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Base-Stock Models for Lost Sales: A Markovian Approach

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Abstract

We consider the lost sales model with discrete demand that is filled only from inventory on-hand. The inventory is reviewed every T periods and an order is placed to bring the inventory position back to a target base-stock level R. The order is received after a lead time of L periods. Based on the outstanding orders in the pipeline, we represent the state of the system as a Markov chain for a given base-stock level. We develop structural properties of the best stationary base-stock policy. These properties include a stopping rule and a method for finding strictly improving lower-bounds that facilitate the development of an optimization algorithm. We also show that the structure of the transition probability matrix is recursive in R and L and that the matrix is sparse with special structure. This special structure is used to facilitate computation of the stationary distribution. In turn this distribution is used to compute the long-run average cost for a given base-stock from which a search yields the best, minimum cost, base-stock value. Analytical results complemented by numerical examples reveal that neither the best stationary base-stock nor its average cost is monotone in L for a given T.

Keywords: Stochastic inventory system; Base-stock policy; lost sales; lead time

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1 Introduction

In a stochastic inventory system, there will be instances when there is no inventory on-hand. For such instances, a demand fulfillment policy must be specified. One strategy is to accept all demand and fill it from inventory that will arrive from a future delivery, so that demand is fully *backordered*. The other polar strategy is to turn away all customers that arrive during such instances so that demand is *lost*. While the backorder model is relatively well understood, the lost sales model with positive lead time, as noted by Zipkin (2008b), is “notoriously difficult" to analyze even when consideration is restricted to stationary base-stock policies. Nevertheless, by representing demand as a discrete random variable as in Bijvank and Johansen (2012), we are able to exploit the underlying Markovian structure to obtain structural insights that facilitate computation of base-stock policies and their long-run average costs. In particular, using analytical and computational methods, we find non-monotonicity results that were not previously available.

We consider the following infinite horizon variant of the standard single-item lost sales model. In each period $t, t = 1, 2, \dots$, discrete random demand drawn from an independent identical distribution, is realized. Demand that exceeds inventory on-hand is lost. Every T periods the inventory position is reviewed and a restock order is placed that brings the inventory position to the target base-stock level R, which may not be independent of the state of the system; making R independent of the steady-state results in a *stationary* base stock model. This order is received after a lead time of L periods, where without loss of generality, we let $L = (m-1)T + n$, $m \geq 1$ and $0 < n \leq T$, with integers m and n. When $m = 1$ and $n < T$, the lead time is less than the review period, yielding the case of so called “fractional" lead times as in Kapalka et al. (1999). When $m > 1$ and $n < T$, the lead time is greater than the review period. And, when $n = T$, the lead time is a multiple of the review period.

While the case when lead time is a multiple of the review period is predominant in the literature, this is not the case in increasingly more and more business settings. A classic example is that of a franchised hardware store that may place a weekly order to its distributor to replenish its many stock-keeping units; the distributor would typically fill high demand items from its inventory on-hand and schedule delivery for sometime within the week, making the lead time fractional. For other items, after consolidating demand from stores it serves, the distributor would place an order with the company’s central facility and would fill orders for such items in a subsequent week with the regularly scheduled consolidated weekly shipment. In such cases, the lead time would obviously be greater than the review period but not necessarily its multiple. Similar delivery systems are used to manage spare parts at auto dealerships and slow moving items in drugstores.

An example of recent vintage is the way Pyxis® machines are used to manage medical products on hospital floors. Every morning items are restocked to its base-stock targets, so called PAR levels; in the event that an item is out of stock and is needed immediately, an expedited delivery is executed to get that unit immediately from hospital stores; in such an instance, this item is effectively 'a lost sale' since it is filled from outside the system and so not counted in the inventory cycle at the Pyxis® machine. Remarkably, with the increasing penetration of internet retailers into same day and even two-hour delivery, fractional lead times are becoming more and more prevalent in distribution systems, warranting the need to study their impact on inventory control systems.

Literature Review

We begin with the seminal work by Karlin and Scarf (1958) whose analysis of the problem focused on cases with $T = 1$. Since the state of the system at each review epoch depends on the inventory on-hand and the ordered sequence of pipeline orders, the combinatorial issues appeared insurmountable. Nevertheless, for $T = L = 1$, Karlin and Scarf (1958) are able to find bounds on the cost of the optimal policy for a finite-horizon variant of the problem. Moreover, they made considerably more progress on the less intractable, and more practically appealing, stationary base-stock policy in which the target base-stock level R is independent of the state of the system. Specifically, for the infinite horizon case with a stationary base-stock policy, Karlin and Scarf (1958) find the closed-form stationary distribution of inventory on-hand when $T = L = 1$. For the more general case with lead time $L = m$, they develop the functional equation form of the joint distribution of pipeline inventory. However, because of its complexity, Karlin and Scarf (1958) were only able to derive the closed-form solution for the exponential demand model.

Morton (1969, 1971) revisited the lost sales model with $L = m \geq T = 1$, developed by Karlin and Scarf (1958). In the first paper, he established bounds similar to those in Karlin and Scarf (1958) for the case with lead time $L = m$. In the second paper, as an alternative to the base-stock model, the author proposed a myopic policy and compared it numerically with the optimal one and showed that his myopic policy performs well. Subsequently, Nahmias (1979) generalized the approximation technique of Morton (1971) and applied it to more complex models including the lost sales model with fixed setup cost.

Downs et al. (2001) considered the finite horizon lost sales model and used a sample-path approach to prove that the average conditional lost sales and leftovers are convex functions of a stationary base-stock policy. Later, Janakiraman and Roundy (2004) generalized these results to the case of stochastic lead time and infinite horizon; hence, under quite general conditions a simple search process is sufficient to find the best base-stock when $L = m \geq T = 1$.

More recently, Zipkin (2008a,b) explored the lost sales model with $L = m \geq T = 1$, and extended the work of Karlin and Scarf (1958) and those of Morton (1969, 1971). Zipkin (2008a) transformed the original formulation of the lost sales model to develop new bounds for the optimal policy. Subsequently, Zipkin (2008b) showed that the optimal average cost is increasing in the lead time, and then numerically compared several heuristics, including a dual-balancing policy introduced by Levi et al. (2008) whose cost is guaranteed to be within 50% of the optimal cost; and a constant order rule that is related to the work by Reiman (2004). As summarized in Zipkin (2008a), some heuristics perform better when the lead time is longer while others perform better when the penalty cost is higher. In particular, Zipkin (2008b) reports that the performance of the stationary base-stock model deteriorates as lead time increases but improves as the penalty cost increases. In a more theoretical direction, however, support for the efficacy of the stationary base-stock policy is provided by Huh et al. (2009) who showed that the cost of the stationary base-stock policy converges to the cost of the optimal policy when the ratio of lost sales penalty to inventory holding cost tends to infinity. Additional support for the efficacy of stationary base stock policies is provided by Bijvank et al. (2014).

Despite much attention on elucidating the structure of the lost sales model, we had to wait until 2009 before the “curse of dimensionality" long associated with the periodic review lost-sales model was finally demonstrated theoretically. Halman et al. (2009) proved that the periodic review lost sales problem, whose optimal policy has a non-stationary base-stock structure is NP-complete and so deemed intractable. Xin and Goldberg (2016) develop constant-order heuristic policies in which, as the lead time approaches infinity, the relative error in cost asymptotically approaches zero. While their performance might be acceptable when the lead time is very long, for short and intermediate values of lead time, as in our motivating examples, these heuristics may perform poorly. An excellent summary of the existing contributions is provided in Xin and Goldberg (2016).

Overview and Contributions

While the papers cited above have helped to improve our understanding of the lost sales model for the cases when lead time is a multiple of the review period $T = 1$, limited progress has been made when the review period is greater than 1, and even less progress when the lead time is arbitrary. To this end, Bijvank and Johansen (2012) have studied the case of arbitrary lead time when the demand is modeled by a compound Poisson process. They propose and test a value iteration based approximation procedure for computing stationary base-stock policies for such systems. In contrast, like Karlin and Scarf (1958), we focus on obtaining the distribution of the pipeline inventory and inventory on-hand. However, we use a discrete demand representation to explicitly derive the operating characteristics for any positive integer-valued target base-stock level R, review period T and lead time L. Remarkably, we show that the structure of the problem with arbitrary lead time $L > T$ can be developed by recursively using the representation of the fractional lead time case with $L \leq T$. So, we begin our analysis with the fractional lead time case which has not been well-studied despite the increasing prevalence of fast delivery modes including overnight and same-day delivery services by couriers like Fedex and USPS.

In our analysis of the fractional lead time case, we use the fact that when the lead time $L \leq T$, the review period, no orders are outstanding at a review epoch so that the inventory position and inventory on-hand are identical. Thus, the inventory on-hand at two consecutive review epochs is sufficient to represent state variables in a one-dimensional Markov Chain to describe transitions in the base-stock system. We preview our results in Section 2 by presenting a comprehensive analysis of a simple but incisive problem in which, without loss of generality, the review period is normalized to 1 time unit and the lead time as a fraction of it. As we will see in Section 2 for fractional lead time, and in later sections for arbitrary lead time, the system may incur lost sales early in the review cycle in low inventory on-hand states; yet, end the cycle with positive inventory on-hand. Obviously, this phenomenon cannot occur when lead time is a multiple of the review cycle.

It turns out that this seemingly simple problem with fractional lead time has remarkably complex optimal policy structure, which saliently captures and explains virtually all the crucial findings in our work. Interestingly, the best stationary base-stock (which is also the optimal ordering policy for this problem) may decrease and increase in lead time suggesting that the tradeoff between lost sales penalty and inventory holding costs can become muddled. Even more interestingly, the optimal cost may increase first, then become constant, and then finally decrease in lead time. These structural results sharply contrast with the monotonicity results derived in Zipkin (2008b) where lead time is a multiple of the review period, and consequently, they highlight the theoretical importance and practical relevance of the study of fractional and arbitrary lead time when unmet demand is lost.

Using the simple example as a launching pad, in Section 3 we formalize our representation not only of the fractional lead time case but also of the general case with $L = (m-1)T + n$ and discrete demand. We then turn attention to examining the structure of stationary base-stock policies. Even though the cost function may not be convex or unimodal, adapting arguments as in Downs et al. (2001) and Janakiraman and Roundy (2004), we are able to sufficiently structure the cost function to generate a simple stopping rule that can be embedded in a sequential search algorithm for the best base-stock. A by-product of the algorithm is that at each iteration it generates a strictly improving lower bound, and a non-increasing (feasible) upper bound providing the option to terminate early if a solution sufficiently close to optimal has been found.

Having completed work on structural properties of the cost function of base-stock policies, we turn attention to the underlying Markovian structure. In Section 4, we show that the transitions of the state process vectors from one order epoch to the next can be represented as a Markov chain whose states are represented by m-dimensional vectors where m represents the maximum number of outstanding orders. We also show that in the general case, the transition probability matrix of the Markov chain representing outstanding orders has an elegant recursive structure in R and L. This special structure uses the variant in which $m = 1$ so the lead time $L \leq T$, the review period and facilitates the computation of the stationary distribution of the inventory on-hand. Using the stationary distribution, we show that the best stationary base-stock level and performance measures of interest such as long-run average cost can be computed. As a result, the best base-stock policies can be calculated for the lost sales model with arbitrary lead times.

In Section 5 these computations are illustrated by a series of numerical examples. For the fractional lead time case and bounded demand, we show how to generate all admissible policies for the lost sales models. Then for a test-bed of problems, numerical computations reveal that the best stationary base-stock policy is often the optimal policy, and performs well even when it is not the global optimum. Then, consistent with the motivating example developed in Section 2, we demonstrate that the best base-stock and its long-run average cost are not necessarily non-decreasing in L . To the best of our knowledge, such non-monotonicity has not been reported in the literature on lost sales inventory systems.

Some concluding remarks are presented in Section 6. Proofs of the results, details on expected cost functions, an illustrative example and some of the numerical results are presented in Appendices.

2 A Motivating Example

Before formally introducing the general model with discrete demand and an arbitrary lead time, to preview and explain our results we use a simple base-stock model in which the review period T equals 1 and the fractional lead time L is between 0 and 1. This is without loss of generality since both T and L can be scaled to make the review period any positive value T. If at the start of the review period, the inventory position which equals the inventory on-hand (since the lead time does not exceed T), is below the target base-stock R, an order is placed to bring the inventory position to R. In sub-section 2.3 we will examine the implication on global optimality of making R state-dependent.

The process of demand during the review period is quite simple: with probability $1 - \delta$ demand takes value 0; otherwise demand is exactly 1. When demand is 1, the timing of its realization is uniformly distributed between 0 and 1. Such a demand process would arise if demand during the review period were generated by a truncated Poisson process since in such a case when demand is 1, the timing of its realization would be uniformly distributed between 0 and 1. The unit inventory holding cost is h per period and the unit lost sales penalty cost is l. The unit procurement cost c is normalized to zero. This simple base-stock model allows us to generate sharp insights.

We know immediately that the best base-stock R is either 0 or 1 or 2. When $R = 0$, all demand is lost, and when $R = 2$, all demand is met since the system would have at least one unit on hand at the start of every review period. Finally, when $R = 1$ the system dynamics are more complex. Consider the start of the focal review period t. If the inventory position and inventory on-hand is 1, no order is placed, but the demand, if it is realized, will be filled. Alternatively, if the inventory position and inventory on-hand is 0, an order of size 1 is placed which will be received at time $t + L$. If there is demand and it is realized between time t and $t + L$, it will be lost; if it is realized after $t + L$, it will be filled from the order that arrives at time $t + L$. These dynamics, as well as those for the general model, are elegantly analyzed using the theory of Markov chains.

2.1 Analysis of Stationary Base-stock Cases

We begin the analysis with the degenerate case when the base-stock $R = 0$. Since all demand is lost in this case, it is easily seen that $TC(0) = \delta l$. While this case is trivial, the analysis of the case when $R = 1$ leads to some insight.

To proceed let P_{ij} denote the one-period transition probability, where i and j represent the inventory on-hand of period t and period $t + 1$, respectively. Then, recognizing that when $i = 0$, demand is only filled if it does not arrive before time $t + L$, we can easily see that $P_{00} = \delta(1 - L)$; therefore, we must have $P_{01} = \delta L + (1 - \delta)$. Similarly, when $i = 1$, demand will be filled if it occurs, so we can set $P_{10} = \delta$ and $P_{11} = (1 - \delta)$. Therefore, it is easily verified that $\pi_0 = \frac{\delta}{\delta L+1}$ and $\pi_1 = \frac{1-\delta+\delta L}{\delta L+1}$ are the stationary probabilities that the inventory on-hand is 0 and 1, respectively. Given this stationary distribution, the average cost per period can be represented as:

$$TC(1) \quad = \quad \frac{h[(2-\delta) + \delta^2 L(1-L)]}{2(\delta L + 1)} + \frac{\delta^2 L l}{(\delta L + 1)}. \tag{1}$$

Because we are especially interested in how the optimal R varies in lead time L, it is useful to determine that

$$\frac{dTC(1)}{dL} = \frac{\delta^2 h}{(\delta L+1)^2}\left[\frac{l}{h} - \frac{[2+2\delta L+\delta^2 L^2-2\delta]}{2\delta}\right]. \tag{2}$$

Using $E(\delta, L) = \frac{2+2\delta L+\delta^2 L^2-2\delta}{2\delta}$, we can easily see that when $E(\delta, 0) < \frac{l}{h} < E(\delta, 1)$ holds, TC(1) initially increases in L and then begins falling if $E(\delta, L)$ becomes big enough. Now that we have developed some structural properties of TC(1), we can use them to compare the case $R = 1$ with the case $R = 0$. To this end we can immediately calculate that

$$TC(0) - TC(1) = \frac{\delta h}{(\delta L+1)}\left[\frac{l}{h} - \frac{[(2-\delta)+\delta^2 L(1-L)]}{2\delta}\right]. \tag{3}$$

It is helpful to define $A(\delta, L) = \frac{[(2-\delta)+\delta^2 L(1-L)]}{2\delta}$ and then notice that since $A(\delta, L)$ is quadratic in L, as L increases either $TC(1)$ never crosses (or equals) $TC(0)$, or crosses $TC(0)$ two times, first from below at L_1 then from above at L_3. In the case when $TC(1)$ crosses $TC(0)$ two times, it follows incisively that if we were restricted to choosing between $R = 0$ and 1, the lowest cost policy would be to set $R = 1$ when $L = 0$. As L increases the cost would increase until a threshold(L_1) where R would become 0, and the cost $TC(0)$ would remain constant until L reaches another threshold(L_3) where R would become 1 again and the cost $TC(1)$ would continue to fall after L_3. To complete the analysis, we need to consider the case of $R = 2$ as well.

