

Belief Dispersion in the Stock Market*

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Abstract

There is vast empirical literature with many different, and sometimes conflicting, results on the effects of investors' dispersion of beliefs on asset prices and their dynamics. In this paper, we develop a dynamic model of belief dispersion which simultaneously explains the empirical regularities in a stock price, its mean return, volatility, and trading volume. Our model with a continuum of investors differing in beliefs is tractable and delivers exact closed-form expressions for all quantities. In our analysis, we construct measures for the average bias and dispersion in beliefs, and demonstrate that equilibrium quantities are driven by these two key endogenous variables. Our model has the following implications. We find that the stock price is convex in cash-flow news, and it increases in belief dispersion while its mean return decreases when the economy is optimistic, and vice versa when pessimistic. We also show that the presence of belief dispersion generates excess stock volatility and non-trivial trading volume, and a positive relation between these two quantities. Furthermore, we demonstrate that the more familiar, otherwise identical, finitely-many-investor models of heterogeneous beliefs are not able to generate our main results.

JEL Classifications: D53, G12.

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1 Introduction

The empirical evidence on the effects of investors' dispersion of beliefs on asset prices and their dynamics is vast and mixed. For example, several works find a negative relation between belief dispersion and a stock mean return (Diether, Malloy, and Scherbina (2002), Chen, Hong, and Stein (2002), Goetzmann and Massa (2005), Park (2005), Berkman, Dimitrov, Jain, Koch, and Tice (2009), Yu (2011)). Others argue that the negative relation is only valid for stocks with certain characteristics (e.g., small, illiquid, worst-rated or short sale constrained) and in fact, find either a positive or no significant relation (Qu, Starks, and Yan (2003), Doukas, Kim, and Pantzalis (2006), Avramov, Chordia, Jostova, and Philipov (2009)). Existing theoretical works, on the other hand, do not provide satisfactory answers for these mixed results. In fact, most studies find belief dispersion to be an extra risk factor for investors, and therefore generate only a positive dispersion-mean return relation.¹

In this paper, we develop a tractable model of belief dispersion which is able to simultaneously support the empirical regularities in a stock price, its mean return, volatility, and trading volume. To our knowledge this is the first paper accomplishing this. Towards that, we develop a dynamic general equilibrium model populated by a continuum of constant relative risk aversion (CRRA) investors who differ in their (dogmatic or Bayesian) beliefs. Our model delivers fully closed-form expressions for all quantities of interest. In our analysis, we summarize the wide range of investors' beliefs by two sufficient measures, the average bias and dispersion in beliefs, and demonstrate that equilibrium quantities are driven by these two key endogenous variables. How much an investor's belief contributes to the average bias depends on her wealth and risk attitude. Investors whose beliefs get supported by actual cash-flow news become relatively wealthier through their investment in the stock, and therefore contribute more to the average bias. This leads to fluctuations in the average bias, creating additional uncertainty for investors, so that following good (bad) cash-flow news, the economy becomes more optimistic (pessimistic). Consistently with empirical studies, we construct our belief dispersion measure as the cross-sectional standard deviation of investors' disagreement which also enables us identify its dual role. First, we show that the same level of good (bad) news leads to more optimism (pessimism) in a higher dispersion economy. Hence, belief dispersion amplifies the magnitude of current optimism/pessimism in the economy. Second, belief dispersion indicates how much the average bias fluctuates, and therefore measures the extra uncertainty investors face.

We first find that in the presence of belief dispersion the stock price is convex in cash-flow

¹We discuss the theoretical literature in detail in Section 1.1.

news, indicating that the stock price is more sensitive to news in relatively good states. It also implies that the increase in the stock price following good news is more than the decrease following bad news, as supported by empirical evidence (Basu (1997), Xu (2007)). Convexity arises because, the better the cash-flow news, the higher the extra boost for the stock price coming from elevated optimism. Consequently, the stock price increases with belief dispersion in a relatively optimistic economy, and decreases otherwise, also consistent with empirical evidence (Yu (2011)). The presence of belief dispersion also generates a novel finding that the stock price may actually increase in investors' risk aversion in relatively bad states. This is because bad news generates less pessimism in a more risk averse economy. More risk averse investors have less exposure to the stock, and this reduces the wealth transfers to pessimistic investors in bad times.

We next examine the widely-studied relation between belief dispersion and a stock mean return. Since dispersion represents additional risk for investors, risk averse investors demand a higher return to hold the stock in an economy with more dispersion. However, dispersion also amplifies optimism and pushes up the stock price following good news leading to a lower mean return in those states. In a relatively optimistic economy, the second effect dominates and we find a negative dispersion-mean return relation. As discussed earlier, empirical evidence on this relation is mixed, with some studies finding a negative while others finding a positive or no significant relation. Our model generates both possibilities and demonstrates that this relation is negative when the economy is relatively optimistic, and positive otherwise. Diether, Malloy, and Scherbina (2002) provide supporting evidence to our finding who show that the negative effect of dispersion becomes stronger for more optimistic stocks.

We further find that the presence of belief dispersion generates excess volatility by increasing the stock volatility beyond its fundamental uncertainty. This is because investors face the additional risk of stochastic average bias in beliefs, which then amplifies the stock price fluctuations, inducing excess volatility. Obviously, the higher the belief dispersion, the higher the excess stock volatility, which is also consistent with empirical evidence (Ajinkya and Gift (1985), Anderson, Ghysels, and Juergens (2005), Banerjee (2011)). In addition to belief dispersion, the investors' Bayesian learning process also increases the fluctuations of the average bias in beliefs. This occurs because *all* investors become more optimistic (pessimistic) following good (bad) news due to belief updating. In the literature, both the belief dispersion and learning channels are proposed as explanations for the excess stock volatility, among others. Our closed-form stock volatility expression allows us to disentangle their respective effects, and demonstrate that the effects of learning on excess volatility is reduced in higher dispersion economies.

We also examine the effects of belief dispersion on the stock trading volume and find that the trading volume is increasing in dispersion, consistently with empirical evidence (Ajinkya, Atiase, and Gift (1991), Bessembinder, Chan, and Seguin (1996), Goetzmann and Massa (2005)). This finding is intuitive since in a higher dispersion economy, investors with relatively different beliefs, who also have relatively higher trading demands, are more dominant. We also find a positive relation between the stock volatility and trading volume due to the positive effect of dispersion on both quantities, which is also supported empirically (Gallant, Rossi, and Tauchen (1992)).

Finally, we demonstrate that the most of our results above cannot be obtained in the more familiar, otherwise identical, finitely-many agent economies with heterogeneous beliefs. In particular, we show that in these economies, the stock price is no longer convex in cash-flow news, and a higher belief dispersion has ambiguous effects on the stock volatility and trading volume, in contrast to our model implications. This happens because in these economies, unlike in our model, belief dispersion diminishes in relatively extreme states, which forces these models to be dominated by a particular type of agent and to have implications similar to those in a homogeneous agent economy in those states, yielding irregular behavior for economic quantities across states of the world.

1.1 Related Theoretical Literature

In this paper, we solve a dynamic heterogeneous beliefs model with a continuum of, possibly Bayesian, investors having general CRRA preferences, and obtain fully closed-form solutions for all quantities of interest. Generally, these models are hard to solve for long-lived assets beyond logarithmic preferences (e.g., Detemple and Murthy (1994), Zapatero (1998), Basak (2005)). Our methodological contribution and the tractability of our model is in large part due to the investor types having a Gaussian distribution. This assumption follows from the recent works by Cvitanic and Malamud (2011) and Atmaz (2014). Cvitanic and Malamud does not consider the average bias and dispersion in beliefs and focuses on the survival and portfolio impact of irrational investors, while Atmaz does, but employs logarithmic preferences and focuses on short interest.

The literature on heterogeneous beliefs in financial markets is vast. Most of these works are set in a two-agent framework, and usually consider the overall effects of belief heterogeneity rather than decomposing its effects due to average bias and dispersion in beliefs, as we do. In these models belief heterogeneity is typically defined as the difference in beliefs of the two agents, which cannot readily be extended to an economy with many agents. By taking belief

dispersion to be the cross-sectional standard deviation of investors' disagreement (with unique weights), our measure can be employed for an arbitrarily investor population, but also captures the heterogeneity in beliefs when specialized to a two-agent economy (due to the monotonicity between differences in beliefs and the standard deviation of disagreement). Moreover, differently from earlier works our Gaussian distribution for investor types prevents any one type of agent from dominating the economy in relatively extreme states, which otherwise may lead to irregular behavior for economic quantities (see Sections 4.3 and 5.3). One strand of this extensive literature examines the relation between the belief dispersion and stock mean return. As discussed earlier, most studies find this relation to be positive (e.g., Abel (1989), Anderson, Ghysels, and Juergens (2005), David (2008), Banerjee and Kremer (2010)). On the other hand, Chen, Hong, and Stein (2002) and Johnson (2004) establish a negative relation by imposing short selling constraints for certain type of investors and considering levered firms, respectively. Buraschi, Trojani, and Vedolin (2013) develop a credit risk model and show that an increasing heterogeneity of beliefs has a negative (positive) effect on the mean return for firms with low (high) leverage. However, this result does not hold for unlevered firms. Differently from these works, we show that the belief dispersion-mean return relation is negative in relatively optimistic economies and positive otherwise.

Another strand in the heterogeneous beliefs literature examines the impact of belief dispersion on stock volatility and typically finds a positive effect (e.g., Scheinkman and Xiong (2003), Buraschi and Jiltsov (2006), Li (2007), David (2008), Dumas, Kurshev, and Uppal (2009), Banerjee and Kremer (2010), Andrei, Carlin, and Hasler (2015)). Yet another strand in this literature employs belief dispersion models to explain empirical regularities in trading volume. Early works include Harris and Raviv (1993) and Kandel and Pearson (1995). This strand also includes the works which find a positive relation between belief dispersion and trading volume, as in our work (e.g., Varian (1989), Shalen (1993), Cao and Ou-Yang (2008), Banerjee and Kremer (2010)). Even though our paper differs from each one of these papers in several aspects, one common difference is that none of the above papers generate the stock price convexity as in our model.²

Finally, this paper is also related to the literature on parameter uncertainty and Bayesian learning. In this literature, Veronesi (1999) and Lewellen and Shanken (2002) show that learning leads to stock price overreaction, time-varying expected returns and excess volatility. In

²Other works studying the effects of heterogeneous beliefs in financial markets include Basak (2000), Kogan, Ross, Wang, and Westerfield (2006), Jouini and Napp (2007), Gallmeyer and Hollifield (2008), Yan (2008), Xiong and Yan (2010), Bhamra and Uppal (2014), Chabakauri (2015).

particular, Veronesi shows that the stock price overreaction leads to a convex stock price.³ Timmermann (1993, 1996), Barsky and De Long (1993), Brennan and Xia (2001), Pástor and Veronesi (2003) show that learning generates excess volatility and predictability for stock returns. However, differently from our work, all these works employ homogenous investors setups, and therefore are not suitable for studying the effects of belief dispersion.

The remainder of the paper is organized as follows. Section 2 presents the simpler dogmatic beliefs version of our model which also serves to demonstrate that our results are not driven by Bayesian learning. Section 3 analyzes the average bias and dispersion in beliefs. Section 4 presents our results for the stock price and its mean return, while Section 5 those for the stock volatility and trading volume. Section 6 presents our general model with Bayesian learning and shows that all our results remain valid in this more complex economy. Section 7 concludes the paper. Appendix A.1 contains the proofs of the dogmatic beliefs model. Appendix A.2 introduces the finitely-many-investor version of our model. Appendix A.3 contains the technical Lemmas, and the Internet Appendix B.1 contains the proofs of our general model with Bayesian learning.

2 Economy with Dispersion in Beliefs

We consider a simple and tractable pure-exchange security market economy with a finite horizon evolving in continuous time. The economy is assumed to be large as it is populated by a continuum of investors with heterogeneous beliefs and standard CRRA preferences. In the general specification of our model, investors optimally learn over time in a Bayesian fashion. However, to highlight that our results are not driven by parameter uncertainty and learning, we first consider the economy when all investors have dogmatic beliefs. The richer case when investors update their beliefs over time is relegated to Section 6. We show that all our results hold in this more complex economy.

³In Veronesi (1999) the stock price convexity arises due to parameter uncertainty and the learning process, whereas in our model the convexity follows from the stochastic average bias in beliefs, which is due to the endogenous wealth transfers among heterogeneous investors and obtains even when there is no parameter uncertainty and learning. In more recent work, Xu (2007) develops a model in which the stock price is a convex function of the public signal. However, in his model no-short-sales constraints are needed to obtain this result and he does not investigate the stock mean return and volatility as we do.

