\mathbf{V} : \mathbf{S} \times \mathbb{R}^H \rightrightarrows \mathbb{R}^{H(J-1)} \times \mathbb{R}^J$ and a transition function $\mathbf{F}_\sigma : graph(\mathbf{V}) \to \mathbb{R}^H \times \mathbb{R}^{H(J-1)} \times \mathbb{R}^J$ for all $\sigma \in \mathbf{S}$ satisfying the following two properties:

1. For all $\left(s, \hat{\omega}, \hat{b}_{\backslash 1}, q\right) \in graph(\mathbf{V})$ and all $\sigma \in \mathbf{S}$,

$$\left(\mathbf{F}_\sigma\left(s, \hat{\omega}, \hat{b}_{\backslash 1}, q\right), \phi\left(\sigma, \mathbf{F}_\sigma\left(s, \hat{\omega}, \hat{b}_{\backslash 1}, q\right)\right)\right) \in g_\sigma\left(s, \hat{\omega}, \hat{b}_{\backslash 1}, q, \phi\left(s, \hat{\omega}, \hat{b}_{\backslash 1}, q\right)\right).$$

2. For all $\left(s, \hat{\omega}, \hat{b}_{\backslash 1}, q\right) \in graph(\mathbf{V})$ and all $\sigma \in \mathbf{S}$,

$$\left(\sigma, \mathbf{F}_\sigma\left(s, \hat{\omega}, \hat{b}_{\backslash 1}, q\right)\right) \in graph(\mathbf{V}).$$

Claim 5 *A Markov equilibrium exists.*

Proof. I have previously shown that a $(\mathbf{q})$ −Markov equilibrium exists, and a $(\mathbf{q})$ −Markov equilibrium is a special case of a Markov equilibrium. There exists values for $\mathbf{q}$ that satisfy no arbitrage and for which $\mathbf{Q}_0^{J-1}$ has full column rank J. Claim 4 can then be applied. ■

The next section analyzes when the special case of a $(\mathbf{q})$ −Markov equilibrium generates optimal policy and when the more general concept of a Markov equilibrium is required for optimal policy.

5 Optimality Conditions for Stationary Policy

5.1 Optimal Markov equilibria

To discuss optimality conditions, I first need to introduce the objective function of the monetary authority and verify that the monetary authority variables that maximize this objective function will be such that $\mathbf{Q}_0^{J-1}$ has full rank.

Definition 4 *For a given vector of Pareto weights $\left(\alpha^h\right)_{h\in\mathbf{H}} \in \Delta^{H-1}$, an optimal stationary policy Markov equilibrium is a stationary policy vector $\mathbf{q}$ with a corresponding $(\mathbf{q})-$Markov equilibrium that maximizes the following objective function:*

$$\sum_{t=0}^{\infty} \beta^t \sum_{s^t \succ s_0} \pi\left(s^t | s_0\right) \sum_{h\in\mathbf{H}} \alpha^h u\left(c^h\left(s^t\right)\right). \tag{25}$$

Claim 6 *An optimal stationary policy Markov equilibrium is such that $\mathbf{Q}_0^{J-1}$ has full rank.*

Proof. See Section A.3. ■

Definition 5 *For a given vector of Pareto weights $\left(\alpha^h\right)_{h\in\mathbf{H}} \in \Delta^{H-1}$, an optimal Markov equilibrium is a Markov equilibrium that maximizes (25).*

In any date-event, define the real payout matrix

$$R\left(s^t\right) = \begin{bmatrix} \nu\left(s^t,1\right) & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & \nu\left(s^t,S\right) \end{bmatrix} \begin{bmatrix} 1 & q_1\left(s^t,1\right) & .. & q_{J-1}\left(s^t,1\right) \\ : & : & : & : \\ 1 & q_1\left(s^t,S\right) & .. & q_{J-1}\left(s^t,S\right) \end{bmatrix}.$$

The real payout matrix $R\left(s^t\right)$ has full rank iff the asset price matrix

$$Q_0^{J-1}\left(s^t\right) = \begin{bmatrix} 1 & q_1\left(s^t,1\right) & .. & q_{J-1}\left(s^t,1\right) \\ : & : & : & : \\ 1 & q_1\left(s^t,S\right) & .. & q_{J-1}\left(s^t,S\right) \end{bmatrix}$$

has full rank.