In contrast to the case when $R = 0$ under which we are guaranteed 0 sales, when the base-stock $R = 2$ we are guaranteed to meet all demand. Since the analysis of this case is similar to the case when $R = 1$ it is deferred to the Appendix A. Using developments similar to those that yielded (1) the average cost per period can be easily represented as:

$$TC(2) \quad = \quad h\left[2 - 0.5\delta - \delta L\right]. \tag{4}$$

Because $\frac{\partial TC(2)}{\partial L} = -\delta h < 0$, $TC(2)$ is linear and decreasing in L.

Next, we determine when R = 1 is preferred to R = 2. Specifically,

$$TC(1) - TC(2) = \frac{\delta^2 L h}{(\delta L+1)}\left[\frac{l}{h} - \frac{(1-\delta)}{\delta} - \frac{(2-\delta^2 L^2)}{2\delta^2 L}\right]. \tag{5}$$

Analogous to $A(\delta, L)$, it is helpful to define $B(\delta, L) = \frac{(1-\delta)}{\delta} + \frac{(2-\delta^2 L^2)}{2\delta^2 L}$, and determine that $\frac{\partial B(\delta,L)}{\partial L} = \frac{-1}{\delta^2 L^2} - \frac{1}{2} < 0$, so that given $0 \le \delta \le 1$, $B(\delta, L)$ monotonically decreases in L, and attains its global minimum at $L = 1$. Furthermore, $B(\delta, 1) = \frac{(1-\delta)}{\delta} + \frac{(2-\delta^2)}{2\delta^2} = \frac{(2+2\delta-3\delta^2)}{2\delta^2}$, and $\lim_{L\to 0^+} B(\delta, L) = +\infty$.

Therefore, consistent with intuition, $R = 2$ cannot be optimal when $L = 0$ since instantaneous order delivery assures that we can always start with one unit in inventory on-hand. Thus, we can conclude that as L increase from 0 to 1 it follows that if we were restricted to choosing between $R = 1$ and 2, the lowest cost policy would be to set $R = 1$ when $L = 0$. As L increases, the cost $TC(1)$ first goes up and then down as discussed in sub-section 2.1, but if $TC(2)$ intersects $TC(1)$ it would only be from above at some point L_4 when R would become 2. For higher values of L, the cost $TC(2)$ would continue to fall. To complete the characterization of the stationary base-stock policy, by comparing $A(\delta, L)$ and $B(\delta, L)$ we can provide the following complete characterization of the optimal stationary policy, where R^* is the cost minimizing base-stock.

Proposition 1 *Let* $A(\delta, L) = \frac{[(2-\delta)+\delta^2 L(1-L)]}{2\delta}$ *and* $B(\delta, L) = \frac{(1-\delta)}{\delta} + \frac{(2-\delta^2 L^2)}{2\delta^2 L}$. *Then, in the best stationary base-stock solution* $($*sensitivity to* $\frac{l}{h}$$)$*:*

(a) *if* $\frac{l}{h} \leq A(\delta, L)$, *then* $TC(0) \leq TC(1) \leq TC(2)$ *and* $R^* = 0$*;*

(b) *if* $A(\delta, L) < \frac{l}{h} \leq B(\delta, L)$, *then* $TC(1) \leq TC(0)$, $TC(1) \leq TC(2)$ *and* $R^* = 1$*; and*

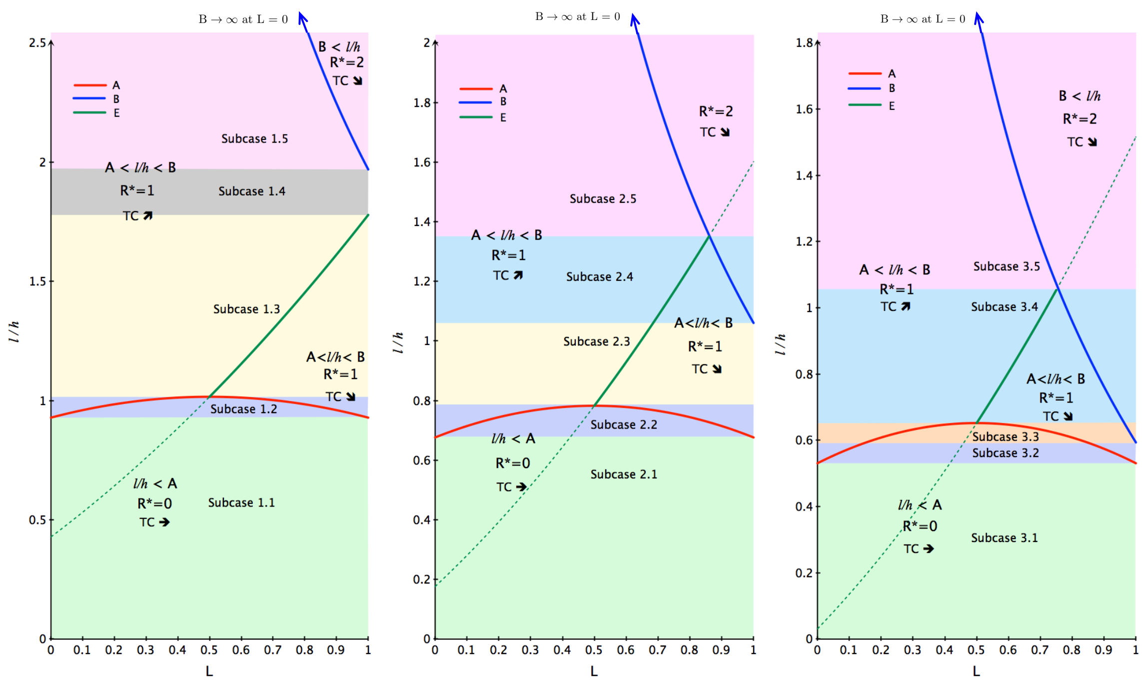
(c) *if* $\frac{l}{h} > B(\delta, L)$, *then* $TC(2) \leq TC(1) \leq TC(0)$ *and* $R^* = 2$.

Proposition 1 gives a full characterization of the best stationary policy by comparing $\frac{l}{h}$ against functions of δ and L. Since $\frac{l}{h}$ is equivalent to the familiar newsvendor fractile, it can be used to provide some intuition. When the lost sales penalty cost is very low, balancing inventory cost with the cost of lost sales is fully tilted in favor of minimizing inventory cost pushing us toward $R = 0$ which we may interpret as advocating low inventory levels. In stark contrast, when the lost sales penalty cost is very high, balancing inventory cost with the cost of lost sales is fully tilted in favor of minimizing lost sales penalty cost pushing us toward $R = 2$ which we may interpret as advocating high inventory levels. In the intermediate range, we select $R = 1$. Proposition 1 is quite intuitive in another sense: it is consistent with the conventional wisdom that suggests that as the lost sales penalty cost increases, the decision maker should respond by raising inventory levels to provide higher service levels. While Proposition 1 is consistent with conventional wisdom, our results, as we have alluded to earlier, are not quite as intuitive when sensitivity to lead time L is considered.

2.2 Sensitivity to L

In this sub-section we focus on examining how R^* and $TC(R^*)$ vary as L increases. We know from Proposition 1 that determining the best base-stock R^* relies on comparing $\frac{l}{h}$ with $A(\delta, L)$ and $B(\delta, L)$. So let us begin by considering the boundary case when $L = 0$. From Proposition 1, we know that $R^* = 0$ if

$\frac{l}{h} \leq A(\delta, 0)$. And, since $A(\delta, L)$ achieves its minimum at $A(\delta, 0) = A(\delta, 1) \leq B(\delta, 1)$, $R^* = 0$ for all L. Moreover $TC(0) = \delta l$, so it is independent of lead time L. The analysis for the case of $\frac{l}{h} > A(\delta, 0)$ is more complex. It turns out that, as shown in Appendix A, there are two numbers δ_{low} and δ_{med} that partition the unit interval for δ into three mutually exclusive intervals. The first constant δ_{low} is determined by equating $E(\delta, 1)$ with $B(\delta, 1)$, where $\delta_{\text{low}} = \sqrt{3} - 1$ is the only positive root of a cubic equation in δ. The other constant δ_{med} is determined by equating $B(\delta, 1)$ with $A(\delta, 0.5)$, where $\delta_{\text{med}} \approx 0.945688$ is the unique positive solution to another cubic equation in δ. Therefore, in what follows, the sensitivity analysis of optimal R^* and $TC(R^*)$ over L is conducted over three intervals of δ : $0 < \delta \leq \delta_{\text{low}}$, $\delta_{\text{low}} < \delta \leq \delta_{\text{med}}$ and $\delta_{\text{med}} < \delta \leq 1$. Graphs of $A(\delta, L)$, $B(\delta, L)$ and $E(\delta, L)$ against L are presented in Figure 1 for three representative values of δ; namely, $\delta = 0.7, 0.85$ and 0.97 since the best base-stock depends on where the point $(L, \frac{l}{h})$ falls on it.



(a) Case 1; δ=0.7

(b) Case 2; δ=0.85

(c) Case 3; δ=0.97

Figure 1: R^* and subcases for low, medium and high value of δ

In each panel, as explained formally in Appendix A and shown in Figure 1, the policy structure can be partitioned into five regions that depend on the value of $\frac{l}{h}$. When $(L, \frac{l}{h})$ is in the region below $A(\delta, L)$, $R^* = 0$ and the minimal cost is constant in L. In contrast, when $(L, \frac{l}{h})$ is in the region above $B(\delta, L)$, $R^* = 2$ and the minimal cost is decreasing in L. In the intermediate regions $R^* = 1$; when $(L, \frac{l}{h})$ is to the left of $E(\delta, L)$, the minimal cost is increasing in L and when $(L, \frac{l}{h})$ is to the right of $E(\delta, L)$, the minimal cost is decreasing in L. While we can easily see that for a given L, consistent with Proposition 1, as $\frac{l}{h}$

increases R^* goes from 0 to 1 to 2, the sensitivity in L is more nuanced.

A rigorous examination of the three panels of Figure 1, as performed in Appendix A, reveals that comparative statics in L depend on the values of $\frac{l}{h}$ to yield five regions drawn from seven policies. As $\frac{l}{h}$ increases, R^* evolves in L as depicted in Table 1 ($R^* = 0, 1$ or 2), where the arrows '$\nearrow$','$\searrow$' and '$\rightarrow$' represent increasing, decreasing and constant costs, respectively; and $x = 1, 2$ and 3 repesent case 1, case 2 and case 3, respectively.

Table 1: Evolution of R*

subcase	Low δ	Med δ	High δ
x.5	$1^{\nearrow}, 2^{\searrow}$	$1^{\nearrow}, 2^{\searrow}$	$1^{\nearrow}, 2^{\searrow}$
x.4	$1^{\nearrow}$	$1^{\nearrow}, 1^{\searrow}, 2^{\searrow}$	$1^{\nearrow}, 1^{\searrow}, 2^{\searrow}$
x.3	$1^{\nearrow}, 1^{\searrow}$	$1^{\nearrow}, 1^{\searrow}$	$1^{\nearrow}, 0 \rightarrow, 1^{\searrow}, 2^{\searrow}$
x.2	$1^{\nearrow}, 0 \rightarrow, 1^{\searrow}$	$1^{\nearrow}, 0 \rightarrow, 1^{\searrow}$	$1^{\nearrow}, 0 \rightarrow, 1^{\searrow}$
x.1	$0 \rightarrow$	$0 \rightarrow$	$0 \rightarrow$

The key insight that we wish to highlight, as presented for representative cases in Figure 1, is that neither R^* nor its cost is monotonically increasing in L. More specifically, consider Subcase 1.2 in Panel (a) of Figure 1. Consistent with Proposition 1(b), $R^* = 1$ when L is near 0; as L increases its cost initially increases; for intermediate values of L, $R^* = 0$ and this cost is constant; finally for higher values of L, $R^* = 1$ but $TC(R^*)$ is decreasing in L. An even more complex case is that of Subcase 3.3 in which the optimal R^* is initially 1, with $TC(R^*)$ increasing in L; then for somewhat higher values of L, R^* becomes 0 and $TC(R^*)$ is constant; and for a subsequent interval of L, R^* is again 1 but $TC(R^*)$ is decreasing in L; finally, R^* becomes 2 and $TC(R^*)$ remains decreasing in L. It is worth noting that such complex pattern has not been reported in the literature on base-stock inventory systems.

We have shown, using this motivating example, that when limited to stationary base-stock policies, neither R^* nor $TC(R^*)$ is necessarily monotone in L. Using these results as building blocks we now examine the implications for the globally optimal policy.

2.3 The Global Optimal Policy

While stationary base-stock policies have analytical tractability since the target base-stock is always a constant R, the optimal policy need not have this structure. In general, the target base-stock $R(x)$ may depend on the value of x, the inventory on-hand at the start of the review cycle. While enumerating all candidate state-dependent policies can be formidable, in this case, since demand during the review period is at most 1 and the lead time is at most 1 period, simple dominance can be used to show that it is not optimal to have x exceed 2 so that $R(x)$ also cannot exceed 2. Thus, other than the three stationary policies, simple enumeration reveals that after taking transient states into account, there is only one admissible state-dependent policy.

Specifically, in this state-dependent policy, $R(0) = 2$, $R(1) = 1$ and $R(2) = 2$. Then, using the approach of the previous sub-sections, the transition probability matrix can be written as:

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & \delta(1-L) & \delta L + (1-\delta) \\ \delta & 1-\delta & 0 \\ 0 & \delta & 1-\delta \end{bmatrix} \end{array}, \tag{6}$$

It is easily verified that $\hat{\pi}_0 = \frac{\delta}{2+\delta L}$, $\hat{\pi}_1 = \frac{1}{2+\delta L}$, and $\hat{\pi}_2 = \frac{1-\delta+\delta L}{2+\delta L}$ are the stationary probabilities that the inventory on-hand is 0, 1, and 2, respectively. Using the subscript sdp to represent state-dependent policy, given this stationary distribution, as shown in Appendix A, the average cost per period can be written as:

$$TC_{sdp} = \frac{\delta^2 Ll}{(2+\delta L)} + \frac{h}{2(2+\delta L)} \left\{6 - 2\delta + 4\delta^2 - 7\delta^2 L + 3\delta^2 L^2\right\}.$$

Since $R(x)$ is never equal to zero, it is sufficient to compare TC_{sdp} with $TC(1)$ and $TC(2)$. After rather involved calculations, we can conclude that $TC_{sdp} > \min\{TC(1), TC(2)\}$ for any $0 \le L \le 1$ and $0 \le \delta \le 1$. Since this is the only possible candidate state-dependent policy for our model, we can claim the following:

Proposition 2 *For our motivating example where: (i) the review period T equals 1 and the fractional lead time L is between 0 and 1, and (ii) demand in a period is either 0 or 1, and when the demand is 1, its timing of realization is uniformly distributed between 0 and 1,*

1) A stationary base-stock policy is globally optimal; and

2) Neither the best base-stock nor the optimal cost is monotone in the lead time L.