2.1 Securities Market

There is a single source of risk in the economy which is represented by a Brownian motion ω defined on the true probability measure \mathbb{P} . Available for trading are two securities, a risky stock and a riskless bond. The stock price S is posited to have dynamics

$$dS_t = S_t [\mu_{S_t} dt + \sigma_{S_t} d\omega_t], \quad (1)$$

where the stock mean return μ_S and volatility σ_S are to be endogenously determined in equilibrium. The stock market is in positive net supply of one unit and is a claim to the payoff D_T , paid at some horizon T , and so $S_T = D_T$. This payoff D_T is the horizon value of the cash-flow news process D_t with dynamics

$$dD_t = D_t [\mu dt + \sigma d\omega_t], \quad (2)$$

where $D_0 = 1$, and μ and σ are constant, and represent the true mean return of the expected payoff and the uncertainty about the payoff, respectively. The bond is in zero net supply and pays a riskless interest rate r , which is set to 0 without loss of generality.⁴

2.2 Investors' Beliefs

There is a continuum of investors who commonly observe the same cash-flow news process D (2), but have different beliefs about its dynamics. The investors are indexed by their type θ , where a θ -type investor agrees with others on the stock payoff uncertainty σ but believes that the mean return of the expected payoff is $\mu + \theta$ instead of μ . This allows us to interpret a θ -type investor as an investor with a bias of θ in her beliefs. Consequently, a positive (negative) bias for an investor implies that she is relatively optimistic (pessimistic) compared to an investor with true beliefs. Under the θ -type investor's beliefs, the cash-flow news process has dynamics

$$dD_t = D_t [(\mu + \theta) dt + \sigma d\omega_t(\theta)],$$

where $\omega(\theta)$ is her perceived Brownian motion with respect to her own probability measure \mathbb{P}^θ , and is given by $\omega_t(\theta) = \omega_t - \theta t / \sigma$. Similarly, the risky stock price dynamics as perceived by the θ -type investor follows

$$dS_t = S_t [\mu_{S_t}(\theta) dt + \sigma_{S_t} d\omega_t(\theta)], \quad (3)$$

⁴Since investors have preferences only over horizon wealth, the interest rate can be taken exogenously. Our normalization of zero interest rate is for expositional simplicity and it is commonly employed in models with no intermediate consumption, see, for example, Pástor and Veronesi (2012) for a recent reference.

which together with the dynamics (1) yields the following consistency relation between the perceived and true stock mean returns for the θ -type investor

$$\mu_{St}(\theta) = \mu_{St} + \sigma_{St} \frac{\theta}{\sigma}. \quad (4)$$

The investor type space is denoted by Θ and it is taken to be the whole real line \mathbb{R} to incorporate all possible beliefs including the extreme ones and to avoid having arbitrary bounds for investor biases. We assume a Gaussian distribution with mean \tilde{m} and standard deviation \tilde{v} for the relative frequency of investors over the type space Θ . This assumption ensures that the investor population has a finite (unit) measure and admits much tractability.⁵ We further assume that all investors are initially endowed with an equal fraction of stock shares. Since a group of investors with the same beliefs and endowments are identical in every aspect, we represent them by a single investor with the same belief and whose initial endowment of stock shares is equal to the relative frequency of that group. This simplifies the analysis and provides the following initial wealth for each distinct θ -type investor

$$W_0(\theta) = S_0 \frac{1}{\sqrt{2\pi\tilde{v}^2}} e^{-\frac{1}{2} \frac{(\theta - \tilde{m})^2}{\tilde{v}^2}}, \quad (5)$$

where S_0 is the (endogenous) initial stock price. We can interpret the (exogenous) constants \tilde{m} and $\tilde{v} > 0$ as the initial wealth-share weighted average bias and dispersion in beliefs in the economy, respectively, since

$$\tilde{m} = \int_{\Theta} \theta \frac{W_0(\theta)}{S_0} d\theta, \quad (6)$$

$$\tilde{v}^2 = \int_{\Theta} (\theta - \tilde{m})^2 \frac{W_0(\theta)}{S_0} d\theta. \quad (7)$$

A higher \tilde{m} implies that the economy is relatively more optimistic initially since optimistic investors have relatively more initial wealth. On the other hand, a higher \tilde{v} implies that the beliefs in the economy are initially more dispersed in the sense that there are relatively more investors with large biases. This specification conveniently nests the benchmark homogenous beliefs economy with no bias when $\tilde{m} = 0$ and $\tilde{v} \rightarrow 0$.

⁵This assumption can be justified on the grounds of the typical investor distribution that is observed in well-known surveys such as the Livingston survey and the survey of professional forecasters conducted by the Philadelphia Federal Reserve. Generally, the observed distributions are roughly symmetric, single-peaked and assign less and less people to the tails. These properties are conveniently captured by our Gaussian distribution assumption, which also follows from the recent works by Cvitanović and Malamud (2011) and Atmaz (2014) as discussed in Section 1.1.

2.3 Investors' Preferences and Optimization

Each distinct θ -type investor chooses an admissible portfolio strategy $\phi(\theta)$, the fraction of wealth invested in the stock, so as to maximize her CRRA preferences over the horizon value of her portfolio $W_T(\theta)$

$$\mathbb{E}^\theta \left[\frac{W_T(\theta)^{1-\gamma}}{1-\gamma} \right], \quad \gamma > 0, \quad (8)$$

where \mathbb{E}^θ denotes the expectation under the θ -type investor's subjective beliefs \mathbb{P}^θ , and the financial wealth of the θ -type investor $W_t(\theta)$ follows

$$dW_t(\theta) = \phi_t(\theta) W_t(\theta) [\mu_{St}(\theta) dt + \sigma_{St} d\omega_t(\theta)]. \quad (9)$$

3 Equilibrium in the Presence of Belief Dispersion

To explore the implications of belief dispersion on the stock price and its dynamics, we first need a reasonable measure of it. In this Section, we define belief dispersion in a canonical way, to be the standard deviation of investors' biases in beliefs. Using the cross-sectional standard deviation of investors' disagreement as belief dispersion is consistent with the commonly employed belief dispersion measures in empirical studies.⁶ However, for this, we first need to determine the average bias in beliefs from which the investors' biases deviate. The average bias is defined to be the bias of the representative investor in the economy. We then summarize the wide range of investors' beliefs in our economy by these two variables, the *average bias* and *dispersion in beliefs*, and determine their values in the ensuing equilibrium. As we also demonstrate in Sections 4–5, the equilibrium quantities are driven by these two key (endogenous) variables, in addition to those in a homogenous-beliefs economy.

Equilibrium in our economy is defined in a standard way. The economy is said to be in equilibrium if equilibrium portfolios and asset prices are such that (i) all investors choose their optimal portfolio strategies, and (ii) stock and bond markets clear. We will often make

⁶See, for example, Diether, Malloy, and Scherbina (2002), Johnson (2004), Boehme, Danielsen, and Sorescu (2006), Sadka and Scherbina (2007), Avramov, Chordia, Jostova, and Philipov (2009), Chatterjee, John, and Yan (2011) who employ the standard deviation of levels in analysts' earnings forecasts, normalized by the absolute value of the mean forecast. Anderson, Ghysels, and Juergens (2005), Moeller, Schlingemann, and Stulz (2007), Yu (2011) employ the standard deviation of (long-term) growth rates in analysts' earnings forecasts as the measure of belief dispersion. Since we define ours as the standard deviation of investors' biases, our belief dispersion measure is similar to those used in the latter works. As Moeller, Schlingemann, and Stulz (2007) argue, there are several advantages of using the standard deviation of growth rates rather than of levels as a measure of belief dispersion, since the timing of the forecasts affect levels but not growth rates, and since growth rates are easily comparable across firms whereas normalization introduces noise for the levels. We discuss the equivalence of these two belief dispersion measures for our purposes in Remark 1.

comparisons with equilibrium in a *benchmark economy* where all investors have unbiased beliefs. We refer to this homogeneous beliefs economy as the economy with no belief dispersion.

Definition 1 (Average bias and dispersion in beliefs). The time- t *average bias in beliefs*, m_t , is defined as the implied bias of the corresponding representative investor in the economy. Moreover, expressing the average bias in beliefs as the weighted average of the individual investors' biases

$$m_t = \int_{\Theta} \theta h_t(\theta) d\theta, \quad (10)$$

with the weights $h_t(\theta) > 0$ are such that $\int_{\Theta} h_t(\theta) d\theta = 1$, we define the *dispersion in beliefs*, v_t , as the standard deviation of investors' biases

$$v_t^2 \equiv \int_{\Theta} (\theta - m_t)^2 h_t(\theta) d\theta. \quad (11)$$

The extent to which an investor's belief is represented in the economy depends on her wealth and risk attitude. In our dynamic economy, the investors whose beliefs are supported by the actual cash-flow news become relatively wealthier. This increases the impact of their beliefs in the determination of equilibrium prices. Our definition of the average bias in beliefs captures this mechanism by equating it to the bias of the representative investor who assigns more weight to an investor whose belief has more impact on the equilibrium prices.⁷ Finding the average bias this way is similar to representing heterogeneous beliefs in an economy by a consensus belief as in Rubinstein (1976), and more recently in Jouini and Napp (2007).⁸

The average bias in beliefs, by construction, implies that when it is positive investors are on average optimistic, and when negative investors are on average pessimistic. The weights, $h_t(\theta)$, are such that the weighted average of individual investors' biases is the bias of the representative investor. We also discuss alternative weights, average bias and dispersion measures in Remark 1. Importantly, it is these weights that allow us to define belief dispersion in an intuitive way. Proposition 1 presents the average bias and dispersion along with the corresponding unique

⁷The representative investor in Definition 1 follows the standard construction, with the representative investor's utility function being given by maximizing a weighted-average of individual investors' utilities adjusted for their beliefs.

⁸The main idea, as elaborately discussed in Jouini and Napp (2007), is to summarize the heterogeneous beliefs in the economy by a single consensus belief so that when the consensus investor has that consensus belief and is endowed with the aggregate consumption in the economy, the resulting equilibrium is as in the heterogeneous-investors economy. Jouini and Napp show that the consensus belief is a risk tolerance-weighted average of the individual investors' beliefs and that the consensus investor's utility has an additional stochastic discount factor taking into account of belief heterogeneity. Our average bias in beliefs measure coincides with their implied bias in the consensus beliefs. However, differently from their analysis, the discount term is not stochastic but deterministic in our economy, and therefore, does not affect the equilibrium. Moreover, in their work the belief dispersion is captured only by the discount term, whereas by defining it as in (11), our model captures belief dispersion even when there is no discount factor in the representative investor's utility.

weights in our economy in closed form.

Proposition 1. *The time- t average bias m_t and dispersion v_t in beliefs are given by*

$$m_t = m + \left(\ln D_t - \left(m + \mu - \frac{1}{2}\sigma^2 \right) t \right) \frac{v_t^2}{\gamma\sigma^2}, \quad (12)$$

$$v_t^2 = \frac{v^2\sigma^2}{\sigma^2 + \frac{1}{\gamma}v^2t}, \quad (13)$$

where their initial values m and v are related to the initial wealth-share weighted average bias \tilde{m} and dispersion \tilde{v} in beliefs as

$$m = \tilde{m} + \left(1 - \frac{1}{\gamma} \right) v^2 T, \quad (14)$$

$$v^2 = \left(\frac{\gamma}{2}\tilde{v}^2 - \frac{\gamma^2}{2T}\sigma^2 \right) + \sqrt{\left(\frac{\gamma}{2}\tilde{v}^2 - \frac{\gamma^2}{2T}\sigma^2 \right)^2 + \frac{\gamma^2}{T}\tilde{v}^2\sigma^2}. \quad (15)$$

The weights $h_t(\theta)$ are uniquely identified to be given by

$$h_t(\theta) = \frac{1}{\sqrt{2\pi v_t^2}} e^{-\frac{1}{2} \frac{(\theta - m_t)^2}{v_t^2}}, \quad (16)$$

where m_t, v_t are as in (12)–(13).

We see that the average bias in beliefs (12) is stochastic and depends on the cash-flow news D .⁹ When there is good news, the relatively optimistic investors' beliefs get supported, and through their investment in the stock they get relatively wealthier. This in turn increases their weight in equilibrium and consequently makes the economy more optimistic. The analogous mechanism makes the economy more pessimistic following bad news. However, the extent of optimism/pessimism in the economy depends crucially on the level of belief dispersion v_t . In particular, dispersion amplifies the effects of cash-flow news on the average bias, and hence the same level of good (bad) news leads to more optimism (pessimism) in a higher dispersion economy. We illustrate this feature in Figure 1, where we plot the weights $h_t(\theta)$ for different levels of dispersion in relatively bad (panel (a)) and good (panel (b)) cash-flow news states. The average bias is given by the point where the respective plot centers. We see that higher dispersion plots are flatter and center at a point further away from the origin, which shows that investors with relatively large biases are indeed assigned higher weights and optimism/pessimism is amplified under higher dispersion. Investors' attitude towards risk, γ , influences the average bias too. As

⁹For notational convenience, we denote the initial values of the average bias and dispersion in beliefs by m and v instead of m_0 and v_0 , respectively. We note that the average bias can also be represented in terms of the initial values by $m_t = \sigma \left(\sigma m + \frac{1}{\gamma} v^2 \omega_t \right) / \left(\sigma^2 + \frac{1}{\gamma} v^2 t \right)$.

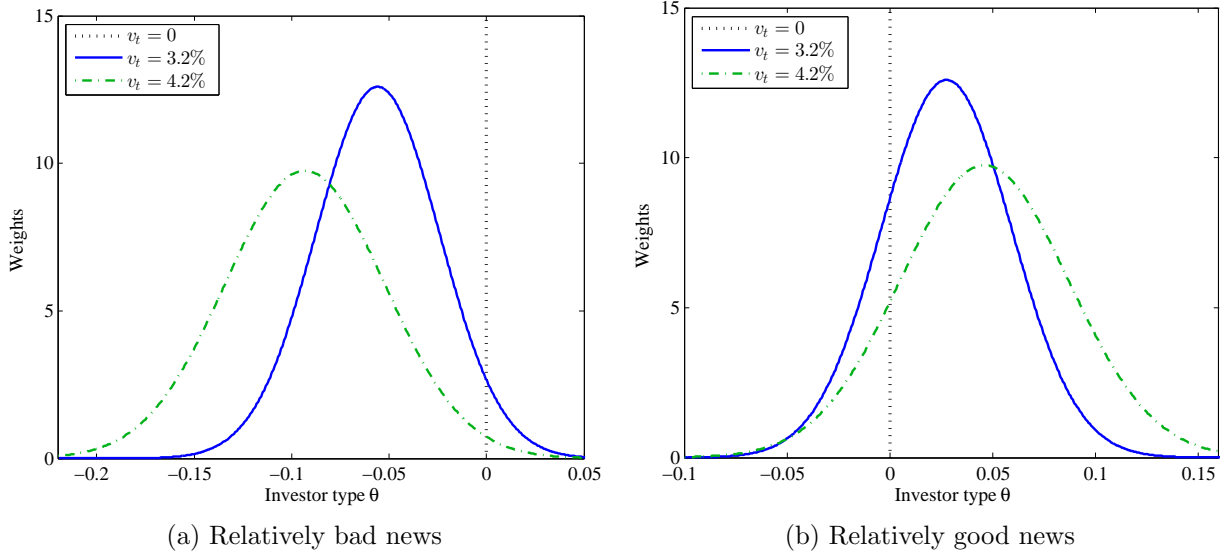


Figure 1: **Investors' weights.** These panels plot the weights $h_t(\theta)$ for each distinct θ -type investor for different levels of current belief dispersion v_t . The belief dispersion is $v_t = 3.2\%$ in solid blue and 4.2% in dashed green lines. The vertical dotted black lines correspond to the benchmark economy with no belief dispersion. The cash-flow news is relatively bad $D_t = 0.5$ in panel (a) and good $D_t = 1.5$ in panel (b). The baseline parameter values are: $\tilde{m} = 0$, $\tilde{v} = 3.23\%$, $\mu = 9.2\%$, $\sigma = 11.5\%$, $\gamma = 1$, $t = 0.5$ and $T = 5$.¹⁰

investors become more risk averse, they invest less in the stock and this limits the wealth transfers in the economy. Consequently, this reduces the sensitivity of the average bias to cash-flow news, leading to less optimism (pessimism) relatively when the cash-flow news is good (bad).