Claim 7 *An optimal Markov equilibrium is such that $Q_0^{J-1}\left(s^t\right)$ has full rank in all date-events.*

Proof. See Section A.4. ■

5.2 Stationary policy is optimal under complete markets

Under complete markets ($J = S$), Claim 7 means that the payout matrix is a square matrix with full rank in all date-events. Under a given vector of Pareto weights $\left(\alpha^h\right)_{h \in \mathbf{H}} \in \Delta^{H-1}$, there exists a unique optimal Markov equilibrium allocation. Refer to this allocation as the complete markets allocation. The set of optimal Markov equilibria is indeterminate as there exists a continuum of variables supporting this complete markets allocation.

In fact, all Markov equilibria are optimal Markov equilibria and support the identical allocation (see Theorem 1 in Hoelle, 2014).

As the set of optimal Markov equilibria includes the set of all $(\mathbf{q})$ −Markov equilibria, then the monetary authority can simply adopt a stationary policy to maximize its objective function.

Theorem 1 *Under Assumptions 1-2 and complete markets ($J = S$), all $(\mathbf{q})$ −Markov equilibria are optimal Markov equilibria.*

Proof. See Theorem 1 in Hoelle (2014). ■

5.3 Optimality conditions for stationary policy under incomplete markets

Since complete markets are uninteresting as a policy exercise, the main results are derived under incomplete markets.

Assumption 3 $J < S$.

The sufficient condition for stationary policy to be optimal under incomplete markets includes restrictions on the number of households and the number of assets.

Assumption 4 $H \leq J$ and $J = S - 1$.

Theorem 2 *Under Assumptions 1-4, all optimal stationary policy Markov equilibria are optimal Markov equilibria.*

Proof. See Section A.5. ■

Under Assumption 4, even with incomplete markets, the complete markets allocation can be supported with an appropriate choice for the stationary policy vector $\mathbf{q}$. As the complete markets allocation is Pareto efficient, the monetary authority will have no incentive to deviate in any future date-events from the stationary policy vector $\mathbf{q}$.

Assumption 4 is also necessary (generically) for the optimality of stationary policy. If the assumption does not hold, then over a generic subset of household endowments, there does not exist a stationary policy vector $\mathbf{q}$ capable of supporting the complete markets allocation. If a stationary policy vector is unable to support the complete markets allocation, then it is optimal for the monetary authority to adopt nonstationary policy.

Theorem 3 *Under Assumptions 1-3 and provided that Assumption 4 does not hold, then over a generic subset of household endowments an optimal stationary policy Markov equilibrium is not an optimal Markov equilibrium.*

Proof. See Section A.6. ■

6 Concluding Remarks

This concludes the theoretical contributions associated with finding the optimal quantitative easing policies in settings with incomplete markets. The sufficient and (generically) necessary condition for the (Pareto) efficiency of stationary policy is restrictive. When this condition is violated, the optimal policy will be nonstationary (generically).

The recursive methodology adopted in this paper for the theoretical results is essential for the computation of optimal policy. In future work, numerical methods will be used to approximate the set of $(\mathbf{q})-$Markov equilibria. Given this set, then both optimal stationary policy Markov equilibria and optimal Markov equilibria can be computed, the former by fixing the vector of asset prices and the latter by updating the vector of asset prices every period.

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A Appendix

A.1 Proof of Claim 1

Given $\left(\sigma, \hat{\omega}'(\sigma), \hat{b}'_{\backslash 1}(\sigma)\right)$, the market clearing condition (22) yields the unique vector $\hat{B}'_{/1}(\sigma) = \left(\hat{B}'_j(\sigma)\right)_{j\in\mathbf{J}\backslash\{1\}}$. Guessing a value for $\nu'(\sigma)$, there exists a unique vector $\left(c'^h(\sigma), \hat{b}'^h_1(\sigma)\right)$ that solves the budget constraint (20) and the Euler equation (21) for all households $h \in \mathbf{H}$. From the market clearing condition (22), a unique value for $\hat{B}'_1(\sigma)$ is found. Finally, the initial guess for $\nu'(\sigma)$ is verified from the monetary authority constraint (19).

The right-hand side of the Euler equation (21) is a strictly monotonic function of the guess $\nu'(\sigma)$, meaning that there exists a unique value for $\nu'(\sigma)$. Strict monotonicity is easily verified. An increase in $\nu'(\sigma)$ increases $\hat{B}'_1(\sigma)$ from the monetary authority constraint (19). An increase in $\hat{B}'_1(\sigma)$ increases $\sum_{h\in\mathbf{H}} \hat{b}'^h_1(\sigma)$ from market clearing (22). An increase in $\hat{b}'^h_1(\sigma)$ for some household means a decrease in $c'^h(\sigma)$ from the budget constraint (20). This means an increase in $\left(c'^h(\sigma)\right)^{-\rho}$, the only other term in the right-hand side of the Euler equation (21) that varies with $\nu'(\sigma)$ (besides $\nu'(\sigma)$ itself).