Thus, we can conclude that for the fractional lead time case, it can happen that the stationary base-stock policy is optimal and the optimal cost can first increase then decrease in L. Now that we have given some preliminary theoretical evidence that stationary base-stock policies can be useful, we next show that their Markovian representation yields sufficient structure when demand is drawn from a discrete distribution and the lead time takes arbitrary integer values. We will use this structure in Section 5.1 to show that under some conditions, the base-stock policies can be optimal and/or perform as effective heuristics for the underlying lost sales model.

3 Model Formulation with Arbitrary Lead Time

Having introduced the key ideas of our work, we now formulate the long-run average cost problem for the infinite horizon inventory system with lost sales. The cost parameters are stationary and the stochastic demand in each period follows an independent and identical discrete distribution. The discrete demand assumption allows us to represent dynamics using stochastic matrices. The system inventory is reviewed every T periods and an order is placed before the realization of the demand to bring the inventory position back to a target base-stock level R. This order is received after a lead time of L periods. Note that the order quantity is zero if no sales occur in the previous cycle. To proceed with the analysis, we will use the following notation:

Demand, Time and Cost Information

D_t = demand in period t, a non-negative integer.

L = order lead time, a positive integer.

T = review cycle, a positive integer.

c = unit purchase cost, may be set to 0 without loss of generality.

h = unit holding cost.

l = unit lost sales penalty cost.

Performance-Related Variables

R = target base-stock level, a non-negative integer.

IL_t = inventory on-hand at the beginning of period t; it includes the order received in period t.

IO_t = order placed in period t.

S_t = sales in period t; $\min\{IL_t, D_t\}$.

P_z^w = probability that the total demand of w consecutive periods equals z;

for w = 1, this is simply P_z and for w = 0, P_0^0 is identically equal to 1.

$\bar{P}_z^w$ = probability that the total demand of w consecutive periods is at least z.

A_R^L = transition probability matrix with base-stock R and lead time L.

LO_t = expected leftovers in period t.

LS_t = expected lost sales in period t.

$PC(R)$ = expected order quantity in a review cycle.

Cost Functions

TC_t = total expected inventory-related cost in period t.

$TC(R)$ = expected average cost in a review cycle with base-stock R and given L and T.

Using the above notation, as shown in Appendix B, we can write the total expected overage and underage cost in period t as $TC_t = h \cdot LO_t + l \cdot LS_t$. The objective is to choose the target base-stock level R that minimizes the long-run average cost per period given by:

$$TC(R) = \frac{c \cdot PC(R) + \sum_{t=1}^{T} TC_t}{T} \tag{7}$$

Note that the expressions of TC_t vary for different values of T and L and are presented in Appendix B.

To analyze the above inventory problem, we will focus on characterizing the performance of the focal review cycle that has the duration of T periods. Based on the outstanding orders in the pipeline, for a given base-stock level, we represent the state of the system as a Markov chain. We begin with the case when the lead time is fractional (i.e., $L = n \leq T$). Given the base-stock R, we first find the transition probabilities of this Markov chain whose transition matrix, A_R^L, is of dimension $(R+1) \times (R+1)$ whose size is independent of n and T. However, the values of the elements in this transition matrix do depend on n, T and R. Then we consider the general case with $L = (m-1)T + n$, $m \geq 1$ and $0 < n \leq T$. Remarkably, for this general case, the transition matrix, A_R^L, can be represented recursively in terms of the elementary matrices for the case $L = n \leq T$. We use this recursive relation together with the sparse structure of the matrix to compute the cost function and find the best stationary base-stock level. While we demonstrate this in Section 5, in the following sub-section and in Section 4, we analyze the model to establish key properties.

3.1 Sample Path Algorithm

The paper develops a search algorithm that computes $TC(R)$ for increasing values of R and stops when it can be ascertained that the best R found so far is also the optimal R or if R has reached an upper bound. Since it is not known whether $TC(R)$ is convex or unimodal in R, a simple stopping rule where you stop once $TC(R)$ starts increasing may terminate the search at a local minimum. However, a sample path argument, adapted from Downs et al. (2001), can be used to develop a sufficient condition to ensure that an optimal base-stock R has been identified when the search terminates.

To this end, we re-write the average cost function (7) as:

$$TC(R) = TCH(R) + TCL(R) \tag{8}$$

where TCH represents the inventory ordering plus holding related costs and TCL represents the lost sale related costs. A sample-path argument yields the following proposition.

Proposition 3 *(Based on Lemma 3.1 of Downs et al. (2001)):*

For $L = (m-1)T + n$, $m \geq 1$, $0 < n \leq T$, assuming $P\{D_r = 0\} > 0$, we have

(a) $M_r^{R+1} - M_r^R \in \{0, 1\}$, *where M_r refers to IL_r, IO_r, LO_r, and S_r; and*

(b) $LS_r^R - LS_r^{R+1} \in \{0, 1\}$.

Proposition 3(a), in particular, implies that for any cycle of T periods, the expected leftover inventory is non-decreasing in R and Proposition 3(b) implies that the expected lost sales is non-increasing in R. Therefore, $TCH(R)$ is non-decreasing in R and $TCL(R)$ is non-increasing in R. A proof for the proposition is provided in Appendix C. Although the result in the proposition is intuitive, the proof for it is quite involved.

We can now state the following corollary that serves as a simple stopping rule for finding the best stationary base-stock policy for our model even though the cost functions in (8) are not necessarily convex or unimodal unlike those in Downs et al. (2001) and Janakiraman and Roundy (2004).

Corollary 1 *Suppose $TC(R)$ has been calculated for $R = 1, 2, ..., U$, and let TC(R_U) be the minimal $TC(R)$ among these candidate U values of R. Then,*

(a) $TC(R) \geq TCH(U)$ *for* $R \geq U$;

(b) *If* $TC(R_U) \leq TCH(U)$, *then R_U is the best stationary base-stock;*

(c) *If* $TC(R_U) > TCH(U)$, *then* $TC(R^*) \leq TC(R_U)$ *and* $\frac{TC(R_U) - TC(R^*)}{TC(R^*)} \leq \frac{TC(R_U) - TCH(U)}{TCH(U)}$, *where R^* is the best stationary base-stock.*

So, as we incrementally increase the target base-stock from some R to $R+1$ and the algorithm terminates at some U where $TCH(U) \geq TC(R_U)$, the best stationary base-stock is given by R_U. Alternatively, if we have pre-set an upper bound U on the candidate base-stock, or the algorithm terminates after U evaluations due to computational limitations before the condition in stopping rule is satisfied, the algorithm returns R_U as the best base-stock found to date, and $TCH(U)$ and $TC(R_U)$ respectively as the highest lower bound and the lowest upper bound on $TC(R^*)$. In this case, the cost error rate of using R_U as the best stationary base-stock is bounded by $\frac{TC(R_U) - TCH(U)}{TCH(U)}$.

4 Analysis of the Model

We begin by considering the cases when the lead time $L = n \leq T$, the so called fractional lead time case. It yields $L = T$ as a special case. Then the structure of this problem is used as the building block for the case $L = (m-1)T + n$ which yields, after setting $n = T$, the special case $L = mT$, in particular $T = 1$ yields the case $L = m$, considered by Zipkin (2008a,b) and others.

4.1 The Case $m = 1$

We begin with the case when there is at most $m = 1$ order outstanding, so that $L = n \leq T$. To conduct the analysis it is sufficient to consider the focal review cycle from period t to period $t + T - 1$. At the beginning of this cycle, an order $IO_t = i, i = 0, 1, ..., R$, is placed to bring the inventory position to R so that the inventory on-hand, $IL_t = R - IO_t$. Therefore, given R, the state of the system at review epochs is fully specified by IO_t. Since the lead time is n and $m = 1$, as in the motivating example, the only order in pipeline inventory will arrive at the beginning of period $t + L$ (i.e., $t + n$) of the focal cycle; thus, it will not be available for satisfying demand in the first n periods so that only $IL_t = R - i$ is available to satisfy demand in the first n periods of the focal cycle. And the unused inventory supplemented by the newly received order is available to satisfy demand during the remaining $T - n$ periods of the focal cycle. Let k be the total sales during the focal review cycle. Then, at the next review epoch, IO_{t+T}, the order placed must be of size k since the inventory position is brought to base-stock R. Analogously to epoch t, at epoch $t + T$, IO_{t+T} fully specifies the state of the system. Therefore, the state transition process at review epochs is fully described as:

$$\cdots \to IO_t \to IO_{t+T} \to \cdots \to IO_{t+kT} \to IO_{t+kT+T} \to \dots \tag{9}$$

Now that we have shown that the evolution of the Markov chain can be characterized by only the amount on order at the review epochs, it remains to define the elements of the transition probability matrix (which, in short, we will call the transition matrix). To this end, suppose that at time t an order of size i was placed and the subsequent order of size k is placed at time $t + T$. Since $R - i$ units were on-hand at the beginning of the review cycle and the remaining i units arrive after lead time n, sales k may take any of the $R + 1$ discrete values from 0 to R; however, if $n = T$, the pipeline inventory i does not become available, so sales may not exceed $R - i$. Thus, the probability that a transition occurs from state i to state k is merely the probability that sales during the focal review cycle are exactly k. As in Section 2, this probability, as shown in Proposition 4 below, can be computed by splitting the review cycle into two complementary intervals, one consisting of the first n periods and the other consisting of the remaining $T - n$ periods.

In the following proposition whose trivial proof is omitted, we give the transition probabilities which are all that are needed to populate the transition matrix.

Proposition 4 *Let j be the inventory on-hand, i be the immediate (or outstanding) order and k be the next order. Then the transition probabilities, denoted by $P_j(i, k)$, of the Markov chain for lead time $L = n \leq T$*

are as follows:

When $i = 0$,

$$(a) \textit{ for } k < j,\ P_j(i,k) = P_k^T \textit{ and for } k = j,\ P_j(i,k) = \bar{P}_j^T; \tag{10a}$$

And, when $i > 0$,

$$(b) \textit{ for } k < j,\ P_j(i,k) = \sum_{z=0}^{k} \left[P_z^n \cdot P_{k-z}^{T-n} \right]; \tag{10b}$$

$$(c) \textit{ for } j \leq k < i + j,\ P_j(i,k) = \sum_{z=0}^{j-1} \left[P_z^n \cdot P_{k-z}^{T-n} \right] + \bar{P}_j^n \cdot P_{k-j}^{T-n}; \tag{10c}$$

$$(d) \textit{ for } k = i + j,\ P_j(i,k) = \sum_{z=0}^{j-1} \left[P_z^n \cdot \bar{P}_{k-z}^{T-n} \right] + \bar{P}_j^n \cdot \bar{P}_i^{T-n}. \tag{10d}$$

Proposition 4(a) provides the transition probabilities for the case when $IL_t = R$ so that all inventory is available to satisfy demand in the entire focal cycle of T periods. Propositions 4(b)-(d) provide the transition probabilities for the case when $IL_t = R - i < R$ so that only a portion of the system inventory is available to satisfy demand in the first n periods of the focal cycle, and any unused inventory as well as the order received after n periods are available to satisfy demand during the remaining $T - n$ periods of the focal cycle. In the second case, depending on the amount of total sales, the transition probabilities are given by equations (10b)-(10d).

While each element of A_R^n is quite complex, as in the motivating example of Section 2, some intuition can be gleaned by considering the pairs $P_z^n P_{k-z}^{T-n}$. Each of these pairs, because demand in each period is independent, represents the probability that exactly z demands occur in the first n periods (lead time n) and exactly $k - z$ demands occur in the complementary interval of $T - n$ periods. For example, if demand in each period is drawn from a geometric distribution, then the term P_z^n is the probability that demand from a negative binomial distribution has the zth success after n trials.

An even more interesting case arises when the demand in each period is from a Poisson distribution with parameter λ. Then, because time increments are independent, the product $P_z^n P_{k-z}^{T-n}$ can be written as the product of the probability that there are exactly k demands from a Poisson distribution with intensity λT and the probability that a random variable with a binomial distribution has exactly z successes in k trials with success probability $\frac{n}{T}$, which is a generalization of the probability δL in Section 2.

Using the transition probabilities given in Proposition 4, we can easily write the transition matrix A_R^n for $R = 0, 1, 2, 3, \ldots$; we refer the readers to Figure D1 in Appendix D. Note that while the values of its elements depend on n, in general each of these matrices A_R^n is dense, distinct and of dimension $(R+1) \times (R+1)$ which is independent of n. An implication is that these matrices do not have a special structure to speed up computation; however, an insightful exception arises when $L = n = T$, as seen in the next sub-section.

4.1.1 The Case $L = T$

When $L = n = T$, the transition probabilities are obtained by simply letting $n = T$ in Proposition 4. In this case, it is directly seen that Proposition 4 is very simple in the sense that the probabilities given in equations (10b)-(10d) are identical to those in (10a). When this is the case, we can write the transition matrix for base-stock $R = 0, 1, 2, \ldots,$ as

$$A_0^T = \begin{array}{cc} & 0 \\ 0 & \left[\begin{array}{c} 1 \end{array}\right] \end{array}; A_1^T = \begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \left[\begin{array}{cc} P_0^T & \bar{P}_1^T \\ 1 & 0 \end{array}\right] \end{array}; A_2^T = \begin{array}{cc} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \left[\begin{array}{ccc} P_0^T & P_1^T & \bar{P}_2^T \\ P_0^T & \bar{P}_1^T & 0 \\ 1 & 0 & 0 \end{array}\right] \end{array}; \cdots \tag{11}$$

Since each of these matrices is upper triangular, each can be trivially inverted in one pass, to find the stationary distribution of the underlying Markov chain. Since $L = T$, is equivalent to $L = T = 1$, this result is the discrete counterpart to the analogous case of continuous demand solved by Karlin and Scarf (1958).

Before we formalize that the transition matrices for $m = 1$ arise repetitively and recursively in the sparse transition matrices $A_R^{(m-1)T+n}$, for the general case $L = (m-1)T + n$, to flesh out intuition, we next consider the case when $m = 2$.

4.2 The Case $m = 2$

In this case, $L = T + n$ and at the beginning of the first period in each review cycle, there are at most m = 2 orders outstanding. Thus, the state transition process now becomes $(IO_t, IO_{t+T}) \to (IO_{t+T}, IO_{t+2T})$. Since the state process is now two-dimensional, the states can be ordered in a variety of ways. Our approach is to order these states in such a way that the structure of the underlying transition matrices can be represented in terms of the transition matrices for the case $L = n \leq T$. We begin with the case $R = 0$.

In this degenerate case, since $R = 0$, there are no orders outstanding so there is only the state (0,0) which occurs with probability 1, so A_0^{T+n} is equal to A_0^n, as depicted in Figure 2. Now consider the case

$R = 1$; since the inventory on-hand can be 0 or 1, the state space takes the values (0,0), (1,0) and (0,1) with the transition probabilities depicted in Figure 2. Unlike in the case when $L = n$, where starting states appear in the same sequence as the ending states, here the ending states are in a different sequence. When the system is in state (0,0) or (1,0), sales would be 0 or 1 so that at the start of the next cycle only the states (0,0) or (0,1) may be reached. And, when the system is in state (0,1), at the start of the next cycle only the state (1,0) can be reached. Also notice that since the second outstanding order is of size 0 for the first two states, the dynamics only depend on the first outstanding order which is why the first square submatrix of size (2×2) in A_1^{T+n} is identical to A_1^n. Moreover, the states are the same as in A_1^n except that each state has been appended by 0 reflecting the size of the latest outstanding order which is 0. The final state and the second square submatrix is identical to A_0^{T+n}, except to keep the inventory on-hand the same, the size of the last order has been incremented by 1. This interpretation suggests that when $R = 1$, the number of states is equal to the number of states in A_1^n plus the number of states in A_0^{T+n}, that is, 2 + 1 = 3.