In the presence of heterogeneity in beliefs, the belief dispersion has a dual role. Besides amplifying the current average bias in beliefs, the belief dispersion also drives the extent to which average bias fluctuates over time and hence represents the riskiness of average bias. Indeed, it can be shown from (12) that the volatility of the average bias in beliefs is $\sigma_{mt} = v_t^2/\gamma\sigma$, where σ_{mt} satisfies $dm_t = \mu_{mt}dt + \sigma_{mt}d\omega_t$. Hence, the higher the dispersion in the economy, the higher the fluctuations in the average bias, and therefore, the greater the uncertainty investors face. As for the dynamics of belief dispersion itself, as (13) highlights, the dispersion is at its highest

¹⁰The initial wealth-share weighted average bias in beliefs \tilde{m} is taken to be zero to prevent any effects arising due to the initial bias. The initial wealth-share weighted belief dispersion \tilde{v} is chosen to match the time-series average of the analysts' forecast dispersion about the earnings growth rate of a market portfolio as reported in Yu (2011) for the period (1981-2005) – this yields $\tilde{v} = 3.23\%$. The parameter value for μ , 9.2% , is obtained by matching it to the realized market excess return in Yu (2011). For σ , we match the initial stock volatility (σ_{S0}) given by Proposition 5 to the average market volatility (16%), also reported in Yu (2011). This gives the quadratic equation $\sigma^2 - \sigma_{S0}\sigma + \tilde{v}^2T = 0$ for σ , which yields the solution as $\sigma = 11.5\%$. The relative risk aversion coefficient is set at 1. The current time is set at $t = 0.5$ and the investment horizon T at 5 years throughout the plots. Substituting above parameter values into (13) gives the current belief dispersion as $v_t = 3.2\%$ after rounding. To highlight the effects of increased dispersion, we also plot the relevant economic quantities when the current belief dispersion is increased to $v_t = 4.2\%$.

level initially and then decreases over time deterministically as investors with extreme beliefs tend to receive less and less weight over time due to their diminishing wealth and impact in equilibrium.

Equation (16) indicates that the time- t weights $h_t(\theta)$, which can be thought of as the time- t “effective” relative frequency of investors, have a convenient Gaussian form with mean m_t and standard deviation v_t as also illustrated in Figure 1. This feature allows us to characterize the wide range of investor heterogeneity in our economy by the average bias and dispersion in beliefs since they are the first two (thus sufficient) central moments of Gaussian weights. Finally, it is worth noting that for logarithmic preferences ($\gamma = 1$), the weights coincide with the wealth-share weights $W(\theta)/S$ as discussed in Remark 1.

Remark 1 (Alternative average bias and dispersion in beliefs measures). In heterogeneous beliefs models with two agents, the dispersion, or the disagreement, in beliefs is typically captured by the difference between the biases of the two agents (e.g., Basak (2005), Banerjee and Kremer (2010), and Xiong and Yan (2010)). With many agents in a dynamic economy, however, there is no immediately obvious alternative. In our setting, the simplest choice would be to use the initial distribution of investors as the weights and compute the corresponding weighted-average bias and dispersion in beliefs. However, this leads to constant average bias and dispersion measures as in (6)–(7), and so it would not be possible to characterize the stochastic equilibrium quantities using these constant measures. For a dynamic economy such as ours, stochastic impact of investors’ beliefs and wealth ought to be taken into account.¹¹

To capture the larger impact of wealthier investors on equilibrium prices, one may use the wealth-share distribution as the weights. This definition does not require the construction of the representative investor and yields alternative average bias and dispersion in beliefs measures denoted by \tilde{m}_t and \tilde{v}_t , respectively, which can be shown to be given by

$$\tilde{m}_t \equiv \int_{\Theta} \theta \frac{W_t(\theta)}{S_t} d\theta = m_t - \left(1 - \frac{1}{\gamma}\right) v_t^2 (T - t), \quad (17)$$

$$\tilde{v}_t^2 \equiv \int_{\Theta} (\theta - \tilde{m}_t)^2 \frac{W_t(\theta)}{S_t} d\theta = \frac{1}{\gamma} v_t^2 + \left(1 - \frac{1}{\gamma}\right) v_T^2, \quad (18)$$

where m_t , v_t are as in (12)–(13).¹² As equations (17)–(18) highlight, our average bias and dispersion in beliefs coincide with their respective wealth-share weighted counterparts when the preferences are logarithmic ($\gamma = 1$) and also at the horizon T . For non-logarithmic preferences,

¹¹This is less of a concern in static models with many agents (e.g., Chen, Hong, and Stein (2002)) since there are no dynamic belief and wealth transfer considerations.

¹²Above expressions are proved in the proof of Proposition 5 in Appendix A.1.

at any point in time, the wealth-share weighted average bias \tilde{m}_t differs from the average bias m_t , but only by a constant. This constant arises since the distinct θ -type investor with the highest wealth is not the same investor whose bias has the highest impact on equilibrium quantities when $\gamma \neq 1$. However, since the difference between the two average bias measures is a constant, we obtain similar results and all the predictions of the model remain valid if, instead of m_t and v_t , we use the wealth-share weighted average bias and dispersion measures as in (17)–(18).

Following Diether, Malloy, and Scherbina (2002), numerous empirical studies use the standard deviation of analysts' earnings forecast levels (normalized by the absolute value of the mean forecast) as a proxy for investors' belief dispersion (see footnote 6). In our setting, this is achieved by considering an alternative belief dispersion measure v_t , defined as the standard deviation of the investors' subjective expectations about the payoff D_T , normalized by the mean expected payoff. This can be shown to be given by

$$v_t^2 = \frac{\int_{\Theta} (\mathbb{E}_t^\theta [D_T] - \int_{\Theta} \mathbb{E}_t^\theta [D_T] h_t(\theta) d\theta)^2 h_t(\theta) d\theta}{\int_{\Theta} \mathbb{E}_t^\theta [D_T] h_t(\theta) d\theta} = e^{v_t^2(T-t)^2} - 1.$$

We see that there is a strictly monotonic positive relation between v_t and our measure v_t , and therefore, all our implications remain valid under either specification.

4 Stock Price and Mean Return

In this Section, we investigate how the stock price and its mean return are affected by the average bias and dispersion in beliefs. In particular, we demonstrate that the stock price is convex in cash-flow news. A higher belief dispersion gives rise to a higher stock price and a lower mean return when the economy is relatively optimistic, and vice versa when pessimistic, consistent with empirical evidence. In Section 4.3, we discuss how the more familiar, otherwise identical heterogeneous beliefs models with two (or finitely many) investors are not capable of generating several of our main findings.

4.1 Equilibrium Stock Price

Proposition 2. *In the economy with belief dispersion, the equilibrium stock price is given by*

$$S_t = \bar{S}_t e^{m_t(T-t) - \frac{1}{2\gamma}(2\gamma-1)v_t^2(T-t)^2}, \quad (19)$$

where the average bias m_t and dispersion v_t in beliefs are as in Proposition 1, and the equilibrium stock price in the benchmark economy with no belief dispersion is given by

$$\bar{S}_t = D_t e^{(\mu - \gamma \sigma^2)(T-t)}. \quad (20)$$

Consequently, in the presence of belief dispersion,

- i) The stock price is higher than in the benchmark economy when $m_t > (1/2\gamma)(2\gamma - 1)v_t^2(T - t)$, and is lower otherwise.
- ii) The stock price is convex in cash-flow news D_t .
- iii) The stock price is increasing in belief dispersion v_t when $m_t > \tilde{m} + (1/2\gamma)(2\gamma - 1)v_t^2(T - t)$, and is decreasing otherwise.
- iv) The stock price is decreasing in investors' risk aversion γ , as in the benchmark economy for relatively good cash-flow news. However, the stock price is increasing in investors' risk aversion for relatively bad cash-flow news and low levels of risk aversion.

The stock price in the benchmark economy is driven by cash-flow news D_t , whereby good news (higher D_t) leads to a higher stock price since investors increase their expectations of the stock payoff D_T . The equilibrium stock price in the presence of belief dispersion has a simple structure, and is additionally driven by the average bias m_t and dispersion v_t in beliefs. The role of the average bias in beliefs is to increase the stock price further following good news, and conversely following bad news. This is because, as discussed in Section 3, following good cash-flow news the economy becomes relatively more optimistic, and so has a higher expectation of the stock payoff, which then leads to a further stock price increase, and vice versa following bad news. Figure 2 plots the equilibrium stock price against cash-flow news for different levels of belief dispersion, illustrating above points. The stock price being increasing in optimism is consistent with empirical evidence (Brown and Cliff (2005)), and also implies that the stock price is eventually higher than in the benchmark economy when the economy is sufficiently optimistic (Property (i)).¹³ Moreover, this property implies a feature, against conventional wisdom, that a moderately optimistic economy can have a stock price that is lower than in the benchmark economy with unbiased beliefs. This happens when the negative effect of belief dispersion, as we discuss below, outweighs the positive effect of optimism.

¹³For logarithmic preferences ($\gamma = 1$), the stock price is higher than in the benchmark economy when $m_t > (1/2)v_t^2(T - t)$. For non-logarithmic preferences, the adjustment for risk aversion generates the $(2\gamma - 1)$ term. This adjustment is intuitive, since the higher the risk aversion, the more investors dislike the uncertainty in the future average bias in beliefs, which is driven by belief dispersion (Section 3), and so the higher the optimism must be for the stock price to be greater than in the benchmark case.

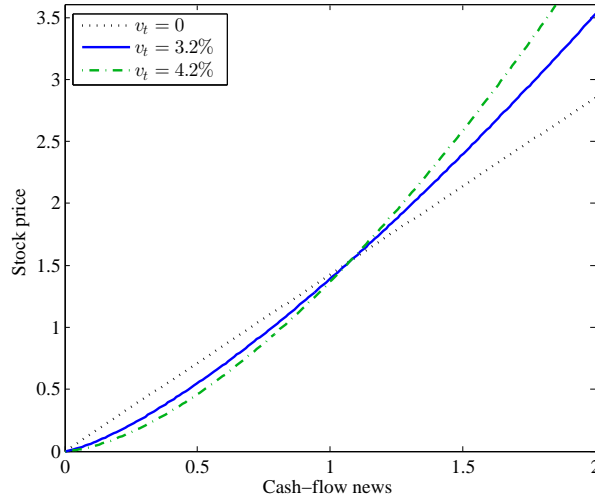


Figure 2: **Stock price convexity and effects of belief dispersion.** This figure plots the equilibrium stock price S_t against cash-flow news for different levels of current belief dispersion v_t . The dotted line corresponds to the equilibrium stock price in the benchmark economy with no belief dispersion. The baseline parameter values are as in Figure 1.

Figure 2 also illustrates the extra boost in the stock price due to increased optimism following good news. The notable implication here is the convex stock price-news relation as opposed to the linear one in the benchmark economy (Property (ii)). The convexity implies that the increase in the stock price following good news is more than the decrease following bad news (all else fixed), which is also supported empirically (Basu (1997), Xu (2007)). It also implies that the stock price is more sensitive to news (good or bad) in relatively good states. Conrad, Cornell, and Landsman (2002) document that bad news decreases the stock price more in good states which is also in line with our finding. As mentioned in the Introduction, a similar convexity property is obtained by Veronesi (1999), but due to parameter uncertainty in a model with homogenous agents.

Turning to the role of belief dispersion v_t , we see that its influence on the stock price (19) enters via two channels: directly (v_t^2 term) and indirectly (via average bias in beliefs m_t). The direct effect always decreases the stock price for plausible levels of risk aversion ($\gamma > 1/2$) since dispersion represents the riskiness of the average bias (as discussed in Section 3). The indirect effect, however, amplifies the average bias (Section 3), thereby increasing the stock price further following relatively good news and decreasing it further following relatively bad news. Since both effects have a negative impact following bad news, the stock price always decreases in relatively bad states due to dispersion. On the other hand, for sufficiently good cash-flow news, the indirect effect of dispersion dominates and the stock price increases. These are also illustrated in Figure 2.