Given $\nu'(\sigma)$, equation (16) yields the unique value for $p'(\sigma)$ and equation (17) yields the unique value for $M'(\sigma)$.

A.2 Proof of Claim 3

Consider any stationary policy vector $\mathbf{q} = (\mathbf{q}_j(s))_{(j,s)\in\mathbf{J}\times\mathbf{S}}$ that satisfied no arbitrage. I will construct a SCE in which $q_j(s^t) = \mathbf{q}_j(s_t)$ for all assets and all date-events. By the definition of no arbitrage, for all date-events s^t, there exists $\mu^T \in \mathbb{R}^S_{++}$ such that

$$(\mathbf{q}_1(s_t), ..., \mathbf{q}_J(s_t)) = \mu\mathbf{Q}_0^{J-1}.$$

Given the entire sequence $\{v(s^t)\}$ and the stationary policy vector $\mathbf{q}$, there exists a unique solution $\left(c^h, b^h, m^h\right)$ to solve the household problem (8). Given the infinite sequences $\left(b^h, m^h\right)_{h\in\mathbf{H}}$, there exists unique vectors $\{B(s^t), M(s^t)\}$ to satisfy the bond market clearing conditions and the money market clearing conditions in the SCE definition. Walras' Law implies that if the constraints for all households and the monetary authority are satisfied, and the bond and money market clearing conditions are satisfied, then the commodity market clearing conditions are satisfied.

The initial conditions s_0, $\left(\omega^h(s_0)\right)_{h\in\mathbf{H}}$, $W(s_0)$, and $p(s_0)$ are held fixed. The scaled wealth distribution is $\left(\hat{\omega}^h(s_0)\right)_{h\in\mathbf{H}}$ such that $\hat{\omega}^h(s_0) = \frac{\omega^h(s_0)}{p(s_0)}$. The vector $(\nu(s_0, \sigma))_{\sigma\in\mathbf{S}}$ is chosen to satisfy (i) the monetary authority constraint (19),

$$\nu(s_0,\sigma)\left(E(s_0)+\sum_{j\in\mathbf{J}}\mathbf{q}_{j-1}(\sigma)\hat{B}_j(s_0)\right)=E(\sigma)+\sum_{j\in\mathbf{J}}\mathbf{q}_j(\sigma)\hat{B}_j(s_0,\sigma) \tag{26}$$

in all date-events $(s_0,\sigma)_{\sigma\in\mathbf{S}}$, and (ii) the no arbitrage condition:

$$(\mathbf{q}_1(s_0),...,\mathbf{q}_J(s_0))=\left(\mu^h(s_0,1),...,\mu^h(s_0,S)\right)\mathbf{Q}_0^{J-1}.$$

The element $\mu^h(s_0,\sigma)$ is defined as:

$$\mu^h(s_0,\sigma)=\beta\pi(s_0,\sigma)\left(\frac{c^h(s_0,\sigma)}{c^h(s_0)}\right)^{-\rho}\nu(s_0,\sigma).$$

Such a vector $(\nu(s_0,\sigma))_{\sigma\in\mathbf{S}}$ exists since changes in $(\nu(s_0,\sigma))_{\sigma\in\mathbf{S}}$ only have minor effects on the consumption choices $\left(c^h(s_0,\sigma)\right)_{h\in\mathbf{H}}$, meaning that the mapping from $(\nu(s_0,\sigma))_{\sigma\in\mathbf{S}}$ to $\left(\mu^h(s_0,\sigma)\right)_{\sigma\in\mathbf{S}}\in\mathbb{R}^S_{++}$ is surjective.

To see why the effects on consumption are minor, consider the household budget constraint (13):

$$c^h(s_0,\sigma)+\sum_{j\in\mathbf{J}}\mathbf{q}_j(\sigma)\hat{b}^h_j(s_0,\sigma)=\nu(s_0,\sigma)\left(\mathbf{e}^h(s_0)+\sum_{j\in\mathbf{J}}\mathbf{q}_{j-1}(\sigma)\hat{b}^h_j(s_0)\right).$$