To flesh out intuition, consider the cases R = 2 and 3. When R = 2, the number of states in A_2^{T+n} is equal to the number of states in A_2^n plus the number of states in A_1^{T+n} , that is, 3 + 3 = 6. And, the states have been generated in a manner similar to that for the case R = 1. Analogously, when $R = 3$, the number of states in A_3^{T+n} is equal to the number of states in A_3^n plus the number of states in A_2^{T+n}, that is, 4 + 6 = 10. Thus, we can conclude, as will be formalized later in this section, when $L = T + n$, there are $(R + 2) \times (R + 1)/2$ states and the transition matrix A_R^{T+n} is block diagonal with blocks A_R^n and A_{R-1}^{T+n}. The first $R + 1$ states in A_R^{T+n} have been augmented by adding a zero vector to the states in A_R^n while the remaining states have been incremented by 1 for the second outstanding order in A_{R-1}^{T+n}. In Figure 2, we illustrate the above rule with R = 0, 1, 2 and 3.

Now that we have made an intuitive connection between the cases when m =1 and $m = 2$, we will formalize in Proposition 5, that for any base-stock level R, we can write A_R^{T+n} in a recursive way as the sparse block diagonal matrix

$$A_R^{T+n} = \begin{bmatrix} A_R^n & 0 \\ 0 & A_{R-1}^{T+n}. \end{bmatrix} \tag{12}$$

In the next sub-section, we show that the transition matrices for the case of general lead time $L = (m - 1)T + n$ can be expressed in a similar way. As in the case of $L = T + n$, the key is to arrange the column and row vectors of the transition matrix in appropriate sequences.

$$A_0^{T+n} = \begin{array}{cc} & \begin{array}{c} 00 \end{array} \\ \begin{array}{c} 00 \end{array} & \begin{bmatrix} 1 \end{bmatrix} \end{array} = A_0^n;$$

$$A_1^{T+n} = \begin{array}{cc} & \begin{array}{ccc} 00 & 01 & 10 \end{array} \\ \begin{array}{c} 00 \\ 10 \\ 01 \end{array} & \begin{bmatrix} P_0^T & \bar{P}_1^T & 0 \\ P_0^{T-n} & \bar{P}_1^{T-n} & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} = \begin{bmatrix} A_1^n & 0 \\ 0 & A_0^{T+n} \end{bmatrix};$$

$$A_2^{T+n} = \begin{array}{cc} & \begin{array}{cccccc} 00 & 01 & 02 & 10 & 11 & 20 \end{array} \\ \begin{array}{c} 00 \\ 10 \\ 20 \\ 01 \\ 11 \\ 02 \end{array} & \begin{bmatrix} P_0^T & P_1^T & \bar{P}_2^T & 0 & 0 & 0 \\ P_0^T & P_0^n P_1^{T-n} + \bar{P}_1^n P_0^{T-n} & P_0^n \bar{P}_2^{T-n} + \bar{P}_1^n \bar{P}_1^{T-n} & 0 & 0 & 0 \\ P_0^{T-n} & P_1^{T-n} & \bar{P}_2^{T-n} & 0 & 0 & 0 \\ 0 & 0 & 0 & P_0^T & \bar{P}_1^T & 0 \\ 0 & 0 & 0 & P_0^{T-n} & \bar{P}_1^{T-n} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} = \begin{bmatrix} A_2^n & 0 \\ 0 & A_1^{T+n} \end{bmatrix};$$

$$A_3^{T+n} = \begin{array}{cc} & \begin{array}{cccccccccc} 00 & 01 & 02 & 03 & 10 & 11 & 12 & 20 & 21 & 30 \end{array} \\ \begin{array}{c} 00 \\ 10 \\ 20 \\ 30 \\ 01 \\ 11 \\ 21 \\ 02 \\ 12 \\ 03 \end{array} & \begin{bmatrix} P_0^T & P_1^T & P_2^T & \bar{P}_3^T & & & & & & \\ P_0^T & P_1^T & \sum_{z=0}^{1} P_z^n P_{2-z}^{T-n} + \bar{P}_2^n P_0^{T-n} & \sum_{z=0}^{1} P_z^n P_{3-z}^{T-n} + \bar{P}_2^n \bar{P}_1^{T-n} & & & & & & \\ P_0^T & P_0^n P_1^{T-n} + \bar{P}_1^n P_0^{T-n} & P_0^n P_2^{T-n} + \bar{P}_1^n P_1^{T-n} & P_0^n \bar{P}_3^{T-n} + \bar{P}_1^n \bar{P}_2^{T-n} & & & & & & \\ P_0^{T-n} & P_1^{T-n} & P_2^{T-n} & \bar{P}_3^{T-n} & & & & & & \\ & & & & P_0^T & P_1^T & \bar{P}_2^T & & & \\ & & & & P_0^T & P_0^n P_1^{T-n} + \bar{P}_1^n P_0^{T-n} & P_0^n \bar{P}_2^{T-n} + \bar{P}_1^n \bar{P}_1^{T-n} & & & \\ & & & & P_0^{T-n} & P_1^{T-n} & \bar{P}_2^{T-n} & & & \\ & & & & & & & P_0^T & \bar{P}_1^T & \\ & & & & & & & P_0^{T-n} & \bar{P}_1^{T-n} & \\ & & & & & & & & & 1 \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} A_3^n & 0 \\ 0 & A_2^{T+n} \end{bmatrix}.$$

Figure 2: Transition matrices for the case $L = T + n$ for different R

4.3 The Case of Arbitrary m, $m \geq 1$

In this, the most general case there are at most m ($m \geq 1$) orders outstanding, the inventory is reviewed every T periods and an order placed at time t is received after a lead time of $L = (m-1)T + n$ periods. Since the lead time exceeds $(m-1)T$, it is clear that at the beginning of the focal review period $t+(m-1)T$, at every review epoch, there are m outstanding orders including the orders placed at the beginning of period $t, t+T, t+2T, \ldots, t+(m-2)T$, and the order placed at the current epoch $t+(m-1)T$. Thus, we have an outstanding order vector $(IO_t, IO_{t+T}, \ldots, IO_{t+(m-1)T})$, where IO_t will arrive at the beginning of period $t+L$ (i.e., $t+(m-1)T+n$) of the focal cycle which consists of T periods starting from $t+(m-1)T$ to $t+mT-1$. Thus, for this Markovian system, we can define the following state transition process:

$$\begin{aligned} &(IO_{t-(qm-1)T}, \ldots, IO_{t-(q-1)mT}) \rightarrow (IO_{t-(qm-2)T}, \ldots, IO_{t-(q-1)mT+T}) \rightarrow \ldots \\ &\cdots \rightarrow (IO_t, IO_{t+T}, \ldots, IO_{t+(m-1)T}) \rightarrow (IO_{t+T}, IO_{t+2T}, \ldots, IO_{t+mT}) \rightarrow \ldots \end{aligned} \tag{13}$$

for a positive integer q. Since R is the target base-stock level, the total outstanding order plus inventory on-hand equals R at the beginning of period $t+(m-1)T$, so that

$$IO_t + IO_{t+T} + \cdots + IO_{t+(m-1)T} + IL_{t+(m-1)T} = R \tag{14}$$

Equation (14) implies that $IL_{t+(m-1)T}$ is fully identified if we have observed the outstanding order vector $(IO_t, IO_{t+T}, \ldots, IO_{t+(m-1)T})$. Consequently, with base-stock R, as in the earlier cases, we can write the transition matrix from state $(IO_t, IO_{t+T}, \ldots, IO_{t+(m-1)T})$ to the next state based on $IL_{t+(m-1)T}$,

IO_t and the realized demand in the two complementary intervals of periods n and $T - n$ of the focal review cycle. For this purpose, we use a constructive approach to recursively develop the structure of the transition matrix and the state space of the underlying Markov chain for any integer R and lead time $L = (m-1)T + n$. The structure can be fully specified in terms of R, m and n. We need some additional notation to streamline the presentation of this rather tedious development. For transition matrix $A_R^{(m-1)T+n}$, we let $NS(R,m)$ denote the number of states (starting or ending), $L(R,m)$ the sequence of the starting states, and $M(R,m)$ the sequence of the ending states. Note that each state is a vector of m outstanding orders of the form $(IO_t, IO_{t+T}, \ldots, IO_{t+(m-1)T})$. The formal statement follows and an example has been included in Appendix D.

Proposition 5 *For $L = (m-1)T + n$, $m \geq 1$, $0 < n \leq T$, and base stock R, $R \geq 1$:*

(a) *The transition matrix is a block diagonal matrix, and, is of the recursive form*

$$A_R^{(m-1)T+n} = \begin{bmatrix} A_R^{(m-2)T+n} & 0 \\ 0 & A_{R-1}^{(m-1)T+n} \end{bmatrix}; \tag{15}$$

(b)*The list of starting states $L(R,m)$ is sequenced so that the first $NS(R,m-1)$ states are identical to those in the $L(R,m-1)$ except that each is appended by $i_m = 0$; and, for the remaining $NS(R-1,m)$ states are identical to those in $L(R-1,m)$ except that each i_m is increased by 1;*

(c) *The list of ending states $M(R,m)$ is in the order given by changing the each state in $L(R,m)$ from $i_1 i_2 \ldots i_m$ to $i_2 i_3 \ldots i_m i_1$;*

(d) *The number of states in $A_R^{(m-1)T+n}$ is $NS(R,m) = \frac{(R+m)!}{R!m!}$;*

(e) *The number of non-zero (positive) entires in $A_R^{(m-1)T+n}$ is $NZ(R,m) = \frac{(R+m)!(2R+m+1)}{R!(m+1)!}$;*

(f) *The density of the transition matrix $A_R^{(m-1)T+n}$ equals $\frac{R!m!(2R+m+1)}{(R+m)!(m+1)}$.*

The logic behind the matrix structure in Proposition 5(a) is the following: for $L = (m-1)T+n$, there are m outstanding orders, namely, $(IO_t, IO_{t+T}, \ldots, IO_{t+(m-1)T})$, where IO_t will arrive at the beginning of period $t+L$ (i.e., $t+(m-1)T+n$) in the focal cycle. This means that it is not available for satisfying demand in the first n periods, while it can be used to satisfy demand in the remaining $T - n$ periods of the focal cycle. Therefore, the role of IO_t in this case is equivalent to IO_t with lead time $L = (m-2)T + n$. Thus, if $IO_{t+(m-1)T} = 0$, then the transition matrix $A_R^{(m-1)T+n}$ will be the same as $A_R^{(m-2)T+n}$. Additionally, if $IO_{t+(m-1)T} > 0$, that is, $IO_{t+(m-1)T} \geq 1$, it is equivalent to adding one unit to $IO_{t+(m-1)T}$ first, and then arranging the remaining $R - 1$ units of inventory to $(IO_t, IO_{t+T}, \ldots, IO_{t+(m-1)T})$ accordingly, so that the transition matrix in this case is equivalent to $A_{R-1}^{(m-1)T+n}$. The above intuitive arguments explain

Propositions 5(a)-(b).

As an explanation of Proposition 5(c), note that since i_1 arrives during the focal cycle, at the beginning of the next cycle the oldest outstanding order is i_2. Therefore, the new state is of the form $i_2 i_3 \ldots i_m k$, where k is the sales in the focal cycle which, like i_1, can take any integer value between 0 to $R - \sum_{l=2}^{m} i_l$, in particular we let $k = i_1$ when generating the list of ending states $M(R, m)$. The implication of this result is that corresponding to each state in $L(R, m)$, we can locate the position of that state in $M(R, m)$ so that the stationary distribution for each state of the Markov chain can be determined. This is explained with an example using Figure D2 in Appendix D. Propositions 5(d)-(e) lead to Proposition 5(f) which shows that the *density* of the transition matrix (i.e., the number of positive entries over the total number of entries in the matrix) decreases as the base-stock R and the value of m increases.

It follows from Proposition 5 that although the state space grows exponentially in R and m, the recursive and block diagonal structure of $A_R^{(m-1)T+n}$ makes its storage requirements frugal. This is because only the $R+1$ square matrices representing $A_0^n, A_1^n, \ldots, A_R^n$ have to be computed once and stored. However, since the starting states $L(R, m)$ and ending states $M(R, m)$ are not in the same sequence, computation of the steady-state probabilities can be challenging. Nevertheless, as will be seen in the next section, these structural properties are sufficient for us to calculate performance measures and to find best base-stocks for problems with lead times longer than those solved previously.

5 Numerical Studies

Now that we have demonstrated that the base-stock model with lost sales and discrete demand can be represented as a Markov chain whose sparse transition matrix can be generated recursively, we use these structural properties to compute the stationary distribution. This steady-state distribution is used to compute the expected cost function (7) using equations (B1)-(B10) from Appendix B. In Section 5.3, we use these properties to consider large-scale problems and then, for some problems related to that in Section 5.2, we re-visit the issue of the efficiency or optimality of using stationary base-stock policies. One important observation comes out from our numerical results is that when L is not an integer multiple of T, neither the best stationary base-stock policy nor the corresponding average cost is necessarily monotone in L.

5.1 Efficiency of Base-stock Policies for Fractional Lead times

Having established that the best stationary base-stock policy is globally optimal for the motivating example of Section 2, we begin our numerical work by examining the performance of the best stationary base-stock policy for more representative demand scenarios for the fractional lead time cases of lost sales model. Since we know from Section 2.5 that the optimal cost function reaches its maximum when L is about $T/2$, we find it adequate to use the case when $T = 2$ and $L = 1$. And we modify the demand process to mimic the general structure that is presented in Sections 3 and 4. In the first set of scenarios, we model demand in each period to be Bernoulli so that demand is either 1 with probability p that corresponds to δ in Section 2, or 0 otherwise. Hence demand realizations in each review cycle can be given by the four familiar doubletons $\{0, 0\}, \{0, 1\}, \{1, 0\}$ and $\{1, 1\}$. In the second set of scenarios, we allow demand in each period to take values 0 or 1 or 2 where these probabilities are drawn from the Binomial distribution with $N = 2$ and success probability p. Here, the demand realization can be described by 9 doubletons. For both sets of these binomial demand scenarios, by carefully selecting p, we can make the demand distribution left- or right-skewed or mimic a symmetric bell-shaped distribution creating a plausible range of representative cases. An alternative approach would have been to use the truncated Poisson distribution, which is a little more difficult to calibrate but can approximate the Binomial distribution.

Since N represents the maximum demand in a period in our process, we know that, in the worst case, an order placed at the start of some period t is all that is available to fulfill demand for the next $T + L$ periods. Therefore, the maximum target inventory, in any inventory management policy, need not exceed $N(T + L)$ which in our scenarios is either 3 or 6. Since it might be best to not be in business which would make $R = 0$, the best stationary base-stock policy is one of $N(T + L) + 1$ values. And if the choice of the target value depends on x, the value of inventory on-hand at time t, then the target $R(x)$ would be one of the $N(T+L)+1-x$ values. Any target value greater than $N(T+L)$ can be eliminated because it would entail unnecessary holding costs. Since x may take any integer value between 0 and $N(T + L)$, it readily follows that consideration for finding the optimal policy can be limited to fewer than $(N(T + L) + 1)!$ policies. Thus in the first set of scenarios, since $N = 1, T = 2$ and $L = 1$, there are 24 admissible policies and in the second set of scenarios there are no more than 82,440 admissible policies. In practice, we were able to eliminate many of these polices by using logical considerations.