Consequently, a notable implication here is that the stock price increases in belief dispersion

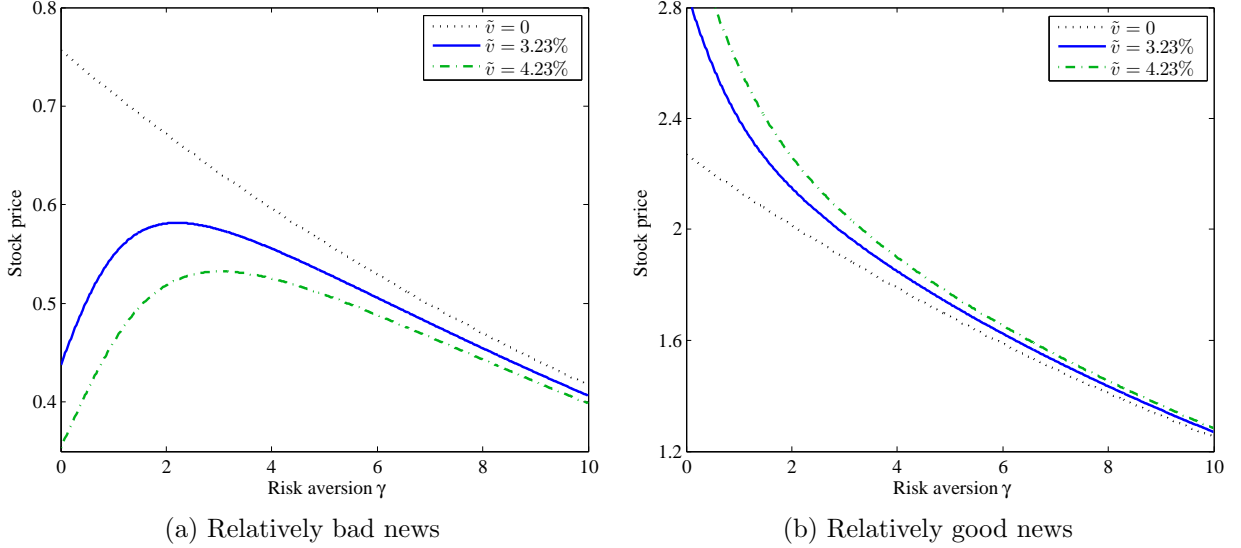


Figure 3: **Effects of risk aversion on stock price.** These figures plot the equilibrium stock price S_t against relative risk aversion coefficient γ for different levels of initial wealth-share weighted belief dispersion \tilde{v} . The dotted lines correspond to the equilibrium stock price in the benchmark economy with no belief dispersion. The cash-flow news is relatively bad $D_t = 0.5$ in panel (a) and good $D_t = 1.5$ in panel (b). The baseline parameter values are as in Figure 1.¹⁴

for a relatively optimistic economy, and decreases otherwise (Property (iii)). A higher belief dispersion leading to a higher stock price is empirically documented by Goetzmann and Massa (2005) and Yu (2011). This is somewhat surprising since, instead of requiring a premium for the extra uncertainty due to belief dispersion, investors appear to pay a premium for it. Our model reconciles with this seemingly counterintuitive finding by demonstrating that a higher dispersion may lead to a higher stock price when the stock price is driven by sufficiently optimistic beliefs. This is supported by evidence in Yu (2011). Yu provides evidence that a higher belief dispersion increases growth stock (low book-to-market) prices more than value stock prices, and associates growth stocks with optimism motivated by the findings of Lakonishok, Shleifer, and Vishny (1994), La Porta (1996). He also finds weak evidence that value stock prices in fact decrease under higher dispersion.

Figure 3 presents the effects of risk aversion on the equilibrium stock price and highlights a novel finding that the stock price in the presence of belief dispersion may actually increase in investors' risk aversion γ (Property (iv)). In the benchmark economy, the stock price always decreases in investors' risk aversion. This is intuitive since more risk averse investors demand

¹⁴We note that unlike earlier Figures, these plots are not for different levels of current belief dispersion v_t but for different levels of initial wealth-share weighted dispersion \tilde{v} , since v_t depends on γ and therefore cannot be fixed across different levels of relative risk aversion.

a higher return to hold the risky stock and so push down its price. In the presence of belief dispersion, the risk aversion has an additional impact on the stock price through the average bias in beliefs. As discussed in Section 3, a higher risk aversion makes the average bias less sensitive to news since it reduces risk taking and hence the magnitude of wealth transfers among investors. Therefore, the same level of bad news generates less pessimism and decreases the stock price less in a more risk averse economy. This creates a range of risk aversion values for which the stock price actually increases in investors' risk aversion. On the other hand, for relatively good news, both the increased risk aversion and the accompanying reduced optimism induce investors to demand a higher return, which leads to the stock price being monotonically decreasing in investors' risk aversion.

Remark 2. The equilibrium stock price in (19) can also be expressed in terms of the volatility of the average bias in beliefs σ_{mt} as

$$S_t = \bar{S}_t e^{m_t(T-t) - \frac{1}{2}(2\gamma-1)\sigma_{mt}(T-t)^2}, \quad (21)$$

where $\sigma_{mt} = v_t^2/\gamma\sigma$ as discussed in Section 3. In Section 6, within a richer setup with Bayesian learning, we show that the stock price expression is also as is in (21).

4.2 Equilibrium Mean Return

In our economy, the mean return perceived by each θ -type investor, $\mu_S(\theta)$, is different than the (observed) true mean return, μ_S , with the relation between them being given by (4). To make our results comparable to empirical studies, in this Section we present our results in terms of the true mean return (as observed in the data), henceforth, simply referred to as the mean return. Proposition 3 reports the equilibrium mean return and its properties.

Proposition 3. *In the economy with belief dispersion, the equilibrium mean return is given by*

$$\mu_{St} = \bar{\mu}_{St} \frac{v_t^4}{v_T^4} - m_t \frac{v_t^2}{v_T^2}, \quad (22)$$

where the average bias m_t and dispersion v_t in beliefs are as in Proposition 1, and the equilibrium mean return in the benchmark economy with no belief dispersion is given by

$$\bar{\mu}_{St} = \gamma\sigma^2. \quad (23)$$

Consequently, in the presence of belief dispersion,

- i) The mean return is lower than in the benchmark economy when $m_t > (v_t^2 + v_T^2)(T-t)$, and is higher otherwise.

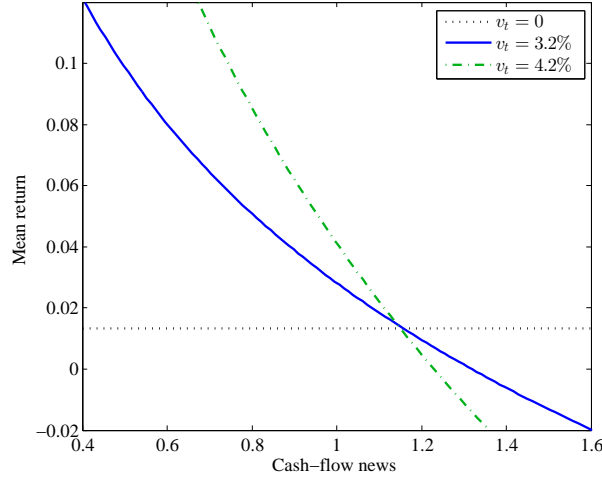


Figure 4: **Effects of belief dispersion on mean return.** This figure plots the equilibrium mean return μ_{S_t} against cash-flow news for different levels of current belief dispersion v_t . The dotted line corresponds to the equilibrium mean return in the benchmark economy with no belief dispersion. The baseline parameter values are as in Figure 1.

- ii) *The mean return is decreasing in belief dispersion v_t when $m_t > v_t^2 (\tilde{m} + 2v_t^2 (T - t)) \times (2v_t^2 - v_T^2)^{-1}$, and is increasing otherwise.*
- iii) *The mean return is increasing in investors' risk aversion γ , as in the benchmark economy for relatively good cash-flow news. However, the mean return is decreasing in investors' risk aversion for relatively bad cash-flow news and low levels of risk aversion.*

The presence of belief dispersion makes the equilibrium mean return stochastic (a constant in benchmark economy) and strictly decreasing in the average bias in beliefs m_t . This is because, the higher the average bias, the higher the stock price (Section 4.1), and therefore, the more optimistic economy receives on average relatively more negative subsequent news, which in turn leads to a lower mean return.¹⁵ The mean return being decreasing in optimism is supported by empirical evidence (La Porta (1996), Brown and Cliff (2005)), and implies that the mean return is lower than in the benchmark economy when the economy is sufficiently optimistic (Property (i)). We have a somewhat surprising feature here in that a moderately optimistic economy ($0 < m_t < (v_t^2 + v_T^2)(T - t)$) may have a higher stock mean return than that in the unbiased benchmark economy. This occurs when the higher return demanded by investors due to the extra risk caused by belief dispersion, as discussed below, outweighs the lower return demanded by investors due to the moderate optimism in the economy.

¹⁵Relatively optimistic economy receiving more negative subsequent news on average is due to the fact that the true data generating process, the cash-flow news, has constant parameters, which imply that the consecutive ratios (D_t/D_{t-h}) and (D_{t+h}/D_t) are i.i.d. lognormal.

Figure 4 plots the equilibrium mean return against cash-flow news for different levels of belief dispersion and illustrates that a higher belief dispersion v_t leads to a lower mean return when the economy is sufficiently optimistic, and to a higher mean return otherwise (Property (ii)). The intuition for this is similar to that for the stock price: dispersion represents additional risk for investors (Section 3), and therefore investors demand a higher return to hold the stock in a higher dispersion economy. However, we see from (22) that dispersion also multiplies the average bias in beliefs, which in turn leads to a lower mean return when the economy is optimistic and to a higher mean return when pessimistic. When the economy is sufficiently optimistic, the latter effect dominates and produces the negative relation between belief dispersion and mean return.

As discussed in the Introduction, the empirical evidence on the relation between belief dispersion and mean return is vast and mixed, and existing theoretical works explain only one side of this relation. Our model generates both the negative and positive effects and implies that the documented negative relation must be due to the optimistic bias and it should be stronger, the higher the optimism. Diether, Malloy, and Scherbina (2002) provide supporting evidence for our implications by finding an optimistic bias in their study overall, and by also showing that the negative effect of dispersion is indeed stronger for more optimistic stocks. Similar evidence is also provided by Yu (2011) who documents that high dispersion stocks earn lower returns than low dispersion ones and this effect is more pronounced for growth (low book-to-market) stocks which tend to represent overly optimistic stocks (see, for example, Lakonishok, Shleifer, and Vishny (1994), La Porta (1996) and Skinner and Sloan (2002)).

Figure 5 plots the equilibrium mean return against investors' risk aversion γ for different levels of belief dispersion and highlights an interesting feature that the equilibrium mean return may decrease in investors' risk aversion for relatively bad news states over a range of risk aversion values (Property (iii)). Analogous to the intuition given for the stock price (Section 4.1), this result is again due to bad news leading to less pessimism in more risk averse economies. We again note that for relatively good news, the mean return monotonically increases in investors' risk aversion as in the benchmark economy. This is because both the increased risk aversion and the accompanying reduced optimism induce investors to demand a higher return.

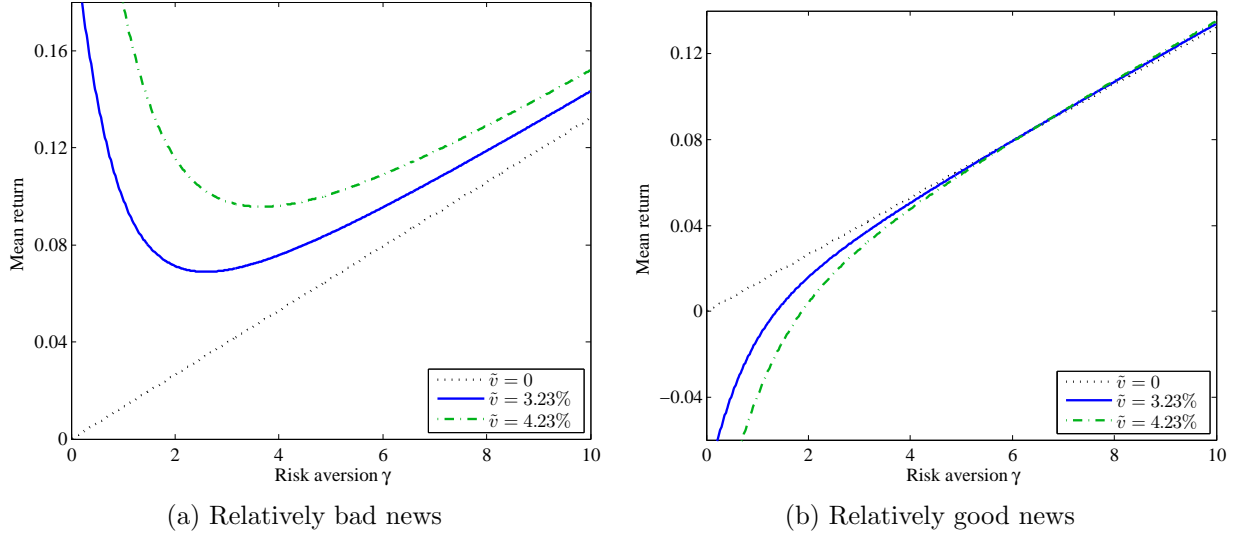


Figure 5: **Effects of risk aversion on mean return.** These figures plot the equilibrium mean return μ_{St} against relative risk aversion coefficient γ for different levels of initial wealth-share weighted belief dispersion \tilde{v} . The dotted lines correspond to the equilibrium mean returns in the benchmark economy with no belief dispersion. The cash-flow news is relatively bad $D_t = 0.5$ in panel (a) and good $D_t = 1.5$ in panel (b). For the choice of dispersion parameter values see footnote 14. The baseline parameter values are as in Figure 1.

4.3 Comparisons with Two-Investor Economy

In our economy, investor heterogeneity does not vary across states of the world (since belief dispersion v_t is deterministic), and hence does not vanish in relatively extreme states. However, this is not the case for otherwise identical heterogeneous beliefs models with two (or finitely many) investors having CRRA preferences over horizon wealth.¹⁶ In these models, the (most) pessimistic investor would control almost all the wealth in the economy in very bad news states, while the (most) optimistic investor would hold all wealth in very good news states. This diminishes the (effective) investor heterogeneity in such states and forces the model to have implications similar to those in a single investor economy, which then yields irregular behavior for economic quantities across states of the world.