The portfolio expenditures $\sum_{j\in\mathbf{J}}\mathbf{q}_j(\sigma)\hat{b}^h_j(s_0,\sigma)$ are directly related to $\nu(s_0,\sigma)$. From the monetary authority constraint (26), the debt outlays $\sum_{j\in\mathbf{J}}\mathbf{q}_j(\sigma)\hat{B}_j(s_0,\sigma)$ are directly related to $\nu(s_0,\sigma)$. From the market clearing condition in the bond markets, the debt $\hat{B}_j(s_0,\sigma)$ is exactly equal to the total household bond positions $\sum_{h\in\mathbf{H}}\hat{b}^h_j(s_0,\sigma)$. Changes in $(\nu(s_0,\sigma))_{\sigma\in\mathbf{S}}$ affect portfolios $\left(\hat{b}^h_j(s_0,\sigma)\right)_{j\in\mathbf{J}}$ and $\left(\hat{B}_j(s_0,\sigma)\right)_{j\in\mathbf{J}}$, but since the total consumption is fixed, the effects on $\left(c^h(s_0,\sigma)\right)_{h\in\mathbf{H}}$ are minor. The variables $\left(\mu^h(s_0,\sigma)\right)_{\sigma\in\mathbf{S}}$ are continuous and strictly increasing functions of $(\nu(s_0,\sigma))_{\sigma\in\mathbf{S}}$.

Suppose that the realization in period $t=1$ is $s_1=\sigma$. The conditions beginning from the date-event (s_0,σ) are $\left(\sigma,\left(\hat{\omega}^h(s_0,\sigma)\right)_{h\in\mathbf{H}}\right)$. These are the state variables for the policy correspondence $\mathbf{V}_\mathbf{q}$. Let's consider any state variables $(s,\hat{\omega})$ as this recursive methodology is applicable in all date-events. The vector $(\nu'(\sigma))_{\sigma\in\mathbf{S}}$ is chosen to satisfy (i) the monetary authority constraint (19) for all shocks $\sigma \in \mathbf{S}$ and (ii) the no arbitrage condition:

$$(\mathbf{q}_1(s), ..., \mathbf{q}_J(s)) = \left(\mu^h(s,1), ..., \mu^h(s,S)\right) \mathbf{Q}_0^{J-1}.$$

The elements of $\mu^h(s,\sigma)$ are defined as in (21):

$$\mu^h(s,\sigma) = \beta\pi(s,\sigma)\left(\frac{c'^h(\sigma)}{c^h}\right)^{-\rho} \nu'(\sigma).$$

Such a vector $(\nu'(\sigma))_{\sigma\in\mathbf{S}}$ exists since the mapping from $(\nu'(\sigma))_{\sigma\in\mathbf{S}}$ to $\left(\mu^h(s,\sigma)\right)_{\sigma\in\mathbf{S}} \in \mathbb{R}^S_{++}$ is surjective.

This finishes the construction of a SCE with the stationary policy vector $\mathbf{q}$.

A.3 Proof of Claim 6

Suppose otherwise, that is the optimal policy rule is such that $\mathbf{Q}_0^{J-1}$ has rank $J-1$ (without loss of generality). With a single commodity traded in each date-event, the allocation will be contrained Pareto efficient with respect to these $J-1$ assets. The addition of any independent asset J leads to a new allocation that is constrained Pareto efficient with respect to the J assets. Unless the former allocation is Pareto efficient, the latter allocation always provides strictly higher aggregate welfare (25).[10] Thus, the monetary authority will select a stationary policy vector such that the asset price matrix $\mathbf{Q}_0^{J-1}$ has full rank.

A.4 Proof of Claim 7

The definition of an optimal Markov equilibrium implies that in any date-event, the vector of asset prices $(q'(\sigma))_{\sigma\in\mathbf{S}}$ is chosen to maximize the following conditional objective function

$$\sum_{\tau>t}\beta^{(\tau-t)}\sum_{\sigma}\sum_{s^\tau\succ(s^t,\sigma)}\pi\left(s^\tau|s^t\right)\sum_{h\in\mathbf{H}}\alpha^h u\left(c^h(s^\tau)\right).$$

This together with Claim 6 imply that an optimal Markov equilibrium is such that the real payout matrix $R(s^t)$ has full rank in all date-events.

[10] If $H < J$, Lemma 1 dictates that the optimal policy selects an appropriate $H-$dimensional linear subspace of $\mathbb{R}^S$ as the asset span and supports the complete markets allocation. With $H < J$, it is then true that the asset span need not be $J-$dimensional, but there are no real effects between an asset span with only H dimensions and one with J, provided that the complete markets allocation continues to be supported.

A.5 Proof of Theorem 2

Define the complete markets allocation as $c^* = \left(c^{*h}\right)_{h \in \mathbf{H}}$. Define the complete markets mapping $\mathbf{c}^{*h} : \mathbf{S} \to \mathbb{R}_+$ such that $c^{*h}\left(s^t\right) = \mathbf{c}^{*h}\left(s_t\right) \forall t, s^t$ and $\forall h \in \mathbf{H}$. Under the assumption that the preferences are identical and homothetic (Assumption 1), there exists $\left(\theta^h\right)_{h \in \mathbf{H}}$ such that $\mathbf{c}^{*h}\left(s\right) = \theta^h E\left(s\right) \forall \left(h, s\right) \in \mathbf{H} \times \mathbf{S}$.