For 6 values of p, for each of these admissible policies, applying Proposition 5 in an adapted form, we generated the transition matrix. The underlying Markov chain was then solved to find the stationary probability distribution. As in Zipkin (2008b) we set $c = 0$, $h = 1$ and $l = 4$, 9, 19 or 39 to calculate the average expected cost per review cycle for each of these scenarios from which the minimal-cost globally optimal policy was found and the best stationary base-stock policy was compared against it.

Table 2: Optimality of Best base-stock policy with Bernoulli demand

Lost sales cost (l)	p	Average Cost Optimal	Average Cost Best Base-stock	% Error
	0.90	1.5000	1.5000	0%
	0.75	2.1400	2.1400	0%
4	0.60	2.0421	2.1118	3.41%
	0.45	2.0469	2.0469	0%
	0.30	1.6975	1.6975	0%
	0.15	1.6102	1.6102	0%
	0.90	1.5000	1.5000	0%
	0.75	2.2500	2.2500	0%
9	0.60	2.9059	2.9059	0%
	0.45	2.6594	2.6594	0%
	0.30	2.5901	2.5901	0%
	0.15	1.8796	1.8796	0%
	0.90	1.5000	1.5000	0%
	0.75	2.2500	2.2500	0%
19	0.60	3.0000	3.0000	0%
	0.45	3.4172	3.4172	0%
	0.30	3.0241	3.0450	0.69%
	0.15	2.4184	2.4184	0%
	0.90	1.5000	1.5000	0%
	0.75	2.2500	2.2500	0%
39	0.60	3.0000	3.0000	0%
	0.45	3.7500	3.7500	0%
	0.30	3.5404	3.5404	0%
	0.15	3.3609	3.3886	0.82%

As can be seen from Tables 2 and 3, these steps yield 24 problem instances each for our 2 scenarios. In the first demand scenario, Binomial(1,p), in 21 out of 24 cases, the best stationary base-stock policy is globally optimal. The percent errors if the best base-stock were used when it is not optimal, are 3.41%, 0.69% and 0.82%. Given the relative coarseness of the demand process, it might not be too surprising or unintuitive to find that the best base-stock performs very well.

Indeed, there is significant more variation under the second set of demand scenarios, Binomial$(2, p)$, as can be seen from Table 3. In 13 of these scenarios the best stationary base-stock policy was found to be globally optimal. In cases, where the base-stock policy was not optimal, consistent with Huh et al. (2009), the percentage error in cost declines as l/h increases. Thus, the base-stock policy is close to optimal for high l/h. And, when the base-stock policy is not optimal, the percentage error in cost tends to be lower when the average demand (as compared to the maximum demand in a period) is either high or low. And, the percentage error tends to be higher when demand is relatively symmetric as for $p=$0.45 and $p=$0.6. It also appears that when the base-stock policy is not optimal, the order when inventory on-hand is low tends to be higher than in the corresponding optimal policy. The effect of over-ordering seems to result in increased

Table 3: Optimality of Best-base stock policy with Binomial demand ($N = 2$)

Lost sales cost (l)	p	Average Cost Optimal	Average Cost Best Base-stock	% Error
	0.90	3.0000	3.0000	0%
	0.75	3.2769	3.4464	5.17%
4	0.60	3.3241	3.4382	3.43%
	0.45	3.0067	3.1652	5.27%
	0.30	2.5712	2.6590	3.41%
	0.15	1.8440	1.8440	0%
	0.90	3.0000	3.0000	0%
	0.75	4.1224	4.1224	0%
9	0.60	4.3635	4.4655	2.34%
	0.45	4.1694	4.2926	2.95%
	0.30	3.7381	3.8248	2.32%
	0.15	2.8532	2.8532	0%
	0.90	3.0000	3.0000	0%
	0.75	4.5000	4.5000	0%
19	0.60	4.9087	4.9087	0%
	0.45	4.9119	4.9531	0.84%
	0.30	4.4806	4.5168	0.81%
	0.15	3.4986	3.4986	0%
	0.90	3.0000	3.0000	0%
	0.75	4.5000	4.5000	0%
39	0.60	5.7347	5.7347	0%
	0.45	5.7965	5.8350	0.66%
	0.30	5.4321	5.4632	0.57%
	0.15	4.4147	4.4147	0%

inventory holding cost without corresponding savings in the lost sales cost. As l/h increases, the optimal fill rate increases and the system operates with high on-hand inventories, and as a result the frequency of such errors goes down. Since the relative holding cost for high l/h values is lower, the increase in the holding cost due to the error is also lower. When combined with Proposition 2, our results indicate that using the best stationary base-stock when lead times are fractional, can be optimal in many problem instances. And, it appears that its heuristic use is likely to be more effective when demand distributions are skewed, rather than symmetric. Now that we have established the viability of using base-stock policies in some problem instances, we next illustrate the application of our algorithm.

5.2 Sample Path Algorithm

Now that we have shown that stationary base-stock policies can be effective, at least in the fractional lead time case, we next want to illustrate the sample path algorithm (see Section 3.1), which supports finding the best policy in the most general setting. To this end, we use the cases of $l = 19$ with $p = .15$ and $p = .5$ that are presented in Tables 4 and 5, respectively. As in Section 5.1, the probabilities are drawn from the Binomial distribution with $N = 2$ and success probability p. We initialize the algorithm with $R = 0$ so that

R_U is necessarily equal to 0. Since no inventory is present, in both cases only lost sales costs are incurred. Thus, as can be seen in the first row of Tables 4 and 5, $TCH(0) = 0$ and $TC(R_U) = TC(0) = TCL(0)$. Then, for each case, we incrementally iterate by trying trial values of U, $U = 1, 2, 3, 4, 5, 6$, for the base-stock R. When $p = .15$, the algorithm terminates when $U = 3$ and returns $R^* = 2$ as the best stationary base-stock value; this would also be the case if $TC(R)$ were known to be unimodal. In contrast, when $p = .5$, the algorithm takes additional iterations before terminating when $U = 6$ and returns $R^* = 4$, as the best stationary base-stock value.

Table 4: Sample Path Algorithm with Binomial Demand $p = 0.15$

U	$TC(U)$	$TCL(U)$	$TCH(U)$	R_U	$TC(R_U)$	Process
0	5.7000	5.7000	0.0000	0	5.7000	Initialize
1	2.4358	1.9174	0.5183	1	2.4358	$TCH(U) < TC(R_U)$, Go to Next U
2	1.7493	0.4351	1.3142	2	1.7495	$TCH(U) < TC(R_U)$, Go to Next U
3	2.3157	0.0569	2.2588	2	1.7495	$TCH(U) \geq TC(R_U)$, STOP $R^* = R_U$
4	3.2545	0.0039	3.2506			
5	4.2501	0.0001	4.2500			
6	5.2500	0.0000	5.2500			

Table 5: Sample Path Algorithm with Binomial Demand $p = 0.5$

U	$TC(U)$	$TCL(U)$	$TCH(U)$	R_U	$TC(R_U)$	Process
0	19.0000	19.0000	0.0000	0	19.0000	Initialize
1	11.6316	11.5000	0.1316	1	11.6316	$TCH(U) < TC(R_U)$, Go to Next U
2	7.0000	6.5330	0.4670	2	7.0000	$TCH(U) < TC(R_U)$, Go to Next U
3	4.0326	3.0623	0.9703	3	4.0326	$TCH(U) < TC(R_U)$, Go to Next U
4	2.6193	0.9667	1.6526	4	2.6193	$TCH(U) < TC(R_U)$, Go to Next U
5	2.6618	0.1397	2.5221	4	2.6193	$TCH(U) < TC(R_U)$, Go to Next U
6	3.5000	0.0000	3.5000	4	2.6193	$TCH(U) \geq TC(R_U)$, STOP $R^* = R_U$

Now that we have shown that even with the coarseness of the data in small problems, the algorithm can terminate as soon as the first minimum is found, we use this algorithm to assist in the search for the best stationary base-stock policies in the more general demand cases considered in this paper.

5.3 Numerical Results

As discussed above, although the transition matrix is generated in such a way that it appears block diagonal, since the lists of starting and ending states are not in the same sequence except when lead times are less than or equal to the review period, this is not sufficient to speed up computations. Nevertheless, we only store the $R + 1$ square submatrices $A_0^n, A_1^n, \ldots, A_R^n$, which have the only non-zero entries in A_R^{mT+n}, we can save substantially on storage. Even though the state space grows rapidly, we did not use sophisticated computing strategies, like parallel processing to speed up computations. Given our goal of gleaning insight into the lost sales base-stock problem using exact methods, we found it sufficient to use an iterative scheme to solve $\pi A_R^{(m-1)T+n} = \pi$, where π is the row vector of size $NS(R, m)$ that represents the stationary distribution of the Markov chain. Computations were run using MATLAB 2013a, on a MacBook Pro OS X 10.9 with 2.4 GHz dual-core Intel core i5 processor and 8 GB memory.

To make our computational results comparable with those of Zipkin (2008b), who has recently made significant advances in exact computational methods by solving problems with $T = 1$ and L up to 4, we developed our test bed of problems from his design. Like him, we modeled demand as either Poisson or geometric with mean demand per period equal to 5 and set the purchase cost $c = 0$ and holding cost $h = 1$. Since the results for the geometric demand case are qualitatively similar, we report here results only for the Poisson demand case with $T = 1$ and 2, which are presented in Tables 6 and 7, respectively. The results for the Poisson demand case with $T = 3$, 4, 5 and 6, are given in Appendix E. We also report results for the geometric demand case in Appendix E.

Table 6: Numerical results for $T = 1$ with Poisson demand, up to $L = 6$

Lost sales cost(l)		L					
		1	**2**	**3**	**4**	**5**	**6**
4	Best base-stock R	12	16	20	25	29	33
	Ave. Cost	4.163	4.639	4.975	5.198	5.372	5.512
	No. of states(R^*)	13	153	1,771	23,751	278,256	3,262,623
	Comp. time(sec)	0.07	0.13	0.36	3.69	143.50	9678.80
9	Best base-stock R	13	19	23	28	33	37
	Ave. Cost	5.547	6.316	6.864	7.271	7.607	7.893
	No. of states(R^*)	14	210	2,600	35,960	501,942	6,096,454
	Comp. time(sec)	0.08	0.19	0.54	6.56	476.45	32257.06
19	Best base-stock R	15	21	26	31	36	41
	Ave. Cost	6.728	7.842	8.604	9.232	9.753	10.194
	No. of states(R^*)	16	253	3,654	52,360	749,398	10,737,573
	Comp. time(sec)	0.10	0.24	0.76	11.19	1020.99	89532.64
39	Best base-stock R	16	22	28	33	39	44
	Ave. Cost	7.863	9.190	10.218	11.062	11.776	12.374
	No. of states(R^*)	17	276	4,495	66,045	1,086,008	15,890,700
	Comp. time(sec)	0.12	0.26	0.93	15.22	2073.84	183065.78

As can be observed from Table 6, when the review period $T = 1$ and the order lead time L = 1, 2, 3 and 4, as in Zipkin (2008b), our average costs for best base-stocks closely match with those in Zipkin (2008b). While Zipkin (2008b) reports only the average costs for the base-stocks, we report the best base-stock policies as well. The results for $L = 5$ and 6 have not appeared previously. Also consistent with Zipkin (2008b), the minimum average cost of the base-stock policy, like that of the optimal policy, increases in L, corresponding to lead times which are integer multiples of the review period T. And, consistent with Proposition 1 and intuition, costs and quantities rise with increasing l, the unit lost sales cost.

In contrast, a close examination of Table 7 and Tables E1 - E4 in Appendix E reveals that when L is not

Table 7: Numerical results for $T = 2$ with Poisson demand, up to $L = 8$

Lost sales cost(l)		L							
		1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
4	Best base-stock R	15	19	23	27	31	35	39	43
	Ave. Cost	6.1536	6.2684	6.7297	6.6215	7.0506	6.8355	7.2560	6.9807
	No. of states (R^*)	16	20	300	406	5,984	8,436	123,410	178,365
	Comp. time(sec)	0.27	0.18	0.38	0.47	1.83	2.26	40.33	50.89
9	Best base-stock R	18	22	27	32	36	41	46	50
	Ave. Cost	7.9292	8.3793	8.9166	9.0843	9.5487	9.5641	9.9807	9.9140
	No. of states (R^*)	19	23	406	561	9,139	13,244	230,300	316,251
	Comp. time(sec)	0.32	0.26	0.55	0.74	3.53	4.19	158.04	198.53
19	Best base-stock R	20	25	30	35	40	45	50	55
	Ave. Cost	9.4786	10.2083	10.9095	11.3469	11.8841	12.1556	12.6045	12.7815
	No. of states (R^*)	21	26	496	666	12,341	17,296	316,251	455,126
	Comp. time(sec)	0.42	0.35	0.74	0.93	5.44	5.67	289.12	393.29
39	Best base-stock R	21	27	32	38	43	48	53	58
	Ave. Cost	10.8788	11.8645	12.7689	13.4130	14.0699	14.5634	15.1155	15.4897
	No. of states (R^*)	22	28	561	780	15,180	20,825	395,010	557,845
	Comp. time(sec)	0.48	0.43	0.90	1.16	7.35	6.91	438.93	568.55

an integer multiple of T, as anticipated from the motivating example, the minimum average cost as well as the optimal R with the base-stock policy for a larger value of L can be lower than that for a smaller value of L. For example, in Tables 7 and E1 - E4, when lost sales cost $l = 4$, the minimum average cost goes up and down as L increases, whereas in Table E4, when $l = 4$, the optimal R for $L = 8$ is smaller than that for $L = 6$ and 7. Interestingly, Bijvank and Johansen (2012) who also consider the same problem but use an approximate method, do not report observing such non-monotonicity in their work. Analogously, Kapalka et al. (1999) who use an approximate procedure to find (s, S) policies for cases when $L < T$, also do not report non-monotonic behavior of the optimal policy.

This seemingly unintuitive and surprising behavior is more easily seen in Figure 3 which shows performance of a series of problems with $T = 6$ and $l = 4$. As can be seen as L increases, the minimum average cost and corresponding best base-stock rise and fall in L. As discussed in Section 2, one possible explanation is that since only inventory on-hand is available to fill demand early in the focal review cycle, there can be instances when there are stockouts. In those cases, the incoming order that arrives after n periods into the cycle comes too late but it may also become an underutilized asset since it is only available to serve a portion of demand. It is as if there too is a tradeoff between cost of lost sales and inventory carrying cost. A managerial implication of this result is that in some instances, a retailer may be better off with accepting a longer lead time since such an action may retain the financing cost of pipeline inventory with the supplier.