To better highlight the effects of diminishing heterogeneity, Figure 6 plots the equilibrium stock price and mean return against cash-flow news in an otherwise identical two-investor economy, with one investor being optimistic and the other pessimistic. The equilibrium expressions are provided in Appendix A.2 and are for logarithmic preferences to be consistent with earlier

¹⁶Without loss of generality, we consider the more familiar two-investor economy, $\Theta = \{\theta_1, \theta_2\}$, in our discussion here. However, the arguments and the plots presented here are valid for the more general otherwise identical heterogeneous beliefs models with finitely many investors, $\Theta = \{\theta_1, \dots, \theta_N\}$, as discussed in Appendix A.2. Examples of other works using this economic setup include Kogan, Ross, Wang, and Westerfield (2006) (two investors) and Cvitanic and Malamud (2011) (finitely many investors).

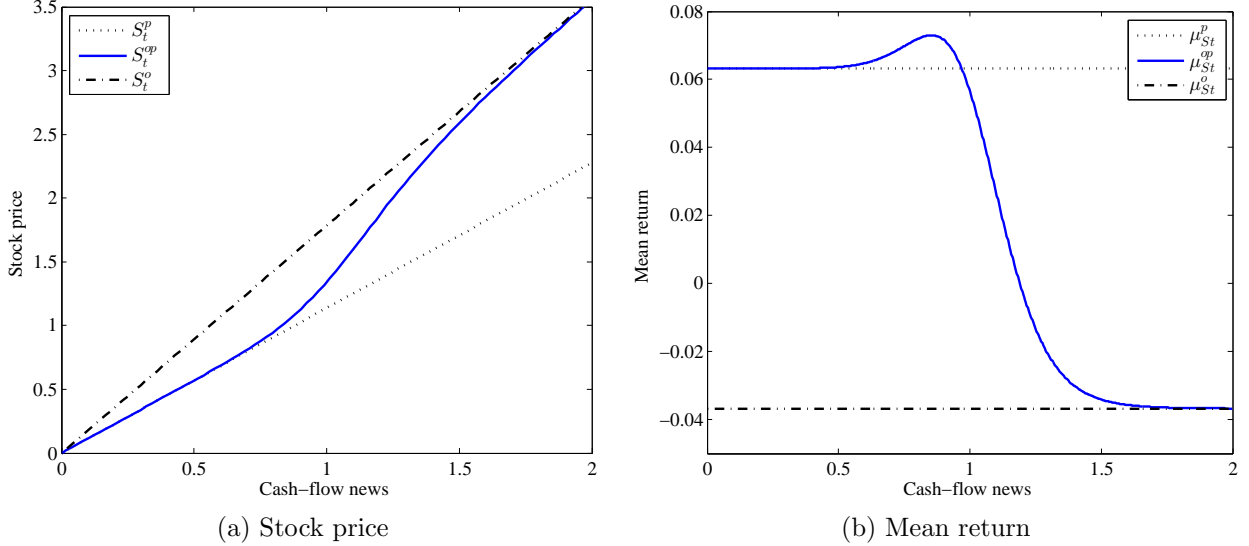


Figure 6: **Stock price and its mean return in a two-investor economy.** These figures plot the equilibrium stock price S_t^{op} in panel (a) and the mean return $\mu_{S_t}^{op}$ in panel (b) against cash-flow news in an otherwise identical two-investor economy with an optimistic and a pessimistic investor. The dotted lines correspond to the stock price S_t^p and mean return $\mu_{S_t}^p$ in an economy with a single pessimistic investor with a bias in beliefs $\theta = -5\%$. The dashed lines correspond to the stock price S_t^o and mean return $\mu_{S_t}^o$ in an economy with a single optimistic investor with a bias in beliefs $\theta = +5\%$. The other applicable parameter values are as in Figure 1.

figures and for tractability as discussed in the Introduction. We see that for very good (bad) news states, the equilibrium stock price and mean return behaviors are similar to their respective counterparts in single optimistic (pessimistic) investor economies. When this is combined with the behavior in moderate news states for which the effect of heterogeneity is prevalent, the stock price and mean return generate strikingly different implications than our model. In particular, we see that the stock price is no longer convex in cash-flow news across all states as opposed to that in our model (as discussed in Section 4.1). Likewise, we also see that the mean return does not always strictly decrease in cash-flow news, and in fact it may even increase during transitional states. In Section 5.3, we further discuss how such familiar two-investor models also do not generate uniform relations for the stock volatility and trading volume with respect to belief dispersion, in contrast to our model.

5 Stock Volatility and Trading Volume

In our economy, investors manifest their differing beliefs by taking diverse stock positions, which in turn generate trade and wealth transfers among investors. As discussed in Section 3, these wealth transfers make the average bias in beliefs stochastic, which then leads to extra

uncertainty for investors. In this Section, we demonstrate how this extra uncertainty and investors' trading motives give rise to excess stock volatility and trading volume. We further show that a higher belief dispersion leads to a higher stock volatility and trading volume in our model. These findings are well supported by empirical evidence. However, in Section 5.3 (analogously to Section 4.3), we show that a higher belief dispersion has ambiguous effects on the stock volatility and trading volume in the more familiar, otherwise identical heterogeneous beliefs models with two (or finitely many) investors.

5.1 Equilibrium Stock Volatility

Proposition 4. *In the economy with belief dispersion, the equilibrium stock volatility is given by*

$$\sigma_{St} = \bar{\sigma}_{St} + \frac{v_t^2}{\gamma\sigma} (T - t), \quad (24)$$

where the dispersion in beliefs v_t is as in Proposition 1, and the equilibrium stock volatility in the benchmark economy with no belief dispersion is given by

$$\bar{\sigma}_{St} = \sigma. \quad (25)$$

Consequently, in the presence of belief dispersion,

- i) *The stock volatility is higher than in the benchmark economy.*
- ii) *The stock volatility is increasing in belief dispersion v_t .*
- iii) *The stock volatility is decreasing in investors' risk aversion γ .*

The key implication of Proposition 4 is that the belief dispersion v_t generates excess volatility in our economy by amplifying the stock volatility beyond the fundamental payoff uncertainty σ .¹⁷ This is because, as discussed in Section 4, the stock price increases more than in the benchmark economy following good cash-flow news and decreases more following bad cash-flow news. This additional fluctuation in the stock price across news states generates the excess stock volatility (Property (i)). This is consistent with the well-documented empirical evidence that stock volatilities are too high to be justified by fundamental uncertainty (Leroy and Porter (1981), Shiller (1981)).

Naturally, the higher the belief dispersion, the higher the excess stock volatility (Property (ii)). This is because in a higher belief dispersion economy, the average bias in beliefs fluctuates

¹⁷The stock volatility can also be written as $\sigma_{St} = \bar{\sigma}_{St} v_t^2 / v_T^2$ where the ratio $v_t^2 / v_T^2 > 1$ for all $t < T$.

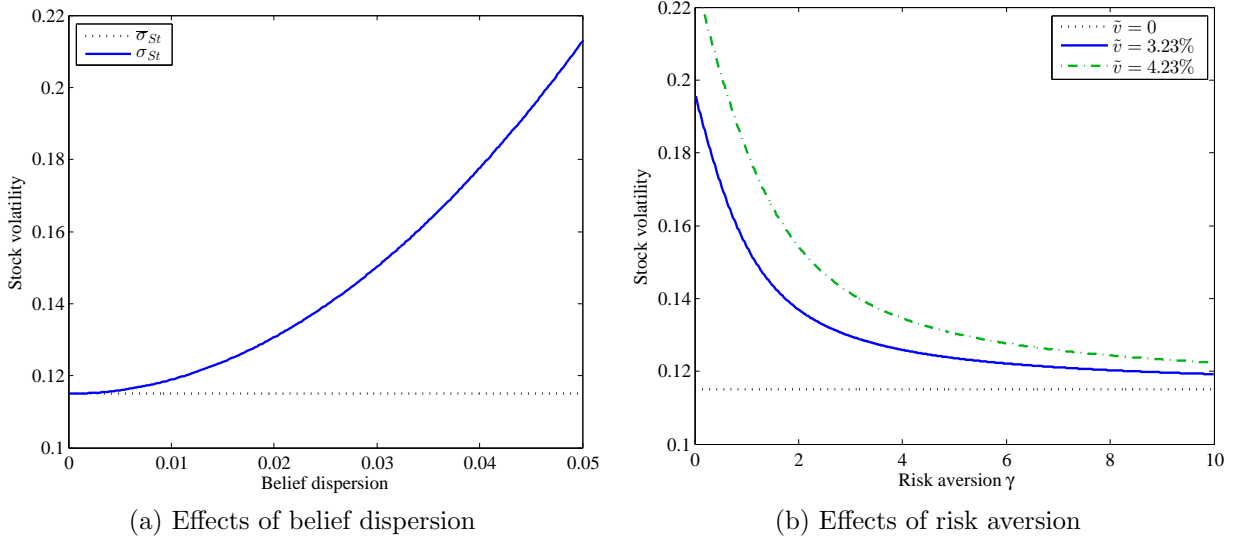


Figure 7: **Effects of belief dispersion and risk aversion on stock volatility.** These figures plot the equilibrium stock volatility σ_{St} against current belief dispersion v_t in panel (a) and against investors' relative risk aversion coefficient γ for different levels of initial wealth-share weighted belief dispersion \tilde{v} in panel (b). The dotted lines correspond to the equilibrium stock volatility in the benchmark economy with no belief dispersion. For the choice of dispersion parameter values in panel (b) see footnote 14. The baseline parameter values are as in Figure 1.

more (Section 3), and hence so does the stock price. Figure 7a illustrates this feature by plotting the equilibrium stock volatility against belief dispersion. This result is also consistent with the empirical evidence that the stock volatility increases with investors' belief dispersion (Ajinkya and Gift (1985), Anderson, Ghysels, and Juergens (2005) and Banerjee (2011)).¹⁸ In Section 5.3, we show that this simple, intuitive result does not necessarily hold in an otherwise identical two (or finitely many) investor economy.

Property (iii) reveals that the stock volatility decreases in investors' risk aversion γ . This is intuitive in light of the discussion in Section 3 that a higher risk aversion reduces the fluctuations in the average bias in beliefs, and therefore in the stock price. Consequently, the (excess) stock volatility is lower in a more risk averse economy as illustrated in Figure 7b.

5.2 Equilibrium Trading Volume

In order to explore the aggregate trading activity in our economy, we first express each θ -type investor's portfolio holdings in terms of the number of shares held in the stock, $\psi(\theta) =$

¹⁸In a recent study Carlin, Longstaff, and Matoba (2014) document that a higher belief dispersion leads to a higher volatility for mortgage-backed securities.

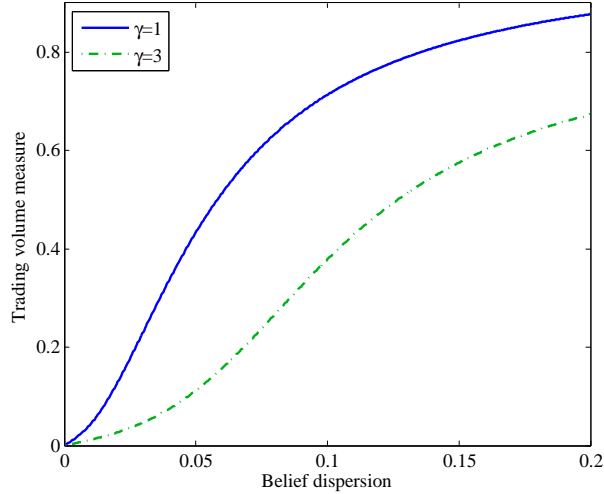


Figure 8: **Effects of belief dispersion on trading volume measure.** This figure plots the equilibrium trading volume measure V_t against current belief dispersion v_t for different relative risk aversion coefficients γ . The baseline parameter values are as in Figure 1.

$\phi(\theta) W(\theta) / S$, with dynamics

$$d\psi_t(\theta) = \mu_{\psi t}(\theta) dt + \sigma_{\psi t}(\theta) d\omega_t,$$

where $\mu_{\psi}(\theta)$ and $\sigma_{\psi}(\theta)$ are the drift and volatility of θ -type investor's portfolio process $\psi(\theta)$, respectively. Following recent works in continuous-time settings (e.g., Xiong and Yan (2010), Longstaff and Wang (2012)), we consider a trading volume measure V that sums over the absolute value of investors' portfolio volatilities,

$$V_t \equiv \frac{1}{2} \int_{\Theta} |\sigma_{\psi t}(\theta)| d\theta, \quad (26)$$

where the adjustment of $1/2$ is to prevent double summation of the shares traded across investors.¹⁹ Proposition 5 reports the equilibrium trading volume measure in closed form and its properties.