The set containing all $J-$dimensional linear subspaces of $\mathbb{R}^S$ is the $J(S-J)-$dimensional Grassmanian space $Gr\left(J, S\right)$.

Given the state variables $\left(s, \hat{\omega}, \hat{b}_{\backslash 1}\right)$, the asset payout matrix is $\left[v'\right] \mathbf{Q}_0^{J-1}$, where

$$\left[v'\right] = \begin{bmatrix} \nu'\left(1\right) & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & \nu'\left(S\right) \end{bmatrix}.$$

The asset span is defined as

$$\left\langle \left[v'\right] \mathbf{Q}_0^{J-1} \right\rangle = \left\{ x \in \mathbb{R}^S : x = \left[v'\right] \mathbf{Q}_0^{J-1} b \text{ for some } b \in \mathbb{R}^J \right\} \in Gr(J, S).$$

The proof will be constructed so that the asset span $\left\langle \left[v'\right] \mathbf{Q}_0^{J-1} \right\rangle \in Gr\left(J, S\right)$ remains constant across all periods. If the asset span remains constant across all time periods and the asset price vector is stationary, the vector $\left(v'\left(\sigma\right)\right)_{\sigma \in \mathbf{S}}$ belongs to a 1-dimensional subspace $\Psi \subseteq \mathbb{R}^S_{++}$ in all time periods. Specifically, if $\left(v'\left(\sigma\right)\right)_{\sigma \in \mathbf{S}} \in \Psi$, then $\Psi = \left\{ \kappa \left(v'\left(\sigma\right)\right)_{\sigma \in \mathbf{S}} : \ \kappa > 0 \right\}$.

Hold the vector $\left(v'\left(\sigma\right)\right)_{\sigma \in \mathbf{S}} \in \mathbb{R}^S_{++}$ fixed. Under the complete markets allocation, there exists a unique vector $\mathbf{q} = \left(\mathbf{q}_j\left(s\right)\right)_{(j,s) \in \mathbf{J} \times \mathbf{S}} \in \mathbb{R}^{SJ}_+$ satisfying the asset price equations (21). Specifically, given $\Gamma \in \mathbb{R}^{S,S}_{++}$ with elements $\Gamma\left(s, \sigma\right) = \beta \pi\left(s, \sigma\right) \left(\frac{E(\sigma)}{E(s)}\right)^{-\rho} v'\left(\sigma\right)$, there exists a unique vector $\mathbf{q}$ such that $\mathbf{Q}_1^J = \Gamma \mathbf{Q}_0^{J-1}$. The procedure works by backward induction as first the unique vector $\left(\mathbf{q}_1\left(s\right)\right)_{s \in \mathbf{S}}$ is determined, which then means that the unique vector $\left(\mathbf{q}_2\left(s\right)\right)_{s \in \mathbf{S}}$ can be determined, and so forth.

The space $Gr(J, S)$ has $J(S-J)$ dimensions. The vector $\left(v'\left(\sigma\right)\right)_{\sigma \in \mathbf{S}} \in \mathbb{R}^S_{++} \backslash \Psi$ determines the asset span. Any non-proportional changes in $\left(v'\left(\sigma\right)\right)_{\sigma \in \mathbf{S}}$ will change the asset span $\left\langle \left[v'\right] \mathbf{Q}_0^{J-1} \right\rangle$. The dimension of $\mathbb{R}^S_{++} \backslash \Psi$ is $S - 1$.

Since $\left[v'\right] \mathbf{Q}_0^{J-1}$ only contains strictly positive elements, then with $J = 1$, all asset spans must satisfy $\left\langle \left[v'\right] \mathbf{Q}_0^{J-1} \right\rangle \subset \left(\mathbb{R}^S_+ \cup \mathbb{R}^S_-\right)$. This set does not include orthants with a mixture of positive and negative values. This is merely an artifact of the $J = 1$ case. Under Assumption 2 ($J > 1$), any constant asset span $\Phi \in Gr(J, S)$ is obtainable provided that

$\dim\left(\mathbb{R}^S_{++}\backslash\Psi\right) \geq \dim\left(Gr(J,S)\right)$. This requires

$$S - 1 \geq J(S - J). \tag{27}$$

Under Assumption 2 ($J > 1$) and Assumption 3 ($J < S$), the inequality (27) is only satisfied provided that $J = S - 1$. This is part of Assumption 4, meaning that any constant asset span $\Phi \in Gr\left(J, S\right)$ is obtainable.