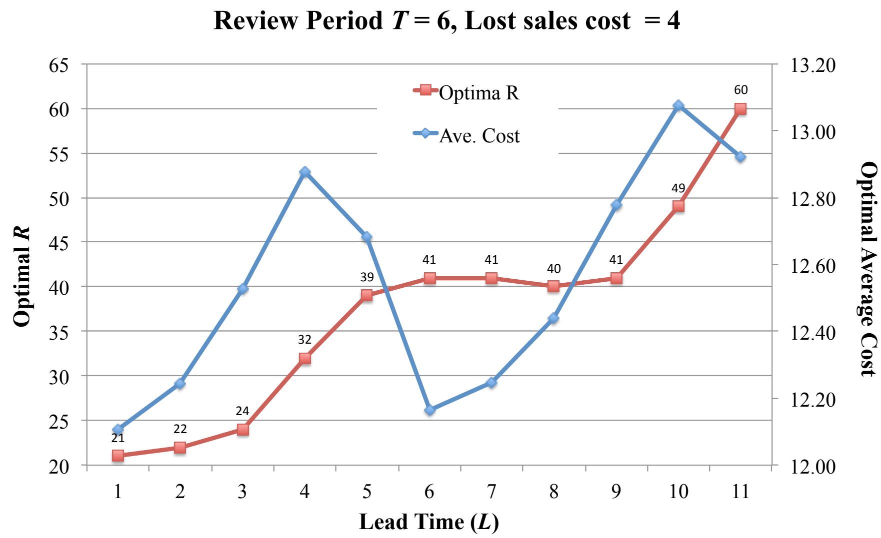


Figure 3: Optimal R and average costs for T = 6 with Poisson demand for different L

6 Extensions and Conclusion

Almost six decades have elapsed since Karlin and Scarf (1958) published their seminal work on base-stock models with lost sales. While many efforts were made to find good approximate policies, only recently have researchers made progress on elucidating the underlying problem structure. While some progress has been made on understanding the structure, the focus of this stream of research has been on the case when $L = m$ and $T = 1$. Our study on lost sales models with arbitrary lead time is remarkable given that there are many real world applications in which lead times are not integer multiples of the review period. These practical problems span retail, health care and manufacturing. And, their prevalence is increasing with the advent of innovations in fast delivery.

To explore the impact of arbitrary lead time, we have taken a different approach to this classic problem. Like Bijvank and Johansen (2012), we use a discrete demand representation and, in our most comprehensive setting, without loss of generality, we model lead time as discrete by representing it as $L = (m-1)T + n$, where m is the maximum number of orders outstanding. The key to our analysis is a full understanding of the fractional lead time case, the special case when the lead time does not exceed the review period.

Using analytical results for the motivating example of Section 2 and supporting numerical work in Section 5, for the fractional lead time case, we are able to: 1) demonstrate analytically and computationally that the best stationary base-stock policy can be globally optimal and even when it is not, the best stationary base-stock policy typically provides performance close to optimal; and, 2) explain intuitively and analytically why the minimum average cost and the optimal policy need not be increasing in the lead time L. These findings run contrary to the conventional wisdom that shortening delivery lead times in stochastic inventory system results in lower operating costs so that it can be economical to pay more for faster delivery. Our results seem to indicate that there are instances where shorter lead times can increase both operating expense and inventory making such a tactical choice un-economical.

After having demonstrated that stationary base-stock policies can be effective, we focus on studying stationary base-stock policies when $L = (m-1)T + n$. Using a sample-path analysis we are able to show that the cost function can be decomposed into two parts: one that is increasing in R, the base-stock, and the other that is decreasing in R. This provides sufficient structure to develop an algorithm with a simple stopping-rule, which also provides upper and lower bounds that get tighter at each iteration.

To facilitate computation of the cost function used in a search algorithm for the best stationary base-stock, we develop an exact Markovian representation of the underlying system with arbitrary lead time. Using the fractional lead time case, we are able to represent the Markovian structure model with arbitrary lead time in terms of a sparse matrix that uses this fractional lead time case as a building block. This allows us to solve problems larger than those solved previously; for example those solved by Zipkin (2008b).

Furthermore our numerical experiments demonstrate that the non-monotonicity in L persists as the lead time increases to become longer than the review period. For example, as can be seen from Figure 3, while R^* and $TC(R^*)$ are not monotone in $n > 0$ for a given m with $L = (m-1)T + n$, it appears that for a fixed n, $TC(R^*)$ may be monotone in m.

While establishing monotonicity for the best stationary base-stock remains elusive, for the corresponding optimal cost, establishing monotonicity in m for a fixed n is straightforward. Adapting the argument in the first paragraph of Section 5 of Zipkin (2008b), consider two systems that are otherwise identical except one has lead time $L = mT + n$ and the other has lead time $L = (m-1)T + n$. Then, at each review epoch $t + T$ for the system with the shorter lead time, order the quantity that was ordered at epoch t for the system with the longer lead time. Then, on every sample path the two systems would have the same cost. If the optimal control was used for the system with $L = mT + n$, then it follows immediately that the optimal policy for the system with $L = (m-1)T + n$ cannot cost more. The monotonicity in m follows directly and generalizes the analogous result in Zipkin (2008b), since he only considers the case equivalent to $n = T = 1$.

While we have been able to show that at least some structural properties are preserved when the lead time is general, it remains to be determined if other theoretical results can be reproduced. For instance, is the dual-balancing heuristic still effective or is the best stationary base-stock policy still asymptotically optimal?

Appendix A Technical Details for Section 2

Appendix A.1 The Structure of Optimal Policy

As discussed in Section 2, the structure of the optimal policy depends on the ratio $\frac{l}{h}$, and the values of δ and L. In particular, we know that given δ, $E(\delta, L)$ monotonically increases in L, and $B(\delta, L)$ monotonically decreases in L, while $A(\delta, L)$ increases and then decreases in L and attains its global maximum at $L = 0.5$. Therefore, by arranging the three extreme values $E(\delta, 1)$,$B(\delta, 1)$ and $A(\delta, 0.5)$ in their proper order, the spatial relationship between $A(\delta, L)$, $B(\delta, 1)$ and $E(\delta, L)$ can be discerned, thereby helping determine the best base-stock. Given that $A(\delta, 0.5) = E(\delta, 0.5) < E(\delta, 1)$, there are exactly three admissible orders:

Case 1. $B(\delta, 1) \geq E(\delta, 1) > A(\delta, 0.5) = E(\delta, 0.5)$;

Case 2. $E(\delta, 1) > B(\delta, 1) > A(\delta, 0.5) = E(\delta, 0.5)$;

Case 3. $E(\delta, 1) > A(\delta, 0.5) = E(\delta, 0.5) \geq B(\delta, 1)$.

We start with Case 1.

Case 1. $B(\delta, 1) \geq E(\delta, 1) > A(\delta, 0.5) = E(\delta, 0.5)$.

We proceed by defining

$$W(\delta, L) = B(\delta, L) - E(\delta, L) = \left\{ \frac{(1-\delta)}{\delta} + \frac{(2-\delta^2 L^2)}{2\delta^2 L} \right\} - \frac{[2 + 2\delta L + \delta^2 L^2 - 2\delta]}{2\delta}$$

Then,

$$\begin{aligned} \frac{\partial W(\delta, L)}{\partial \delta} &= -\frac{1}{\delta^2} - \frac{2}{\delta^3 L} + \frac{1}{\delta^2} - \frac{L^2}{2} \\ &= -\frac{2}{\delta^3 L} - \frac{L^2}{2} < 0. \end{aligned}$$

Now let

$$W(\delta, 1) = \left\{ \frac{(1-\delta)}{\delta} + \frac{(2-\delta^2)}{2\delta^2} \right\} - \frac{[2 + 2\delta + \delta^2 - 2\delta]}{2\delta} = 0,$$

which simplifies to $\delta^3 + 3\delta^2 - 2 = (\delta + 1)(\delta - \sqrt{3} + 1)(\delta + \sqrt{3} + 1) = 0$. It is easy to verify that its unique positive root is $\delta_{low} = \sqrt{3} - 1$.

Therefore, we conclude that: for $0 \leq \delta \leq \sqrt{3} - 1 = \delta_{\text{low}}$, $B(\delta, L) \geq E(\delta, L)$ holds for all $L \in [0, 1]$, and, that for $\sqrt{3} - 1 < \delta \leq 1$, there exists a unique threshold L_δ ($L_{\text{low}} \leq L_\delta \leq 1$), such that $W(\delta, L_\delta) = 0$; and for $L < L_\delta$, $B(\delta, L) > E(\delta, L)$, and for $L > L_\delta$, $B(\delta, L) < E(\delta, L)$. Now that we know for $0 \leq \delta \leq \sqrt{3} - 1 = \delta_{\text{low}}$, $A(\delta, 0) < A(\delta, 0.5) = E(\delta, 0.5) < E(\delta, 1) < B(\delta, 1)$ holds, we can fully characterize R^* into 5 mutually exclusive policies:

<u>Subcase 1.1</u>: $0 < \frac{l}{h} \le A(\delta, 0) = A(\delta, 1)$

In this subcase, we have $TC(0) \le TC(1) \le TC(2)$ so that $R^* = 0$ for all $0 \le L \le 1$.

<u>Subcase 1.2</u>: $A(\delta, 0) < \frac{l}{h} < A(\delta, 0.5)$

As discussed in Section 2, in this subcase we know that as L increases from 0 to 1, $\frac{l}{h}$ will intersect $A(\delta, L)$ two times, first from above at L_1 and then from below at L_3, and in between, it intersects $E(\delta, L)$ at L_2. As a result, optimal $R^* = 1$ for $L \in [0, L_1]$ and $L \in [L_3, 1]$, and optimal $R = 0$ for $L \in (L_1, L_3)$. Furthermore, $TC(1)$ increases in L in $[0, L_1]$, and then decreases in $[L_3, 1]$; while in between $TC(0)$ is constant in L.

<u>Subcase 1.3</u>: $A(\delta, 0.5) = E(\delta, 0.5) < \frac{l}{h} < E(\delta, 1)$

In this subcase, as L increases from 0 to 1, $\frac{l}{h}$ never intersects $A(\delta, L)$ or $B(\delta, L)$, namely, $A(\delta, L) \le \frac{l}{h} \le B(\delta, L)$. However, $\frac{l}{h}$ intersects $E(\delta, L)$ exactly once at $L = L_2$, so that optimal $R^* = 1$ and for $L \in [0, L_2]$, $TC(1)$ increases in L, and then for $L \in (L_2, 1]$, $TC(1)$ decreases in L.

<u>Subcase 1.4</u>: $E(\delta, 1) \le \frac{l}{h} \le B(\delta, 1)$

In this subcase, as L increases from 0 to 1, $\frac{l}{h}$ does not intersect $A(\delta, L)$, $B(\delta, L)$ or $E(\delta, L)$, so that $R^* = 1$, and $TC(1)$ increases in $L \in [0, 1]$.

<u>Subcase 1.5</u>: $\frac{l}{h} > B(\delta, 1)$

In this subcase, as L increases from 0 to 1, $\frac{l}{h}$ intersects $B(\delta, L)$ once at $L = L_4$, and does not intersect $A(\delta, L)$, so for $L \in [0, L_4]$, $R^* = 1$, and $TC(1)$ is increasing in L; then for $L \in (L_4, 1]$, $R^* = 2$ and $TC(2)$ is decreasing in L. Now that we have fully characterized the optimal policy for Case 1, $0 \le \delta \le \sqrt{3} - 1 = \delta_{\text{low}}$, we move on to analyze Case 2.

Case 2. $E(\delta, 1) > B(\delta, 1) > A(\delta, 0.5) = E(\delta, 0.5)$.

From the analysis of case 1 we have already established that for the first inequality to hold, δ must be greater than δ_{low}. To establish the second inequality we define

$$\begin{aligned} H_1(\delta) &= B(\delta, 1) - A(\delta, 0.5) \\ &= \frac{(2 + 2\delta - 3\delta^2)}{2\delta^2} - \frac{2 - \delta + 0.25\delta^2}{2\delta} \\ &= \frac{2 - 2\delta^2 - 0.25\delta^3}{2\delta^2}. \end{aligned}$$

It is obvious that the numerator of $H_1(\delta)$ is decreasing in δ, hence $H_1(\delta)$ has only one positive root. Numerically solving $H_1(\delta) = 0$, we get $\delta_{\text{med}} \approx 0.945688$.

And we can conclude that:

for $0 \leq \delta \leq \delta_{\text{med}}$, $B(\delta, 1) \geq A(\delta, 0.5)$;

for $\delta_{\text{med}} < \delta \leq 1$, $B(\delta, 1) < A(\delta, 0.5)$.

Thus, for $\delta_{\text{low}} = \sqrt{3} - 1 \leq \delta \leq 0.945688 = \delta_{\text{med}}$, $E(\delta, 1) > B(\delta, 1) > A(\delta, 0.5) = E(\delta, 0.5)$ holds. As for Case 1, we can fully characterize optimal R^* into 5 mutually exclusive policies. Since subcases 2.1 and 2.2 are identical to 1.1 and 1.2, we omit their discussion.

Subcase 2.3: $A(\delta, 0.5) \leq \frac{l}{h} < B(\delta, 1)$

This case is analogous to subcase 1.3 except that the upper bound is now $B(\delta, 1)$ since it is not greater than $E(\delta, 1)$.

Subcase 2.4: $B(\delta, 1) \leq \frac{l}{h} < \frac{[2+2\delta L_\delta + \delta^2 L_\delta^2 - 2\delta]}{2\delta} = E(\delta, L_\delta)$

In this subcase, as L increases from 0 to 1, $\frac{l}{h}$ will first intersect $E(\delta, L)$ at L_2, and then it will intersect $B(\delta, L)$ at L_4. Accordingly, the optimal policy is such that for $L \in [0, L_4]$, optimal $R^* = 1$, and as in subcases 1.3 and 2.3, $TC(1)$ increases in L for $L \in [0, L_2]$; then decreases in L for $L \in (L_2, L_4]$; finally, for $L \in (L_4, 1]$, optimal $R^* = 2$ and $TC(2)$ decreases in L.

Subcase 2.5: $\frac{l}{h} \geq \frac{[2+2\delta L_\delta + \delta^2 L_\delta^2 - 2\delta]}{2\delta} = E(\delta, L_\delta)$

Subcase 2.5 is identical to subcase 1.5 except the lower bound is now $E(\delta, L_\delta)$ since it is greater than $B(\delta, 1)$. By comparing Case 1 and Case 2 we see that two intervals have different bounds but only in subcase 2.4 is the optimal policy different from that in the corresponding subcase in Case 1. So the effect of reduced variability has modest incremental effect compared to Case 1. We now turn to the remaining case which has the lowest variability in demand.

Case 3. $E(\delta, 1) > A(\delta, 0.5) = E(\delta, 0.5) \geq B(\delta, 1)$.

From the analysis of Case 1, we know that for this remaining case, $\delta_{\text{med}} = 0.945688 < \delta \leq 1$, so that $A(\delta, 0) < B(\delta, 1) < A(\delta, 0.5) = E(\delta, 0.5) < E(\delta, L_\delta) < E(\delta, 1)$. As for Case 1 and Case 2, we can fully characterize optimal R^* into 5 mutually exclusive policies. Since subcase 3.1 and 3.5 are identical to subcases 2.1 and 2.5 respectively, we omit their discussion.

Subcase 3.2: $A(\delta, 0) < \frac{l}{h} < B(\delta, 1)$

This subcase is analogous to subcase 2.2 (and subcase 1.2) except that the upper bound now is $B(\delta, 1)$ since it is less than $A(\delta, 0.5)$.

Subcase 3.3: $B(\delta, 1) < \frac{l}{h} < A(\delta, 0.5)$

In this subcase as L increases from 0 to 1, $\frac{l}{h}$ first intersects $A(\delta, L)$ at $L = L_1$; then as L increases, it will intersect $E(\delta, L)$ at $L = L_2$; then it will intersect $A(\delta, L)$ again at $L = L_3$; finally, it will intersect $B(\delta, L)$ at L_4. Therefore, in this subcase, the optimal policy is such that for $L \in [0, L_1]$, $R^* = 1$, and $TC(1)$

increases in L; then for $L \in (L_1, L_3]$, $R^* = 0$, and it is obvious that $TC(0)$ is constant as L increases; then for $L \in (L_3, L_4]$, $R^* = 1$ but $L > L_2$ implies that $TC(1)$ decreases in L in this interval. Finally, for $L \in [L_4, 1]$, $R^* = 2$, and $TC(2)$ decreases in L in this interval. Finally we consider:

Subcase 3.4: $A(\delta, 0.5) \leq \frac{l}{h} < E(\delta, L_\delta)$

While the optimal policy in this subcase is identical to that in subcase 2.4, the lower bound in this subcase is $A(\delta, 0.5)$ since it is greater than $B(\delta, 1)$.