Proposition 5. *In the economy with belief dispersion, the equilibrium trading volume measure is given by*

$$V_t = \frac{\sigma}{X_t^2 v_T^2} \left(\frac{1}{2} X_t + \frac{1}{2} \sqrt{X_t^2 + 4} \right) \phi \left(\frac{1}{2} X_t - \frac{1}{2} \sqrt{X_t^2 + 4} \right) - \frac{\sigma}{X_t^2 v_T^2} \left(\frac{1}{2} X_t - \frac{1}{2} \sqrt{X_t^2 + 4} \right) \phi \left(\frac{1}{2} X_t + \frac{1}{2} \sqrt{X_t^2 + 4} \right), \quad (27)$$

where the dispersion in beliefs v_t is as in Proposition 1, and $\phi(\cdot)$ is the probability density

¹⁹As is well recognized, employing the standard definition of trading volume in a continuous-time setting is problematic since the local variation of the driving uncertainty, Brownian motion ω , hence an investor's portfolio, is unbounded.

function of the standard normal random variable, and X is a (positive) deterministic process given by

$$X_t^2 = \gamma^2 \frac{\sigma^4}{v_T^4} \left[\frac{1}{\gamma} v_t^2 + \left(1 - \frac{1}{\gamma} \right) v_T^2 \right].$$

Consequently, in the presence of belief dispersion,

- i) The trading volume measure is increasing in belief dispersion v_t .*
- ii) The trading volume measure is positively related to the stock volatility σ_{St} .*
- iii) The trading volume measure is decreasing in investors' risk aversion γ .*

With belief dispersion, investors take diverse stock positions following cash-flow news, which in turn generate non-trivial trading activity in the economy. Naturally, the aggregate trading activity, which is captured by our trading volume measure V , increases as the belief dispersion increases (Property (i)). This is because, in a higher dispersion economy, investors with relatively different beliefs have more weight and higher trading demand, which increase the trading volume in the economy. Figure 8 illustrates this feature by depicting the equilibrium trading volume measure against belief dispersion. This result is well-supported by empirical evidence (Ajinkya, Atiase, and Gift (1991), Bessembinder, Chan, and Seguin (1996), Bamber, Barron, and Stober (1997) and Goetzmann and Massa (2005)). We again note that this simple, intuitive result does not necessarily hold in an otherwise identical two (or finitely many) investor economies as discussed in Section 5.3.

Figure 9a plots the equilibrium trading volume measure against stock volatility and illustrates the positive relation between these two economic quantities (Property (ii)). This positive relation is intuitive since a higher dispersion leads to both a higher stock volatility (Section 5.1) and a higher trading volume measure. This result is also supported by empirical evidence; for example, Gallant, Rossi, and Tauchen (1992) document a positive correlation between the conditional stock volatility and trading volume. Turning to the effect of investors' risk aversion γ on the trading volume measure, we recall that investors in a more risk averse economy invest less in the stock (Section 3). Therefore, the higher the risk aversion, the lower the trading volume (Property (iii)). This feature is illustrated in Figure 9b, which plots the trading volume measure against investors' risk aversion.

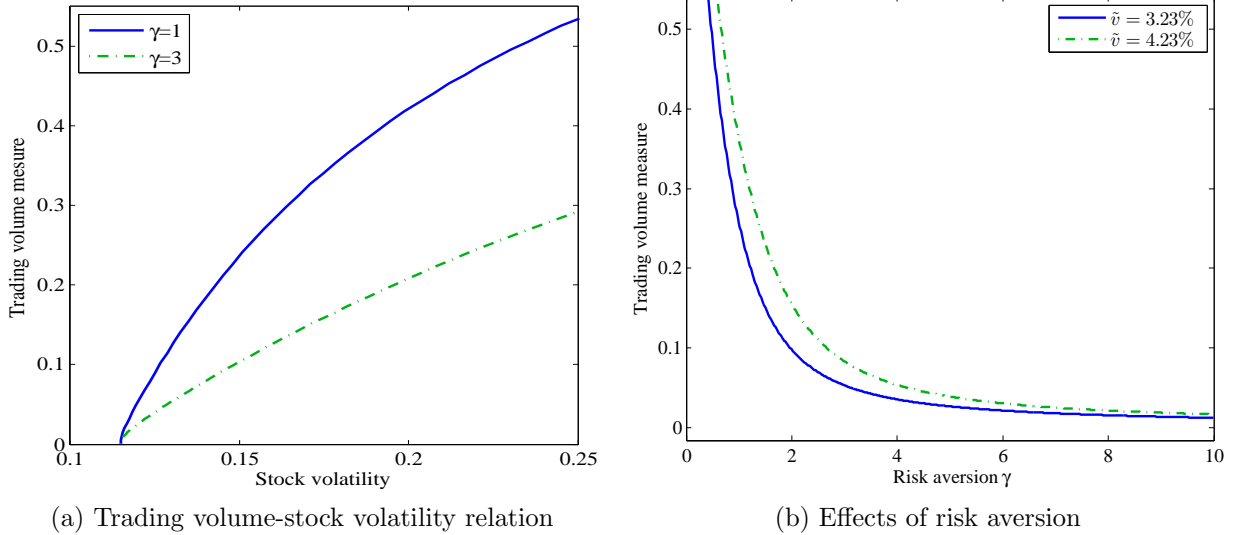
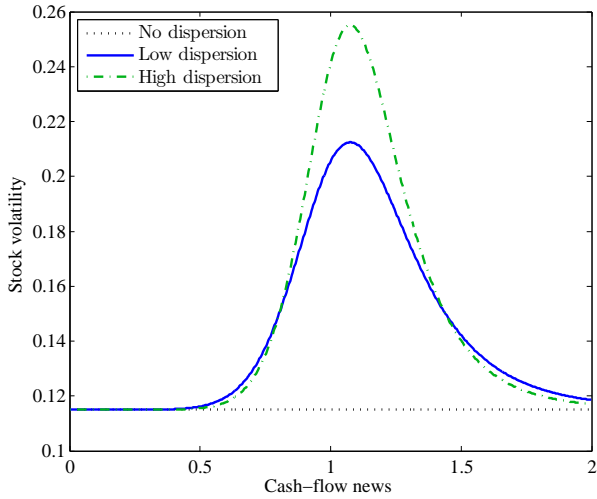


Figure 9: **Trading volume-stock volatility relation and effects of risk aversion.** These figures plot the equilibrium trading volume measure V_t against stock volatility σ_{S_t} in panel (a) and against investors' relative risk aversion coefficient γ for different levels of initial wealth-share weighted belief dispersion \tilde{v} in panel (b). For the choice of dispersion parameter values in panel (b) see footnote 14. The baseline parameter values are as in Figure 1.

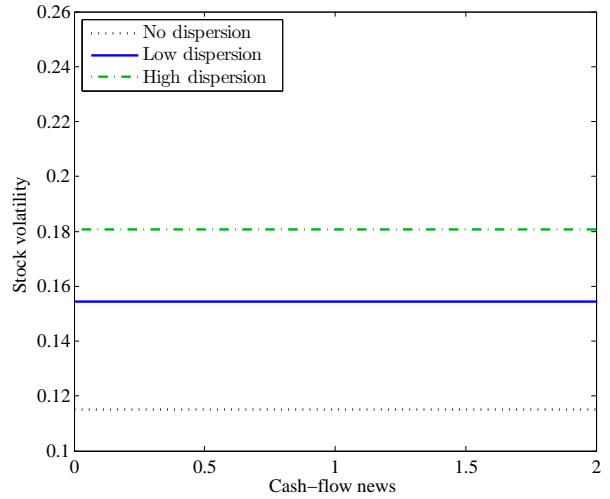
5.3 Comparisons with Two-Investor Economy

We here investigate whether an otherwise identical economy to ours, but with two (or finitely many) investors is able to generate our unambiguous stock volatility and trading volume results. Towards this, Figure 10 plots the stock volatility and the trading volume measure against cash-flow news for a two-investor economy with an optimistic and a pessimistic investor, and for our model.²⁰ In particular, we see from Figure 10a that in a two-investor economy a higher belief dispersion increases the stock volatility only for moderate news states in which the investor heterogeneity is prevalent. However, for relatively extreme states the stock volatility actually decreases with higher belief dispersion, as investor heterogeneity vanishes and the single investor benchmark economy prevails. Similarly, Figure 10c reveals a nonuniform behavior of the trading volume measure under the two-investor economy, where the trading volume may actually decrease with higher belief dispersion, in contrast to our increased-volume result as in Figure 10d. As these Figures illustrate, by keeping the investor heterogeneity (and also the trading volume measure) the same across states, our model is able to generate intuitive and simple results, which are not possible under the more familiar two-investor setting.

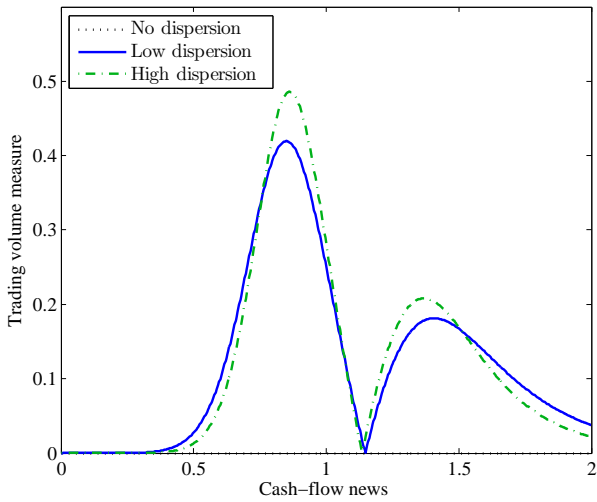
²⁰The expressions for the equilibrium stock volatility and trading volume measure in such a two-investor economy are provided in Appendix A.2.



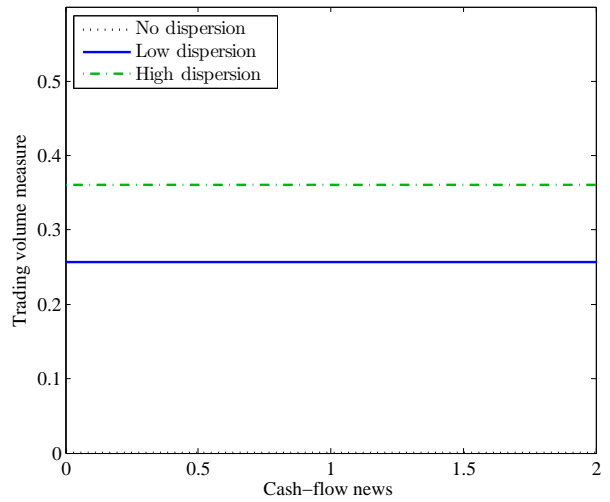
(a) Two-investor economy



(b) Our model



(c) Two-investor economy



(d) Our model

Figure 10: **Stock volatility and trading volume measure in a two-investor economy.** Top panels plot the equilibrium stock volatility against cash-flow news for different levels of belief dispersion in an otherwise identical two-investor economy with an optimistic and a pessimistic investor in panel (a) and in our model in panel (b). Bottom panels plot the equilibrium trading volume measure against cash-flow news for different levels of belief dispersion in an otherwise identical two-investor economy with an optimistic and a pessimistic investor in panel (c) and in our model in panel (d). The other applicable parameter values are as in Figure 1.

6 Economy with Bayesian Learning

So far, we have studied an economy where investors have dogmatic beliefs, which not only simplified the analysis, but also demonstrated that our results are not driven by investors' learning. In this Section, we consider a setting with parameter uncertainty and more rational behavior for investors who optimally update their beliefs in a Bayesian fashion over time as more data becomes available. This setting is also tractable. We again obtain fully-closed form solutions for all quantities of interest, and show that all our results remain valid in this richer, incomplete information economy. This specification also enables us disentangle the effects of belief dispersion and parameter uncertainty on excess volatility, and establish the result that the higher the dispersion, the lower the effect of parameter uncertainty on excess volatility.

To incorporate Bayesian learning, we make the following modification to investors' beliefs. The investors are again indexed by their type θ , but instead of believing the mean return of the expected payoff is $\mu + \theta$ at all times $0 \leq t \leq T$, now the θ -type investor at time 0 believes that the mean return of the expected payoff is normally distributed with mean $\mu + \theta$ and variance s^2 , $\mathcal{N}(\mu + \theta, s^2)$. This allows us to interpret a θ -type investor as an investor with an initial bias of θ in her beliefs. The identical prior variance s^2 for all investors ensures that our results are not driven by heterogeneity in confidence of their estimates. The normal prior and the Bayesian updating rule imply that the time- t posterior distribution of μ , conditional on the information set $\mathcal{F}_t = \{D_s : 0 \leq s \leq t\}$, is also normally distributed, $\mathcal{N}(\mu + \hat{\theta}_t, s_t^2)$, where the time- t bias of θ -type investor $\hat{\theta}_t$ (difference between her mean estimate and the true μ) and the type-independent mean squared error s_t^2 , which represents the level of *parameter uncertainty*, are reported in Proposition 6. Therefore, under the θ -type investor's beliefs, the posterior cash-flow news process has dynamics

$$dD_t = D_t[(\mu + \hat{\theta}_t)dt + \sigma d\omega_t(\theta)],$$

where $\omega(\theta)$ is her perceived Brownian motion with respect to her own probability measure \mathbb{P}^θ , and is given by $d\omega_t(\theta) = d\omega_t - (\hat{\theta}_t/\sigma)dt$. We note that this specification conveniently nests the earlier dogmatic beliefs economy when $s^2 = 0$.

6.1 Equilibrium in the Presence of Bayesian Learning

We construct the average bias and dispersion in beliefs following Definition 1 in Section 3: the time- t *average bias in beliefs*, m_t , is the implied bias of the corresponding representative

investor, expressed as the weighted average of the individual investors' biases

$$m_t = \int_{\Theta} \widehat{\theta}_t h_t(\theta) d\theta, \quad (28)$$

with the weights $h_t(\theta) > 0$ are such that $\int_{\Theta} h_t(\theta) d\theta = 1$, while the *dispersion in beliefs*, v_t , is the standard deviation of investors' biases

$$v_t^2 \equiv \int_{\Theta} (\widehat{\theta}_t - m_t)^2 h_t(\theta) d\theta. \quad (29)$$

Proposition 6 reports the average bias and dispersion in beliefs along with the corresponding unique weights in this economy with belief dispersion and parameter uncertainty in closed form.