Lemma 1 *Provided that $H \leq J$, there exists a $(\mathbf{q})-$Markov equilibrium and corresponding asset span $\Phi = \left\langle [v']\, \mathbf{Q}_0^{J-1} \right\rangle \in Gr(J,S)$ that is capable of supporting the complete markets allocation.*

As any optimal stationary policy Markov equilibrium supports the complete markets allocation, then there is no incentive for the monetary authority to deviate from this stationary policy. The optimal stationary policy Markov equilibrium must then be an optimal Markov equilibrium.

A.5.1 Proof of Lemma 1

Recall from (18) and (20) that the household budget constraints are given by:

$$c'^h(\sigma) + \sum_{j\in\mathbf{J}} \mathbf{q}_j\left(\sigma\right) \hat{b}'^h_j(\sigma) = \nu'\left(\sigma\right) \left(\mathbf{e}^h\left(s\right) + \sum_{j\in\mathbf{J}} \mathbf{q}_{j-1}\left(\sigma\right) \hat{b}^h_j\right). \tag{28}$$

Assume that the bond holdings are stationary and given by $\left(\hat{\mathbf{b}}^h_j\left(\sigma\right)\right)_{j\in\mathbf{J}}$. I later verify that stationary bond portfolios suffice to satisfy the household budget constraints (28) for the given payout matrix.

With the stationary bond portfolios $\left(\hat{\mathbf{b}}^h_j\left(\sigma\right)\right)_{j\in\mathbf{J}}$, define the value of the household expenditure vectors (at the complete markets allocation) as:

$$\mathbf{z}^{*h} = \left(\mathbf{c}^{*h}\left(\sigma\right) + \sum_{j\in\mathbf{J}} \mathbf{q}_j\left(\sigma\right) \hat{\mathbf{b}}^h_j\left(\sigma\right)\right)_{\sigma\in\mathbf{S}} \in \mathbb{R}^S.$$

Define Φ as a $J-$dimensional linear subspace of $\mathbb{R}^S$ such that $\mathbf{z}^{*h} \in \Phi\ \forall h \in \mathbf{H}$. This requires that $H \leq J$ (Assumption 4). Notice that the total value for $\sum_{h\in\mathbf{H}} \mathbf{z}^{*h}$ depends upon the choices of the monetary authority and does not equal a fixed parameter. The degrees of freedom in the excess demand vector $\left(\mathbf{z}^{*h}\right)_{h\in\mathbf{H}}$ is equal to H.

From the analysis above, when $J = S - 1$, there exists a stationary policy vector $\mathbf{q}$ and a stationary vector $(v'(\sigma))_{\sigma \in \mathbf{S}}$ such that $\Phi = \langle [v'] \mathbf{Q}_0^{J-1} \rangle$. Suppose the current shock is s. Since $\mathbf{z}^{*h} \in \Phi$, $\mathbf{z}^{*h} \in \langle [v'] \mathbf{Q}_0^{J-1} \rangle$, implying that there exists $\left(b_j^h\right)_{j \in \mathbf{J}} \in \mathbb{R}^J$ such that

$$\mathbf{z}^{*h} = \left(\mathbf{c}^{*h}(\sigma) + \sum_{j \in \mathbf{J}} \mathbf{q}_j(\sigma) \hat{\mathbf{b}}_j^h(\sigma) \right)_{\sigma \in \mathbf{S}} = [v'] \mathbf{Q}_0^{J-1} b_j^h.$$

This means that for all states $\sigma \in \mathbf{S}$:

$$\mathbf{c}^{*h}(\sigma) + \sum_{j \in \mathbf{J}} \mathbf{q}_j(\sigma) \hat{\mathbf{b}}_j^h(\sigma) = v'(\sigma) \sum_{j \in \mathbf{J}} \mathbf{q}_{j-1}(\sigma) b_j^h.$$

The specification of the stationary portfolios $\left(\hat{\mathbf{b}}_j^h(s)\right)_{j \in \mathbf{J}}$ in terms of $\left(b_j^h\right)_{j \in \mathbf{J}}$ is

$$\begin{aligned} b_j^h &= \mathbf{e}^h(s) + \hat{\mathbf{b}}_1^h(s) \text{ and} \\ b_j^h &= \hat{\mathbf{b}}_j^h(s) \text{ for } j > 1. \end{aligned}$$

This specification yields the household budget constraints (28) for all states $\sigma \in \mathbf{S}$:

$$\mathbf{c}^{*h}(\sigma) + \sum_{j \in \mathbf{J}} \mathbf{q}_j(\sigma) \hat{\mathbf{b}}_j^h(\sigma) = \nu'(\sigma) \left(\mathbf{e}^h(s) + \sum_{j \in \mathbf{J}} \mathbf{q}_{j-1}(\sigma) \hat{\mathbf{b}}_j^h(s) \right).$$

A.6 Proof of Theorem 3

The result is verified from the following two lemmas.