We close by noting that by comparing Case 2 and Case 3 we see that these two cases have different bounds but only in subcase 3.3 is the policy different from that in the corresponding subcase in Case 2. So the effect of even more reduced variability has modest incremental effect compared to Case 2. As we move from Case 1 to Case 2, only subcase 2.4 is different than the corresponding subcase 1.4. Analogously, when we move from Case 2 to Case 3, only subcase 3.3 is different than the corresponding subcase 2.3. All together these seven policies show that the optimal R can increase in L, first increase then decrease in L or remain the same. Similarly, the optimal cost can increase in L, first increase then decrease in L or remain the same. Thus, this simple problem has remarkably complex optimal structure.

Appendix A.2 The State-dependent Policy

It is easily verified that $\hat{\pi}_0 = \frac{\delta}{2+\delta L}$, $\hat{\pi}_1 = \frac{1}{2+\delta L}$, and $\hat{\pi}_2 = \frac{1-\delta+\delta L}{2+\delta L}$ are the stationary probabilities that the inventory on-hand is 0, 1, and 2, respectively. Using the subscript sdp to represent state-dependent policy, given this stationary distribution, the average cost per period can be written as:

$$\begin{aligned} TC_{sdp} &= \hat{\pi}_0 \delta \left\{ L[l + 2h(1-L)] + h \int_L^1 [(1-L) + (t-L)] dt \right\} + 2h\hat{\pi}_0 (1-\delta)(1-L) \\ &\quad + \hat{\pi}_1 \delta h \left[\int_0^1 t dt \right] + \hat{\pi}_1 h (1-\delta) + \hat{\pi}_2 \delta h \left[1 + \int_0^1 t dt \right] + \hat{\pi}_2 (1-\delta) [2h] \\ &= \frac{\delta^2}{(2+\delta L)} \left\{ L[l + 2h(1-L)] + h \int_L^1 [(1-L) + (t-L)] dt \right\} + \frac{\delta h}{2(2+\delta L)} \\ &\quad + \frac{3\delta h [1-\delta+\delta L]}{2(2+\delta L)} + \frac{3h(1-\delta)}{(2+\delta L)} \end{aligned} \tag{A1}$$

$$
\begin{aligned}
&= \frac{\delta^2 Ll}{(2+\delta L)} + \frac{\delta^2 h}{2(2+\delta L)}\left\{7 - 10L + 3L^2\right\} + \frac{\delta h}{2(2+\delta L)} + \frac{3\delta h[1-\delta+\delta L]}{2(2+\delta L)} + \frac{3h(1-\delta)}{(2+\delta L)} \\
&= \frac{\delta^2 Ll}{(2+\delta L)} + \frac{\delta h}{2(2+\delta L)}\left\{\delta[7 - 10L + 3L^2] + 1 + 3[1-\delta+\delta L]\right\} + \frac{3h(1-\delta)}{(2+\delta L)} \\
&= \frac{\delta^2 Ll}{(2+\delta L)} + \frac{\delta h}{2(2+\delta L)}\left\{4 + 4\delta - 7\delta L + 3\delta L^2\right\} + \frac{3h(1-\delta)}{(2+\delta L)} \\
&= \frac{\delta^2 Ll}{(2+\delta L)} + \frac{h}{2(2+\delta L)}\left\{6 - 2\delta + 4\delta^2 - 7\delta^2 L + 3\delta^2 L^2\right\}.
\end{aligned}
$$

To compare TC_{sdp} with $TC(1)$ we compute,

$$
\begin{aligned}
TC_{sdp} - TC(1) &= \frac{\delta^2 Ll}{(2+\delta L)} + \frac{h}{2(2+\delta L)}\left\{6 - 2\delta + 4\delta^2 - 7\delta^2 L + 3\delta^2 L^2\right\} \\
&\quad - \frac{h[(2-\delta) + \delta^2 L(1-L)]}{2(\delta L+1)} - \frac{\delta^2 lL}{(\delta L+1)} \\
&= \frac{-\delta^2 lL}{(\delta L+1)(2+\delta L)} + \frac{h(\delta L+1)\left\{6 - 2\delta + 4\delta^2 - 7\delta^2 L + 3\delta^2 L^2\right\}}{2(2+\delta L)(1+\delta L)} \\
&\quad - \frac{h(\delta L+2)[(2-\delta) + \delta^2 L(1-L)]}{2(2+\delta L)(1+\delta L)} \\
&= \frac{-\delta^2 lL}{(\delta L+1)(2+\delta L)} + \frac{h\left\{2 + 4\delta L + 4\delta^2 - 10\delta^2 L + 5\delta^2 L^2 + 4\delta^3 L - 8\delta^3 L^2 + 4\delta^3 L^3\right\}}{2(2+\delta L)(1+\delta L)} \\
&= \frac{\delta^2 Lh}{(\delta L+1)(2+L\delta)}\left[\frac{\left\{2 + 4\delta L + 4\delta^2 - 10\delta^2 L + 5\delta^2 L^2 + 4\delta^3 L - 8\delta^3 L^2 + 4\delta^3 L^3\right\}}{2\delta^2 L} - \frac{l}{h}\right].
\end{aligned}
\tag{A2}
$$

We can easily see that if

$$
\frac{l}{h} < \frac{\left\{2 + 4\delta L + 4\delta^2 - 10\delta^2 L + 5\delta^2 L^2 + 4\delta^3 L - 8\delta^3 L^2 + 4\delta^3 L^3\right\}}{2\delta^2 L},
$$

then $TC(1) < TC_{sdp}$.

Recall from (5) that

$$
TC(1) - TC(2) = \frac{\delta^2 Lh}{(\delta L+1)}\left[\frac{l}{h} - \frac{(1-\delta)}{\delta} - \frac{(2-\delta^2 L^2)}{2\delta^2 L}\right],
$$

which implies that if $\frac{l}{h} < \frac{(1-\delta)}{\delta} + \frac{(2-\delta^2 L^2)}{2\delta^2 L}$, then $TC(1) < TC(2)$.

Notice that

$$
\begin{aligned}
&\frac{\left\{2+4\delta L+4\delta^2-10\delta^2 L+5\delta^2 L^2+4\delta^3 L-8\delta^3 L^2+4\delta^3 L^3\right\}}{2\delta^2 L}-\left[\frac{(1-\delta)}{\delta}+\frac{(2-\delta^2 L^2)}{2\delta^2 L}\right]\\
&=\frac{\left\{2+4\delta L+4\delta^2-10\delta^2 L+5\delta^2 L^2+4\delta^3 L-8\delta^3 L^2+4\delta^3 L^3\right\}}{2\delta^2 L}-\left[\frac{(2\delta L-2\delta^2 L+2-\delta^2 L^2)}{2\delta^2 L}\right]\\
&=\frac{\left\{2\delta L+4\delta^2-8\delta^2 L+6\delta^2 L^2+4\delta^3 L-8\delta^3 L^2+4\delta^3 L^3\right\}}{2\delta^2 L}\\
&=\frac{\left\{2\delta L+4\delta^2(1-L)^2+4\delta^3 L(1-L)^2+2\delta^2 L^2\right\}}{2\delta^2 L}>0
\end{aligned}
$$

Therefore, we can conclude that if $\frac{l}{h}<\frac{(1-\delta)}{\delta}+\frac{(2-\delta^2 L^2)}{2\delta^2 L}$, then $TC(1)<TC(2)$, and $TC(1)<TC_{sdp}$ hold simultaneously.

Moreover,

$$
\begin{aligned}
TC_{sdp}-TC(2) &= \frac{\delta^2 Ll}{(2+\delta L)}+\frac{h}{2(2+\delta L)}\left\{6-2\delta+4\delta^2-7\delta^2 L+3\delta^2 L^2\right\}-h(2-0.5\delta-\delta L)\\
&= \frac{\delta^2 Ll}{(2+\delta L)}+\frac{h}{2(2+\delta L)}\left\{6-2\delta+4\delta^2-7\delta^2 L+3\delta^2 L^2\right\}-\frac{2h(2-0.5\delta-\delta L)(2+\delta L)}{2(2+\delta L)}\\
&= \frac{\delta^2 Ll}{(2+\delta L)}+\frac{h}{2(2+\delta L)}\left\{6-2\delta+4\delta^2-7\delta^2 L+3\delta^2 L^2\right\}-\frac{h\left\{8-2\delta-\delta^2 L-2\delta^2 L^2\right\}}{2(2+\delta L)}\\
&= \frac{\delta^2 Ll}{(2+\delta L)}+\frac{h[5\delta^2 L^2-6\delta^2 L+4\delta^2-2]}{2(2+\delta L)} \qquad \text{(A3)}\\
&= \frac{\delta^2 Ll}{(2+\delta L)}+\frac{5\delta^2 h(L-0.6)^2+h(2.2\delta^2-2)}{2(2+\delta L)}\\
&= \frac{5\delta^2 h(L-0.6)^2}{2(2+\delta L)}+\frac{\delta^2 Lh}{(2+\delta L)}\left[\frac{l}{h}-\frac{(2-2.2\delta^2)}{2\delta^2 L}\right].
\end{aligned}
$$

We can easily see that if $\frac{l}{h}>\frac{(2-2.2\delta^2)}{2\delta^2 L}$, then $TC_{sdp}>TC(2)$ holds. Also notice that for any $0\leq L\leq 1$ and $0\leq\delta\leq 1$, $\frac{(2-2.2\delta^2)}{2\delta^2 L}<\frac{(1-\delta)}{\delta}+\frac{(2-\delta^2 L^2)}{2\delta^2 L}$ holds, which implies that if $\frac{l}{h}\leq\frac{(2-2.2\delta^2)}{2\delta^2 L}$, then $\frac{l}{h}<\frac{(1-\delta)}{\delta}+\frac{(2-\delta^2 L^2)}{2\delta^2 L}$ holds as well; hence it follows that $TC(1)<TC_{sdp}$. Alternatively, $\frac{l}{h}>\frac{(2-2.2\delta^2)}{2\delta^2 L}$, so that $TC_{sdp}>TC(2)$ holds. Therefore, we can conclude that $TC_{sdp}>\min\{TC(1),TC(2)\}$ for any $0\leq L\leq 1$ and $0\leq\delta\leq 1$. Since this is the only possible candidate state-dependent policy for our model, it completes the proof of proposition 2.

Appendix B Total Expected Cost TC_t

In addition to the notation introduced in Section 3, we will need the following notation:

π_j^R = stationary probability that the inventory on-hand equals j at the beginning of a review cycle.

Then, the expression of TC_t , the total expected inventory-related cost in period t appearing in equation (11) for different values of L are as given below:

a) The Case when $m = 1, \quad 0 < n < T$

In this case, the total expected cost for the first period of the review cycle, i.e., period t, is given by

$$TC_t = h \cdot LO_t + l \cdot LS_t = \sum_{j=0}^{R} \pi_j^R \left[h \sum_{z=0}^{j} P_z \cdot (j-z) + l \sum_{z=j+1}^{+\infty} P_z \cdot (z-j) \right] \tag{B1}$$

for period $t+s$ $(1 \le s \le L)$,

$$\begin{aligned} TC_{t+s} &= h \cdot LO_{t+s} + l \cdot LS_{t+s} \\ &= h \sum_{j=0}^{R} \pi_j^R \left[\sum_{z=0}^{j} P_z^{s+1} \cdot (j-z) \right] + l \sum_{j=0}^{R} \pi_j^R \left[\sum_{z=j+1}^{+\infty} \left(P_z^{s+1} - P_z^{s} \right) \cdot (z-j) \right] ; \end{aligned} \tag{B2}$$

for period $t+L$,

$$\begin{aligned} TC_{t+L} &= h \cdot LO_{t+L} + l \cdot LS_{t+L} \\ &= h \sum_{j=1}^{R} \pi_j^R \left[\sum_{z=0}^{j-1} P_z^L \sum_{k=0}^{R-z} P_k \cdot (R-z-k) + \bar{P}_j^L \sum_{k=0}^{R-j} P_k \cdot (R-j-k) \right] \\ &\quad + l \sum_{j=0}^{R} \pi_j^R \left[\sum_{z=0}^{j-1} P_z^L \sum_{k=R-z+1}^{+\infty} P_k \cdot (z+k-R) + \bar{P}_j^L \sum_{k=R-j+1}^{+\infty} P_k \cdot (k+j-R) \right] ; \end{aligned} \tag{B3}$$

and, for period $t+s$ $(L < s < T-1)$,

$$\begin{aligned} TC_{t+s} &= h \cdot LO_{t+s} + l \cdot LS_{t+s} \\ &= h \sum_{j=1}^{R} \pi_j^R \left[\sum_{z=0}^{j-1} P_z^L \sum_{k=0}^{R-z} P_k^{s-L+1} \cdot (R-z-k) + \bar{P}_j^L \sum_{k=0}^{R-j} P_k^{s-L+1} \cdot (R-j-k) \right] \\ &\quad + l \sum_{j=0}^{R} \pi_j^R \left[\sum_{z=0}^{j-1} P_z^L \sum_{k=R-z+1}^{+\infty} \left(P_k^{s-L+1} - P_k^{s-L} \right) \cdot (z+k-R) \right] \\ &\quad + l \sum_{j=0}^{R} \pi_j^R \left[\bar{P}_j^L \sum_{k=R-j+1}^{+\infty} \left(P_k^{s-L+1} - P_k^{s-L} \right) (j+k-R) \right] . \end{aligned} \tag{B4}$$

b) The Case when $m > 1, \quad 0 < n < T$

In this case, the total expected cost for the first period of the review cycle, i.e., period t, is given by

$$TC_t = h \cdot LO_t + l \cdot LS_t = \sum_{j=0}^{R} \pi_j^R \left[h \sum_{z=0}^{j} P_z \cdot (j-z) + l \sum_{z=j+1}^{+\infty} P_z \cdot (z-j) \right] \tag{B5}$$

and, for period $t+s$ ($1 \le s < n$),

$$\begin{aligned} TC_{t+s} &= h \cdot LO_{t+s} + l \cdot LS_{t+s} \\ &= h \sum_{j=0}^{R} \pi_j^R \left[\sum_{z=0}^{j} P_z^{s+1} \cdot (j-z) \right] + l \sum_{j=0}^{R} \pi_j^R \left[\sum_{z=j+1}^{+\infty} \left(P_z^{s+1} - P_z^s \right) \cdot (z-j) \right] ; \end{aligned} \tag{B6}$$

for period $t+n$, the formulation is different because the first outstanding order arrives within the cycle. In order to calculate the expected lost sales in period $t+n$, not only do we need to know the distribution of inventory on-hand at the beginning of the cycle (i.e., period t), but also we need to know the joint distribution of the first outstanding order and the rest of m outstanding orders, which we denote as $\tilde{\pi}_{uv}^R$, $u+v \le R$, where u denotes the first outstanding order, and v denotes the sum of the rest of m outstanding orders, respectively. Therefore, we have $\pi_j^R = \sum_{u+v=R-j} \tilde{\pi}_{uv}^R$, so that