Proposition 6. *The time- t average bias m_t and dispersion v_t in beliefs are given by*

$$m_t = m + \left(\ln D_t - \left(m + \mu - \frac{1}{2}\sigma^2 \right) t \right) \left(\frac{1}{\gamma}v^2 + s^2 \right) \frac{1}{\sigma^2} \frac{v_t^2}{v^2} \frac{s_t^2}{s^2}, \quad (30)$$

$$v_t^2 = \frac{v^2 \sigma^2}{\sigma^2 + \left(\frac{1}{\gamma}v^2 + s^2 \right) t} \frac{s_t^2}{s^2}, \quad (31)$$

where the investors' time- t parameter uncertainty s_t is given by

$$s_t^2 = \frac{s^2 \sigma^2}{\sigma^2 + s^2 t}, \quad (32)$$

and the initial values m and v are related to the initial wealth-share weighted average bias \tilde{m} and dispersion \tilde{v} in beliefs as

$$m = \tilde{m} + \left(1 - \frac{1}{\gamma} \right) v^2 T, \quad (33)$$

$$v^2 = \left(\frac{\gamma}{2} \tilde{v}^2 - \frac{\gamma^2}{2T} (\sigma^2 + s^2 T) \right) + \sqrt{\left(\frac{\gamma}{2} \tilde{v}^2 - \frac{\gamma^2}{2T} (\sigma^2 + s^2 T) \right)^2 + \frac{\gamma^2}{T} \tilde{v}^2 (\sigma^2 + s^2 T)}. \quad (34)$$

The weights $h_t(\theta)$ are uniquely identified to be given by

$$h_t(\theta) = \frac{1}{\sqrt{2\pi v_t^2}} e^{-\frac{1}{2} \frac{(\widehat{\theta}_t - m_t)^2}{v_t^2} \frac{s_t^2}{s^2}}, \quad (35)$$

where m_t , v_t and s_t are as in (30)–(32) and the time- t bias of θ -type investor $\widehat{\theta}_t$ is given by

$$\widehat{\theta}_t = \frac{s_t^2}{s^2} \theta + \frac{s_t^2}{\sigma} \omega_t. \quad (36)$$

Consequently, in the presence of belief dispersion and parameter uncertainty, for economies with the same initial average bias m and dispersion v ,

i) The average bias in beliefs is increasing in parameter uncertainty s_t when

$D_t > \exp\left(\left(m + \mu - \frac{1}{2}\sigma^2\right) t\right)$, and is decreasing otherwise.

ii) The dispersion in beliefs is decreasing in parameter uncertainty s_t .

The average bias and dispersion in beliefs (30)–(31) are generalizations of the earlier dogmatic beliefs case and are now additionally driven by parameter uncertainty s_t .²¹ We see that, similarly to the effect of dispersion, a higher parameter uncertainty leads to a relatively more optimistic (pessimistic) economy following good (bad) news (Property (i)), and this then amplifies the volatility of the average bias relative to the dogmatic beliefs case. However, the underlying mechanisms of belief dispersion and parameter uncertainty are notably different. In the case of dispersion, the economy becomes more optimistic following good news, because the optimistic investors, whose beliefs are supported, become wealthier and this increases their impact on the average bias in beliefs. In the case of parameter uncertainty, the economy becomes more optimistic following good news, because all Bayesian investors increase their estimates of the mean return of the expected payoff. Proposition 6 also reveals that a higher parameter uncertainty leads to a lower dispersion in beliefs (Property (ii)). This is intuitive because investors' estimates of the mean return is a weighted average of their prior and the data (cash-flow news). The higher the parameter uncertainty, the more weight the investors place on the data, which in turn reduces the differences in their optimal estimates and the belief dispersion.

We remark that in this Section, we consider the effects of parameter uncertainty s_t only for economies with the same initial average bias m and dispersion v , as highlighted in Proposition 6. This way, economies only differ in their initial level of parameter uncertainty s and our results are not driven by the indirect effects through the initial average bias and dispersion.²²

²¹Note that when $s^2 = 0$, the ratio $s_t^2/s^2 = 1$ for all t , and the expressions in Proposition 6 collapse down to the dogmatic beliefs economy expressions in Proposition 1.

²²This is established by letting the initial indirect effect of parameter uncertainty fall on the initial wealth-share weighted average bias \tilde{m} and dispersion in beliefs \tilde{v} , using the monotonic relations between m and \tilde{m} , and v and \tilde{v}

$$\tilde{m} = m - \left(1 - \frac{1}{\gamma}\right) v^2 T, \quad \tilde{v}^2 = v^2 \frac{(\sigma^2 + s^2 T) + \frac{1}{\gamma^2} v^2 T}{(\sigma^2 + s^2 T) + \frac{1}{\gamma} v^2 T}.$$

For the effects of belief dispersion v_t and investors risk aversion γ , comparing economies with the same initial average bias m and dispersion v is not necessary, as in our earlier analysis with dogmatic beliefs, since letting these effects fall on either the initial average bias m and dispersion v , or on their wealth-share weighted counterparts, yields similar results.

6.2 Equilibrium Stock Price and Mean Return

Proposition 7. *In the economy with belief dispersion and parameter uncertainty, the equilibrium stock price and mean return are given by*

$$S_t = \bar{S}_t e^{m_t(T-t) - \frac{1}{2}(2\gamma-1)\left(\frac{1}{\gamma}v^2+s^2\right)\frac{v_t^2 s_T^2}{v^2 s_t^2}(T-t)^2}, \quad (37)$$

$$\mu_{St} = \bar{\mu}_{St} \frac{v_t^4 s_T^4}{v_T^4 s_t^4} - m_t \frac{v_t^2 s_T^2}{v_T^2 s_t^2}, \quad (38)$$

where the average bias m_t , dispersion v_t in beliefs and parameter uncertainty s_t are as in Proposition 6 and the equilibrium stock price \bar{S}_t and mean return $\bar{\mu}_{St}$ in the benchmark economy with no belief dispersion are as in Propositions 2–3, respectively.

Consequently, in the presence of belief dispersion and parameter uncertainty, in addition to the properties in Propositions 2–3, for economies with the same initial average bias m and dispersion v ,

i) The stock price is increasing in parameter uncertainty s_t when

$$m_t > m + (1/2)(2\gamma - 1)\left((v^2/\gamma) + s^2\right)\left(v_t^2 s^2/v^2 s_t^2\right)(T - t), \text{ and is decreasing otherwise.}$$

ii) The mean return is decreasing in parameter uncertainty s_t when

$$m_t > [m + 2\gamma\sigma^2\left((v_t^2 s_T^2/v_T^2 s_t^2) - 1\right)]\left[2 - (v_T^2 s_t^2/v_t^2 s_T^2)\right]^{-1}, \text{ and is increasing otherwise.}$$

Proposition 7 confirms that our earlier implications for the stock price and mean return remain valid with Bayesian learning. However, the stock price and mean return are now also driven by the parameter uncertainty s_t . Similarly to the effects of belief dispersion, the stock price increases while its mean return decreases in parameter uncertainty when the economy is sufficiently optimistic, and vice versa when pessimistic (Properties (i)–(ii)). More notably, the additional effect due to the investors’ learning now makes the stock price more convex as compared with the dogmatic beliefs case. These effects are illustrated in Figure 11 which depicts the equilibrium stock price and mean return against the cash-flow news. The empirical evidence on the effects of parameter uncertainty on asset prices are limited. Massa and Simonov (2005) show that both the investors’ learning and the dispersion in beliefs are conditionally priced, whereas Ozoguz (2009) provides evidence that parameter uncertainty has a negative impact on the stock price in bad times, which is consistent with our finding.

²³The initial belief dispersion v is set as $v = 3.23\%$ as in Figure 1. However, due to Bayesian learning, the time- t dispersion now becomes $v_t = 3.14\%$ instead of $v_t = 3.20\%$. The parameter value for s is obtained by using the belief dispersion data in Graham and Harvey (2013) who report average dispersion values of 2.92% and 2.36% for the years 2001 and 2006, respectively. Substituting these (along with the earlier parameter values) into the belief dispersion (31) and backing out s yields the initial parameter uncertainty $s = 1.51\%$, which after substituting into (32) yields the time- t value of $s_t = 1.49\%$.

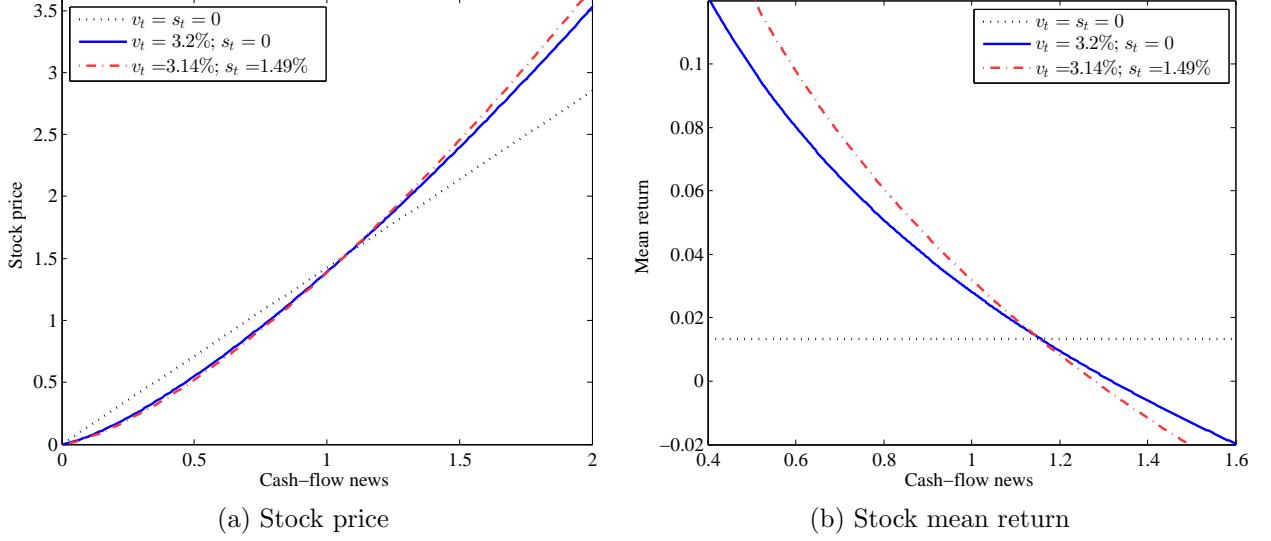


Figure 11: **Effects of parameter uncertainty on stock price and mean return.** These figures plot the equilibrium stock price S_t in panel (a) and mean return μ_{St} in panel (b) against cash-flow news. The dotted black (solid blue) lines correspond to the equilibrium stock price and mean returns in the benchmark economy with no belief dispersion (economy with belief dispersion and no parameter uncertainty). The dashdot red lines correspond to the equilibrium stock price and mean returns in the economy with belief dispersion and parameter uncertainty. The baseline parameter values are as in Figure 1.²³

6.3 Equilibrium Stock Volatility and Trading Volume

Proposition 8. *In the economy with belief dispersion and parameter uncertainty, the equilibrium stock volatility and the trading volume measure are given by*

$$\sigma_{St} = \bar{\sigma}_{St} + \frac{1}{\sigma} \left(\frac{1}{\gamma} v^2 + s^2 \right) \frac{v_t^2 s_T^2}{v^2 s_t^2} (T - t), \quad (39)$$

$$V_t = \frac{\sigma}{X_t^2} \frac{v_t^2 s_T^2}{v_T^2 s_t^2} \left(\frac{1}{2} X_t + \frac{1}{2} \sqrt{X_t^2 + 4} \right) \phi \left(\frac{1}{2} X_t - \frac{1}{2} \sqrt{X_t^2 + 4} \right) - \frac{\sigma}{X_t^2} \frac{v_t^2 s_T^2}{v_T^2 s_t^2} \left(\frac{1}{2} X_t - \frac{1}{2} \sqrt{X_t^2 + 4} \right) \phi \left(\frac{1}{2} X_t + \frac{1}{2} \sqrt{X_t^2 + 4} \right), \quad (40)$$

where the dispersion in beliefs v_t and the parameter uncertainty s_t are as in Proposition 6, and the equilibrium stock volatility $\bar{\sigma}_{St}$ in the benchmark economy with no belief dispersion is as in Proposition 4, and $\phi(\cdot)$ is the probability density function of the standard normal random variable, and X is a (positive) deterministic process given by

$$X_t^2 = \frac{\gamma^2 \sigma^4}{v_T^4} \left[\frac{1}{\gamma} v_t^2 \frac{s_T^4}{s_t^4} + \left(1 - \frac{1}{\gamma} \right) v_T^2 \right]. \quad (41)$$

Consequently, in the presence of belief dispersion and parameter uncertainty, in addition to the properties in Propositions 4–5, for economies with the same initial average bias m and

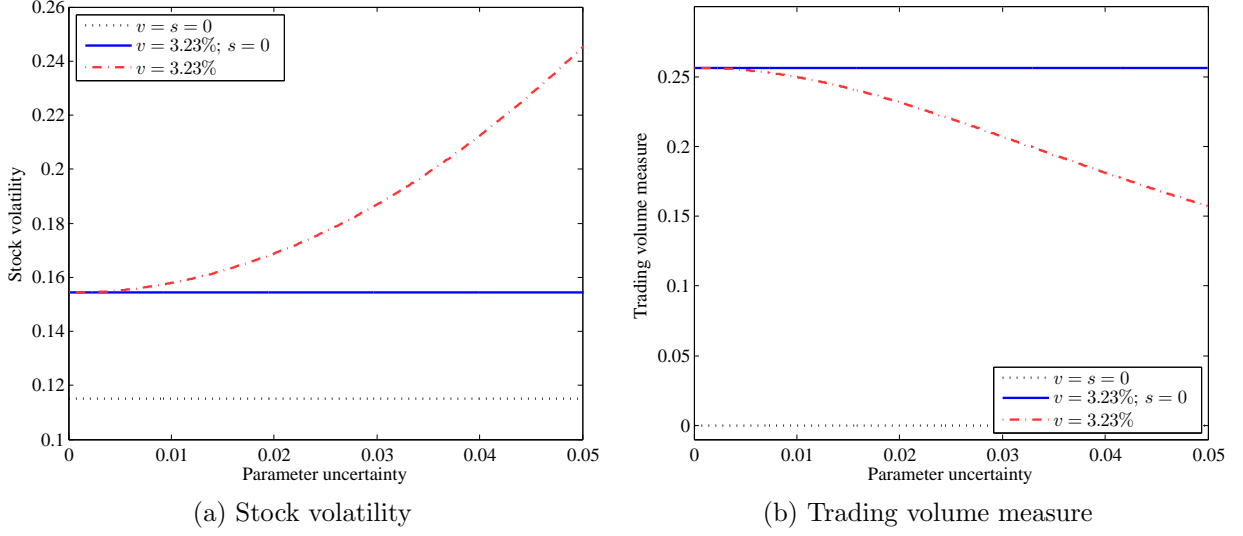


Figure 12: **Effects of parameter uncertainty on stock volatility and trading volume measure.** These figures plot the equilibrium stock volatility σ_{S_t} in panel (a) and trading volume measure V_t in panel (b) against parameter uncertainty s_t . The dotted black (solid blue) lines corresponds to the equilibrium stock volatility and trading volume measure in the benchmark economy with no belief dispersion (economy with belief dispersion and no parameter uncertainty). The dashdot red lines correspond to the equilibrium stock volatility and trading volume measure in the economy with belief dispersion and parameter uncertainty. The baseline parameter values are as in Figure 1.

dispersion v ,

- i) The stock volatility is increasing in parameter uncertainty s_t .
- ii) The effect of parameter uncertainty s_t on stock volatility is decreasing in belief dispersion v_t .
- iii) The trading volume measure is decreasing in parameter uncertainty s_t when investors' risk aversion $\gamma \geq 1$, and it may increase or decrease otherwise.