Lemma 2 *If Assumption 3 ($J < S$) holds and Assumption 4 does not hold (either $J < H$ or $J < S - 1$), then over a generic subset of household endowments there does not exist a $(\mathbf{q})$ $-$Markov equilibrium capable of supporting a Pareto efficient allocation.*

Lemma 3 *Under Assumption 3 ($J < S$), an optimal stationary policy Markov equilibrium that does not support a Pareto efficient allocation cannot be an optimal Markov equilibrium.*

A.6.1 Proof of Lemma 2

This proof refers to the steps used in the proof of Lemma 1.

If $J < H$, then over a generic subset of household endowments, there does not exist $\Phi \in Gr(J, S)$, a $J-$dimensional linear subspace of $\mathbb{R}^S$, such that $\mathbf{z}^{*h} \in \Phi \; \forall h \in \mathbf{H}$. If $J < S - 1$, then varying the vector $(v'(\sigma))_{\sigma \in \mathbf{S}} \in \mathbb{R}_{++}^S \backslash \Psi$ only allows for a measure zero subset of the possible asset spans $\Phi \in Gr(J,S)$ to be supported. Over a generic subset of household endowments, there would then not exist $\Phi \in Gr(J,S)$ such that $\mathbf{z}^{*h} \in \Phi\ \forall h \in \mathbf{H}$.

As the equilibrium asset span $\Phi \in Gr(J,S)$ does not contain $\left(\mathbf{z}^{*h}\right)_{h\in\mathbf{H}}$, then the complete markets allocation (or any other Pareto efficient allocation, for that matter) cannot be supported with a constant asset span as in the proof of Lemma 1.

Can a non-constant asset span be found to support the complete markets allocation?

The asset span for assets purchased in date-event s^{t-1} must then be $\Phi\left(s_{t-1}\right) \in Gr(J,S)$ as it depends upon the realization s_{t-1}. With a non-constant asset span, it must be such that $\Phi\left(s\right) \neq \Phi\left(s'\right)$ for some realizations $s, s' \in \mathbf{S}$. With $J < H$, then even if $\mathbf{z}^{*h} \in \Phi\left(s\right)$ $\forall h \in \mathbf{H}$, it can't be the case that $\mathbf{z}^{*h} \in \Phi\left(s'\right)\ \forall h \in \mathbf{H}$. The excess demand vectors must then be defined conditional on the realization s so that both $\mathbf{z}^{*h}(s) \in \Phi\left(s\right)\ \forall h \in \mathbf{H}$ and $\mathbf{z}^{*h}(s') \in \Phi\left(s'\right)\ \forall h \in \mathbf{H}$. The definition of the excess demand vectors, given by

$$\mathbf{z}^{*h}(s) = \left(\mathbf{c}^{*h}\left(\sigma\right) + \sum_{j\in\mathbf{J}} \mathbf{q}_j\left(\sigma\right)\hat{\mathbf{b}}_j^h\left(s,\sigma\right)\right)_{\sigma\in\mathbf{S}},$$

then implies that $\mathbf{z}^{*h}(s) \neq \mathbf{z}^{*h}(s')$ iff the stationary portfolios $\left(\hat{\mathbf{b}}_j^h(s,\sigma)\right)_{j\in\mathbf{J}} \neq \left(\hat{\mathbf{b}}_j^h(s',\sigma)\right)_{j\in\mathbf{J}}$ for the realizations $s, s' \in \mathbf{S}$. This results in a violation of the budget constraints as $\mathbf{z}^{*h}(s) \in \Phi\left(s\right)$ implies that there exists $\left(b_j^h(s)\right)_{j\in\mathbf{J}} \in \mathbb{R}^J$ such that

$$\mathbf{z}^{*h}(s) = \begin{bmatrix} \nu\left(s,1\right) & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & \nu\left(s,S\right) \end{bmatrix} \mathbf{Q}_0^{J-1} b_j^h(s).$$

This equation requires that $b_j^h(s)$ can only depend upon the realization s and not any prior realization. This contradiction finishes the claim that the complete markets allocation can only be supported with a constant asset span $\Phi \in Gr(J,S)$.