$$\begin{aligned} TC_{t+n} &= h \cdot LO_{t+n} + l \cdot LS_{t+n} \\ &= h \sum_{u+v=0}^{R} \tilde{\pi}_{uv}^R \left[\sum_{z=0}^{R-u-v-1} P_z^n \sum_{k=0}^{R-v-z} P_k (R-v-z-k) + \bar{P}_{R-u-v}^n \sum_{k=0}^{u} P_k (u-k) \right] \\ &\quad + l \sum_{u+v=0}^{R} \tilde{\pi}_{uv}^R \left[\sum_{z=0}^{R-u-v-1} P_z^n \sum_{k=R-v-z+1}^{+\infty} P_k (v+z+k-R) + \bar{P}_{R-u-v}^n \sum_{k=u+1}^{+\infty} P_k \cdot (k-u) \right] ; \end{aligned} \tag{B7}$$

and, for period $t+s$ ($n < s \le T-1$),

$$\begin{aligned} TC_{t+s} &= h \cdot LO_{t+s} + l \cdot LS_{t+s} \\ &= h \sum_{u+v=0}^{R} \tilde{\pi}_{uv}^R \left[\sum_{z=0}^{R-u-v-1} P_z^n \sum_{k=0}^{R-v-z} P_k^{s-n+1} \cdot (R-v-z-k) + \bar{P}_{R-u-v}^n \sum_{k=0}^{u} P_k^{s-n+1} \cdot (u-k) \right] \\ &\quad + l \sum_{u+v=0}^{R} \tilde{\pi}_{uv}^R \left[\sum_{z=0}^{R-u-v-1} P_z^n \sum_{k=R-v-z+1}^{+\infty} \left(P_k^{s-n+1} - P_k^{s-n} \right) \cdot (v+z+k-R) \right] \\ &\quad + l \sum_{u+v=0}^{R} \tilde{\pi}_{uv}^R \left[\bar{P}_{R-u-v}^n \sum_{k=u+1}^{+\infty} \left(P_k^{s-n+1} - P_k^{s-n} \right) (k-u) \right] . \end{aligned} \tag{B8}$$

c) The Case when $L = mT, \quad m \ge 1$

In this case, the total expected cost for the first period of the review cycle, i.e., period t, is given by

$$TC_t = h \cdot LO_t + l \cdot LS_t = \sum_{j=0}^{R} \pi_j^R \left[h \sum_{z=0}^{j} P_z \cdot (j-z) + l \sum_{z=j+1}^{+\infty} P_z \cdot (z-j) \right] \tag{B9}$$

and, for period $t+s$ ($1 \le s \le T-1$),

$$\begin{aligned} TC_{t+s} &= h \cdot LO_{t+s} + l \cdot LS_{t+s} \\ &= h \sum_{j=0}^{R} \pi_j^R \left[\sum_{z=0}^{j} P_z^{s+1} \cdot (j-z) \right] + l \sum_{j=0}^{R} \pi_j^R \left[\sum_{z=j+1}^{+\infty} \left(P_z^{s+1} - P_z^s \right) \cdot (z-j) \right]. \end{aligned} \tag{B10}$$

Appendix C Proof of Proposition 3

Proposition 6 *3 (Based on Lemma 3.1 of Downs et al. (2001)): For $L = (m-1)T+n$, $m \ge 1$, $0 < n \le T$, assuming that demand in each period r is discrete and $P\{D_r = 0\} > 0$, we have for any sample path:*
(a) $M_r^{R+1} - M_r^R \in \{0,1\}$, where M_r refers to IL_r, IO_r, LO_r, and S_r; and
(b) $LS_r^R - LS_r^{R+1} \in \{0,1\}$.

Note : The assumption $P\{D_t = 0\} > 0$, is identical to Assumption 3 in Jankiraman and Roundy (2004) and is needed to guarantee that as time r goes to infinity, the system inventory on-hand IL_r will reach base stock R with a positive probability. Otherwise, on-hand inventory $IL_{r\to+\infty} = R$ becomes a reducible state when the lost sales inventory system becomes stable after a long-run.

Proof.

We first prove Proposition 3 for $m = 1$; namely, $L = n < T$. We let t be the first period of the problem horizon. Without loss of generality, we assume that at the beginning of period t, $IL_t = R$, such that $IO_t = 0$. Subsequently, demand in period t realizes, and sales $S_t = \min\{IL_t, D_t\} = \min\{R, D_t\}$, which is non-decreasing in R; specifically, when R goes up by one unit to $R+1$, S_t either goes up by one unit if $D_t \ge R+1$, or remains the same if $D_t \le R$, hence is non-decreasing in R. Notice that $LO_t = IL_t - S_t = R - S_t = (R - D_t)^+$, it is also non-decreasing in R, namely, if we increase R by one unit, LO_t will either increase by one unit or remain unchanged. Later in period $t+1$, since no pipeline order is received, $IL_{t+1} = LO_t$; subsequently, $S_{t+1} = \min\{IL_{t+1}, D_{t+1}\} = \min\{IL_t - S_t, D_{t+1}\} = \min\{LO_t, D_{t+1}\}$, which is again non-decreasing in R since LO_t is non-decreasing in R; specifically, if R increases by one unit, then S_{t+1} either increases by one unit or remains unchanged. It follows immediately that $LO_{t+1} = IL_{t+1} - S_{t+1} = IL_t - S_t - S_{t+1} = (IL_t - D_t - D_{t+1})^+$ is non-decreasing in R and increases at most by one unit when R does. Similarly, in period $t+2, t+3, \ldots, t+T+L-1$, since no pipeline inventory is delivered, as demands realize, the on-hand inventory level at the beginning of these periods is reduced till it reaches zero; specifically, for $1 \le j \le T+L-1$,

$$IL_{t+j} = LO_{t+j-1} = IL_{t+j-1} - S_{t+j-1} = IL_t - \sum_{i=t}^{t+j-1} S_i = \max\{R - \sum_{i=t}^{t+j-1} D_i, 0\}. \quad \text{(C1)}$$

Therefore, it is easy to verify that as R increases by one unit, IL_{t+j}, LO_{t+j}, and S_{t+j} either increase by one unit or do not change for all realization of $\{D_t, D_{t+1}, \dots, D_{t+T+L-1}\}$. Moreover, considering that $\sum_{i=t}^{t+j-1} S_i = \max\{R, \sum_{i=t}^{t+j-1} D_i\}$ for $1 \le j \le T+L-1$, $\sum_{i=t}^{t+j-1} S_i$ may increase by up to one unit at most if R goes up by one unit. Also, notice that a replenishment order, IO_{t+T} is placed at the beginning of period $t+T$, and $IO_{t+T} = S_t + \dots + S_{t+T-1} = \min\{IL_t, \sum_{i=t}^{t+j-1} D_i\} = \min\{R, \sum_{i=t}^{t+T-1} D_i\}$, namely, the total sales incurred from period t to period $t+T-1$, and it is obvious that if R goes up by one unit, IO_{t+T} will either increase by one unit or remain unchanged. Since IO_{t+T} is delivered at the beginning of period $t+T+L$,

$$IL_{t+T+L} = LO_{t+T+L-1} + IO_{t+T} = IL_t - \sum_{i=t}^{t+T+L-1} S_i + \sum_{i=t}^{t+T-1} S_i = R - \sum_{i=t+T}^{t+T+L-1} S_i, \quad \text{(C2)}$$

which is also non-decreasing in R, and the increase of IL_{t+T+L} is still upper-bounded by the increment of R, or $IL_{t+T+L}^{R+1} - IL_{t+T+L}^{R} \in \{0, 1\}$. In the following T periods, since no order is received from period $t+T+L+1$ to period $t+2T+L-1$, it is easy to verify that IL_{t+j}, LO_{t+j}, S_{t+j} and are all non-decreasing in R, but each increases more slowly than R increases. Again the next replenishment order is placed at the beginning of period $t+2T$, and

$$\begin{aligned}
IO_{t+2T} &= S_{t+T} + \dots + S_{t+2T-1} \\
&= \min\left\{IL_{t+T}, \sum_{i=t+T}^{t+T+L-1} D_i\right\} + \min\left\{\left(IL_{t+T} - \sum_{i=t+T}^{t+T+L-1} D_i\right)^+ + IO_{t+T}, \sum_{i=t+T+L}^{t+2T-1} D_i\right\} \quad \text{(C3)} \\
&= \min\left\{IL_{t+T}, \sum_{i=t+T}^{t+T+L-1} D_i\right\} + \min\left\{\left(IL_{t+T} - \sum_{i=t+T}^{t+T+L-1} D_i\right)^+ + R - IL_{t+T}, \sum_{i=t+T+L}^{t+2T-1} D_i\right\}
\end{aligned}$$

From previous analysis, we know that both IL_{t+T} and IO_{t+T} are non-decreasing in R; hence if R goes up by one unit to $R+1$, we either have $IL_{t+T}^{R+1} = IL_{t+T}^{R} + 1$ and $IO_{t+T}^{R+1} = IO_{t+T}^{R}$; or $IL_{t+T}^{R+1} = IL_{t+T}^{R}$ and $IO_{t+T}^{R+1} = IO_{t+T}^{R} + 1$, given that $IO_{t+T}^{R+1} + IL_{t+T}^{R+1} = R+1$. To complete the proof, consider the following two cases.

Case 1: $IL_{t+T} \geq \sum_{i=t+T}^{t+T+L-1} Di$. The expression for IO_{t+2T} simplifies to

$$IO_{t+2T} = \min \left\{ R, \sum_{i=t+T}^{t+2T-1} D_i \right\}$$

Case 2: $IL_{t+T} < \sum_{i=t+T}^{t+T+L-1} Di$. The expression for IO_{t+2T} simplifies to

$$IO_{t+2T} = \min \left\{ R, IL_{t+T} + \sum_{i=t+T+L}^{t+2T-1} D_i \right\}$$

It is easy to see that in either case, $IO_{t+2T}^{R+1} - IO_{t+2T}^{R} \in \{0, 1\}$. Similarly, we can show that Proposition 3(a) holds for any period over the infinite horizon. The proof can also be completed using induction as in Downs et al. (2001). Since $LS_i = D_i - S_i$, Proposition 3(b) follows from $S_t^{R+1} - S_t^R \in \{0, 1\}$.

For $L = (m-1)T + n$, $m > 1$, we will prove the proposition by induction. As in the case of $m = 1$, we still assume that at the beginning of period t, the first period of a review cycle, $IL_t = R$, such that $IO_t = 0$. And the next order is placed at the beginning of period $t + T$, and is delivered at the beginning of period $t + mT + n$. It is easy to verify that the Proposition holds for periods t to $t + mT + n - 1$. Also, within these time periods, there are total m replenishment orders placed at the beginning of periods $t + T$, $t + 2T$, $\ldots$, and $t + mT$.

With a stationary base stock policy, at the beginning of period $t + mT$, we must have $IL_{t+mT} + \sum_{i=1}^{m} IO_{t+iT} = R$. Since IL_{t+mT} and $IO_{t+iT} (i = 1, 2, \ldots, m)$ are all non-decreasing in R, when R goes up by one unit, only one of them can go up by one unit, and all other terms remain unchanged. Hence, we get: $(IL_{t+mT}^{R+1} + IO_{t+T}^{R+1}) - (IL_{t+mT}^{R} + IO_{t+T}^{R}) \in \{0, 1\}$. Then, at the beginning of period $t + T + L = t + mT + n$, $IL_{t+mT+n} = (IL_{t+mT} - \sum_{i=t+mT}^{t+mT+n-1} D_i)^+ + IO_{t+T}$, it follows immediately that $IL_{t+mT+n}^{R+1} - IL_{t+mT+n}^{R} \in \{0, 1\}$. In following periods $t + mT + n + 1, \ldots, t + (m+1)T - 1$, since no pipeline inventory is delivered, we still have $IL_{t+mT+j}^{R+1} - IL_{t+mT+j}^{R} \in \{0, 1\}$, for $j = n+1, \ldots, T-1$. Similarly, we can show that other parts of the Proposition hold for periods $t+mT+n, \ldots, t+(m+1)T-1$.

Now assume the Proposition holds from period t to period $t + kT$, for $k > m$, then the proposition also holds in period $t + kT$ through $t + kT + n - 1$ since no outstanding order is delivered. Then at the beginning of period $t + kT + n$, outstanding order $IO_{t+(k-m+1)T}$ is delivered, and $IL_{t+kT+n} = (IL_{t+kT} - \sum_{i=t+kT}^{t+kT+n-1} D_i)^+ + IO_{t+(k-m+1)T}$. When R goes up by 1, we either have $IL_{t+kT}^{R+1} = IL_{t+kT}^{R} + 1$ and $IO_{t+(k-m+1)T}^{R+1} = IO_{t+(k-m+1)T}^{R}$; or $IL_{t+kT}^{R+1} = IL_{t+kT}^{R}$ and $IO_{t+(k-m+1)T}^{R+1} = IO_{t+(k-m+1)T}^{R} + 1$ or $IL_{t+kT}^{R+1} = IL_{t+kT}^{R}$ and $IO_{t+(k+m-1)T}^{R+1} = IO_{t+(k+m-1)T}^{R}$. We get $(IL_{t+kT}^{R+1} + IO_{t+(R-m+1)T}^{R+1}) - (IL_{t+kT}^{R} + IO_{t+(R-m+1)T}^{R}) \in \{0, 1\}$.

To complete the proof for IL_{t+kT+n}, we consider the following two cases.

Case 3: $IL_{t+kT} \geq \sum_{i=t+kT}^{t+kT+n-1} D_i$. We get $IL_{t+kT+n} = IL_{t+kT} - \sum_{i=t+kT}^{t+kT+n-1} D_i + IO_{t+(k-m+1)T}$.

Case 4: $IL_{t+kT} < \sum_{i=t+kT}^{t+kT+n-1} D_i$. We get $IL_{t+kT+n} = IO_{t+(k-m+1)T}$.

It follows $IL^{R+1}_{t+kT+n} - IL^{R}_{t+kT+n} \in \{0,1\}$ in both cases.

Next we prove that $IO^{R+1}_{t+(k+1)T} - IO^{R}_{t+(k+1)T} \in \{0,1\}$. Recall that in a base stock policy $IO_{t+(k+1)T}$ simply equals the total sales of previous review cycle, namely,

$$\begin{aligned}
IO_{t+(k+1)T} &= \sum_{i=t+kT}^{t+(k+1)T-1} S_i = \sum_{i=t+kT}^{t+kT+n-1} S_i + \sum_{i=t+kT+n}^{t+(k+1)T-1} S_i \\
&= \min\left\{IL_{t+kT}, \sum_{i=t+kT}^{t+kT+n-1} D_i\right\} + \min\left\{IL_{t+kT+n}, \sum_{i=t+kT+n}^{t+(k+1)T-1} D_i\right\} \\
&= \min\left\{IL_{t+kT}, \sum_{i=t+kT}^{t+kT+n-1} D_i\right\} \\
&\quad + \min\left\{\max\left\{0, IL_{t+kT} - \sum_{i=t+kT}^{t+kT+n-1} D_i\right\} + IO_{t+(k-m+1)T}, \sum_{i=t+KT+n}^{t+(k+1)T-1} D_i\right\}.
\end{aligned} \tag{C4}$$

From our analysis above, we know when R goes up by 1, we either have $IL^{R+1}_{t+kT} = IL^{R}_{t+kT} + 1$ and $IO^{R+1}_{t+(k-m+1)T} = IO^{R}_{t+(k-m+1)T}$; or $IL^{R+1}_{t+kT} = IL^{R}_{t+kT}$ and $IO^{R+1}_{t+(k-m+1)T} = IO^{R}_{t+(k-m+1)T} + 1$ or $IL^{R+1}_{t+kT} = IL^{R}_{t+kT}$ and $IO^{R+1}_{t+(k-m+1)T} = IO^{R}_{t+(k-m+1)T}$. In all these three case, it is easy to verify that when R goes up by 1, $IO_{t+(k+1)T}$ can either goes up by 1 or remain unchanged.

Similarly, we can show that other parts of Proposition 3 hold for any period over the infinite horizon. □

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