A.6.2 Proof of Lemma 3

Consider an optimal stationary policy Markov equilibrium and consider the incentive for the monetary authority to deviate from stationary policy. The current state variable is $\left(s, \hat{\omega}, \hat{b}_{\backslash 1}, q\right)$. Under stationary policy, $q'\left(s\right) = q$ is the next period asset price vector. Under a deviation from stationary policy, the monetary authority chooses the entire vector $\left(q'\left(\sigma\right)\right)_{\sigma\in\mathbf{S}}$

to maximize the conditional objective function

$$\sum_{\tau>t}\beta^{(\tau-t)}\sum_{\sigma}\sum_{s^{\tau}\succ(s^{t},\sigma)}\pi\left(s^{\tau}|s^{t}\right)\sum_{h\in\mathbf{H}}\alpha^{h}u\left(c^{h}\left(s^{\tau}\right)\right). \tag{29}$$

The payout matrix is

$$R\left(s\right)=\begin{bmatrix}\nu'\left(1\right) & 0 & 0\\ 0 & ... & 0\\ 0 & 0 & \nu'\left(S\right)\end{bmatrix}\begin{bmatrix}1 & q_1'\left(1\right) & .. & q_{J-1}'\left(1\right)\\ : & : & : & :\\ 1 & q_1'\left(S\right) & .. & q_{J-1}'\left(S\right)\end{bmatrix}.$$

From Claims 3 and 4, if the price vector $\left(q'\left(\sigma\right)\right)_{\sigma\in\mathbf{S}}$ satisfies no arbitrage, then there exists a corresponding SCE and a corresponding $\left(\left(q'\left(\sigma\right)\right)_{\sigma\in\mathbf{S}}\right)$ $-$Markov equilibrium. The vector $\left(q'\left(\sigma\right)\right)_{\sigma\in\mathbf{S}}$, chosen from the set of no arbitrage prices, determines the asset span $\Phi(s)=\langle R(s)\rangle\in Gr(J,S)$. The only additional requirement that must be satisfied for a Markov equilibrium is that the vector $\left(q'\left(\sigma\right)\right)_{\sigma\in\mathbf{S}}$ satisfies the backward-looking Euler equations (23):

$$q_j\left(s\right)=\beta\sum_{\sigma\in\mathbf{S}}\pi\left(s,\sigma\right)\left(\frac{c'^h(\sigma)}{c^h}\right)^{-\rho}\nu'\left(\sigma\right)q_{j-1}'\left(\sigma\right)\quad\forall j\in\mathbf{J}.$$

The price vector $\left(q'\left(\sigma\right)\right)_{\sigma\in\mathbf{S}}$ uniquely determines the asset span $\Phi(s)$. Proportional changes in $\left(q_{j-1}'\left(\sigma\right)\right)_{\sigma\in\mathbf{S}}$ (an elementary column operation on the payout matrix $R\left(s\right)$) do not change the asset span $\Phi(s)$, but do allow for the Euler equation (23) to be satisfied for asset j. Similar elementary column operations can be applied to all assets $j\in\mathbf{J}$. Thus, any asset span $\Phi(s)$ can be achieved by an appropriate choice of the stationary policy vector $\left(q'\left(\sigma\right)\right)_{\sigma\in\mathbf{S}}$ satisfying no arbitrage, since elementary column operations to satisfy (23) do not change the span.

Given that the optimal stationary policy Markov equilibrium is not Pareto efficient, then there exists an asset span $\Phi^{*}(s)$ for which the objective function (29) is maximized. I term this the optimal asset span. If the wealth distribution $\hat{\omega}'\left(\sigma\right)\neq\hat{\omega}$, then the optimal asset span $\Phi^{*}(s)\neq\Phi'^{*}(\sigma)$, where $\Phi'^{*}(\sigma)$ is the optimal asset span chosen for the state variables $\left(\sigma,\hat{\omega}'(\sigma),\hat{b}_{\setminus 1}'(\sigma),q'(\sigma)\right)$. The distribution $\hat{\omega}'\left(\sigma\right)\neq\hat{\omega}$ provided that $\sigma\neq s$. Under the assumption that no single state is absorbing, there always exists some shock $\sigma\neq s$ that occurs with strictly positive probability. This means that the optimal asset span in the current period (with state variables $\left(s,\hat{\omega},\hat{b}_{\setminus 1},q\right)$) is distinct from the optimal asset span in the next period occurring with strictly positive probability (with state variables $\left(\sigma,\hat{\omega}'(\sigma),\hat{b}_{\setminus 1}'(\sigma),q'(\sigma)\right)$). Since $\Phi^{*}(s)\neq\Phi'^{*}(\sigma)$, then the asset price vectors are distinct. The optimal policy is nonstationary, meaning that an optimal stationary policy Markov equilibrium cannot be an optimal Markov equilibrium.