To better highlight our results in Proposition 8, Figure 12 plots the equilibrium stock volatility and trading volume measure against the parameter uncertainty s_t . We see that excess volatility is generated not only by belief dispersion as in our earlier analysis, but also by parameter uncertainty, with the stock volatility being increasing in parameter uncertainty (Property (i)). This is because a higher parameter uncertainty makes the average bias more volatile (Section 6.1), which leads to a higher stock price following relatively good news, and to a lower stock price otherwise, thereby increasing the stock volatility.

The effect of parameter uncertainty on stock volatility, however, decreases as belief dispersion increases (Property (ii)). This is because fluctuations in the average bias due to parameter uncertainty is lower in a higher belief dispersion economy. Importantly, our stock volatility

expression in (39) allows us to disentangle the effects of belief dispersion from those of parameter uncertainty.²⁴ In the literature, both channels are proposed to explain the excess stock volatility, among other channels. Bulkley and Tonks (1989), Barsky and De Long (1993), Timmermann (1993, 1996) show that the excess volatility in the stock market is broadly consistent with models in which investors are uncertain about parameters and estimate them from observable processes. We discussed the belief dispersion channel in Section 5. By disentangling these effects, our result may help future works to measure the relative contributions of parameter uncertainty and belief dispersion in excess volatility better.

Finally, we find that the trading volume measure decreases in parameter uncertainty s_t for plausible levels of risk aversion $\gamma \geq 1$ (Property (iii)). This is illustrated in Figure 12b and happens because a higher parameter uncertainty leads to a lower belief dispersion (Section 6.1), and if there are no opposing effects, as in the case of $\gamma \geq 1$, the lower dispersion unambiguously decreases the trading volume as in our earlier analysis. For $\gamma < 1$, there is an opposing effect coming from the wealth-share weighted dispersion (ratio v_t^2/\tilde{v}_t may increase in parameter uncertainty) in addition to the effect of lower dispersion. When this effect is significant, the trading volume measure may increase in parameter uncertainty.

7 Conclusion

In this paper, we have developed a dynamic model of belief dispersion which simultaneously explains the empirical regularities in a stock price, its mean return, volatility, and trading volume. In our analysis, we have determined two sufficient measures, the average bias and dispersion in beliefs, to summarize the wide range of investors' beliefs and have demonstrated that the equilibrium quantities are driven by these two key variables. Our model is tractable and delivers exact closed-form expressions for all quantities, implying the following. We have found that the stock price increases in cash-flow news in a convex manner. We have also shown that the stock price increases and its mean return decreases in belief dispersion when the economy is optimistic, and vice versa when pessimistic. We have found that the presence of belief dispersion produces excess stock volatility, trading volume, and a positive relation between these two quantities. Furthermore, we have disentangled the effects of belief dispersion from the effects of learning, which also contributes to the excess stock volatility, and found that the

²⁴The parameter uncertainty channel is shut down by setting $s^2 = 0$ in (39), which implies $s_t^2/s^2 = 1$, and yields $\sigma_{St} = \bar{\sigma}_{St} + (v_t^2/\sigma\gamma)(T-t)$. Similarly, the belief dispersion channel is shut down by setting $v^2 = 0$ in (39), which implies $v_t^2/v^2 = s_t^4/s^4$, and yields $\sigma_{St} = \bar{\sigma}_{St} + (s_t^2/\sigma)(T-t)$.

effects of the latter is reduced in higher dispersion economies. Finally, we have demonstrated how the more familiar, otherwise identical, finitely-many-investor heterogeneous beliefs models are not able to generate most of our main results.

Our model is set in a finite horizon framework to highlight the effects of the presence of belief dispersion on asset prices and dynamics. This setting allows us to abstract away from issues of long-run survival of investors. In our model, the investor heterogeneity is captured by the belief dispersion measure, which monotonically decreases over time. However, it is important to note that it takes longer and longer for dispersion to halve as it gets smaller. This indicates that even though the investor heterogeneity disappears in the long-run, it may take thousands of years to do so.

In this paper, we have carried out our analysis within a single-stock framework. It turns out this simple setting is sufficient to demonstrate our major insights and results. In ongoing work we extend this framework to a multi-stock setting and our preliminary analysis reveal that our main results, including those on the stock price, its mean return and volatility, hold in this economy as well. Due to its tractability, our model has also other potential applications. For instance, in our model investors disagree with each other only in one dimension. By incorporating additional dimensions of heterogeneity and taking advantage of the well-known properties of multivariate Gaussian distribution, one can still maintain tractability and obtain potentially valuable insights. Other interesting but not straightforward extensions of our model involves incorporating portfolio constraints (short-selling, borrowing, etc.) to analyze market frictions in the presence of belief dispersion. We leave these extensions for future research.

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A Appendix

A.1 Proofs

Proof of Proposition 1. We proceed by first solving each θ -type investor's problem, and determining the equilibrium horizon prices and investors' Lagrange multipliers, which are used in the representative investor construction. We then construct the representative investor to infer her implied bias and to hence define the average bias in beliefs, m_t , in the economy. We next identify the unique weights $h_t(\theta)$ so that the $h_t(\theta)$ -weighted average of individual biases is indeed the average bias m_t in the economy. Finally, we determine the belief dispersion, v_t , from the average bias and weights.

We begin by first solving each θ -type investor's optimization problem. Dynamic market completeness implies a unique state price density process ξ under \mathbb{P} , such that the time- t value of a payoff X_T at time T is given by $\mathbb{E}_t[\xi_T X_T] / \xi_t$. Accordingly, the dynamic budget constraint (9) of each θ -type investor under \mathbb{P} can be restated as

$$\mathbb{E}_t[\xi_T W_T(\theta)] = \xi_t W_t(\theta). \quad (\text{A.1})$$

We also rewrite each θ -type investor's expected utility function (8) under the objective measure \mathbb{P} as

$$\mathbb{E} \left[\eta_T(\theta) \frac{W_T(\theta)^{1-\gamma}}{1-\gamma} \right], \quad (\text{A.2})$$

where $\eta_T(\theta)$ is the Radon-Nikodým derivative of the subjective measure \mathbb{P}^θ with respect to the true measure \mathbb{P} given by

$$\eta_T(\theta) = \frac{d\mathbb{P}^\theta}{d\mathbb{P}} = e^{\frac{\theta}{\sigma}\omega T - \frac{1}{2}\frac{\theta^2}{\sigma^2}T}.$$

Maximizing each (distinct) θ -type investor's expected objective function (A.2) subject to (A.1) evaluated at time $t = 0$ leads to the optimal horizon wealth of each θ -type as

$$W_T(\theta) = \left(\frac{\eta_T(\theta)}{y(\theta) \xi_T} \right)^{\frac{1}{\gamma}}, \quad (\text{A.3})$$

where the Lagrange multiplier $y(\theta)$ solves (A.1) evaluated at time $t = 0$

$$y(\theta)^{-\frac{1}{\gamma}} = \mathbb{E} \left[\eta_T(\theta)^{\frac{1}{\gamma}} \xi_T^{1-\frac{1}{\gamma}} \right]^{-1} \frac{\xi_0 S_0}{\sqrt{2\pi\tilde{v}^2}} e^{-\frac{1}{2}\frac{(\theta-\tilde{m})^2}{\tilde{v}^2}}. \quad (\text{A.4})$$

We next determine the time- T equilibrium state price density ξ_T . Substituting (A.3) into the market clearing condition $\int_{\Theta} W_T(\theta) d\theta = D_T$ yields $\xi_T^{-\frac{1}{\gamma}} \int_{\Theta} y(\theta)^{-\frac{1}{\gamma}} \eta_T(\theta)^{\frac{1}{\gamma}} d\theta = D_T$, which

after rearranging we obtain the time- T equilibrium state price density

$$\xi_T = D_T^{-\gamma} M_T^\gamma, \quad (\text{A.5})$$

where the auxiliary process M and the likelihood ratio process $\eta(\theta)$ are given by

$$M_t \equiv \int_{\Theta} \left(\frac{\eta_t(\theta)}{y(\theta)} \right)^{\frac{1}{\gamma}} d\theta, \quad (\text{A.6})$$

$$\eta_t(\theta) = \mathbb{E}_t[\eta_T(\theta)] = e^{\frac{\theta}{\sigma}\omega_t - \frac{1}{2}\frac{\theta^2}{\sigma^2}t}. \quad (\text{A.7})$$

As we show below, $y(\theta)^{-\frac{1}{\gamma}}$ is (scaled) Gaussian over the type space Θ for some mean α_o , variance β_o^2 and a constant K :

$$y(\theta)^{-\frac{1}{\gamma}} = K \frac{1}{\sqrt{2\pi\beta_o^2}} e^{-\frac{1}{2}\frac{(\theta-\alpha_o)^2}{\beta_o^2}}. \quad (\text{A.8})$$

Substituting (A.7)–(A.8) into the definition of M (A.6) yields

$$M_t = K \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\beta_o^2}} e^{-\frac{1}{2}\frac{(\theta-\alpha_o)^2}{\beta_o^2}} e^{\frac{1}{\gamma}\frac{\theta}{\sigma}\omega_t - \frac{1}{2}\frac{1}{\gamma}\frac{\theta^2}{\sigma^2}t} d\theta = K \frac{\beta_t}{\beta_o} e^{-\frac{1}{2}\frac{\alpha_o^2}{\beta_o^2} + \frac{1}{2}\frac{\alpha_t^2}{\beta_t^2}}, \quad (\text{A.9})$$

where the last equality follows by completing the square and integrating, and the processes α and β are

$$\alpha_t = \sigma \frac{\sigma\alpha_o + \frac{1}{\gamma}\beta_o^2\omega_t}{\sigma^2 + \frac{1}{\gamma}\beta_o^2t}, \quad (\text{A.10})$$

$$\beta_t^2 = \frac{\beta_o^2\sigma^2}{\sigma^2 + \frac{1}{\gamma}\beta_o^2t}, \quad (\text{A.11})$$

with their initial values given by $\alpha_0 = \alpha_o$ and $\beta_0 = \beta_o$, respectively.

We now verify $y(\theta)^{-\frac{1}{\gamma}}$ is as in (A.8). Substituting (A.5) into (A.4) gives

$$y(\theta)^{-\frac{1}{\gamma}} = \left(\mathbb{E} \left[\eta_T(\theta)^{\frac{1}{\gamma}} D_T^{1-\gamma} M_T^{\gamma-1} \right] \right)^{-1} \frac{\xi_0 S_0}{\sqrt{2\pi\tilde{v}^2}} e^{-\frac{1}{2}\frac{(\theta-\tilde{m})^2}{\tilde{v}^2}}, \quad (\text{A.12})$$

where M_T is equal to (A.9) evaluated at time T . From Lemma 2 in the technical Appendix A.3, evaluated at $t = 0$, the expectation in (A.12) is equal to

$$\mathbb{E} \left[\eta_T(\theta)^{\frac{1}{\gamma}} D_T^{1-\gamma} M_T^{\gamma-1} \right] = \xi_0 S_0 K^{-1} \left(\frac{1}{\sqrt{2\pi\beta_o^2}} e^{-\frac{1}{2}\frac{(\theta-\alpha_o)^2}{\beta_o^2}} \right)^{-1} \frac{1}{\sqrt{2\pi\beta_o^2 A^2}} e^{-\frac{1}{2}\frac{(\theta - [\alpha_o - (1-\frac{1}{\gamma})\beta_o^2 T])^2}{\beta_o^2 A^2}}, \quad (\text{A.13})$$

where the constant A^2 is given by $A^2 = \left(\sigma^2 + \frac{1}{\gamma^2}\beta_o^2 T \right) / \left(\sigma^2 + \frac{1}{\gamma}\beta_o^2 T \right)$. Substituting (A.13) into

(A.12) and manipulating terms yields (A.8) with

$$\alpha_o = \tilde{m} + \left(1 - \frac{1}{\gamma}\right) \beta_o^2 T, \quad (\text{A.14})$$

$$\beta_o^2 = \left(\frac{\gamma}{2} \tilde{v}^2 - \frac{\gamma^2}{2T} \sigma^2\right) + \sqrt{\left(\frac{\gamma}{2} \tilde{v}^2 - \frac{\gamma^2}{2T} \sigma^2\right)^2 + \frac{\gamma^2}{T} \tilde{v}^2 \sigma^2}. \quad (\text{A.15})$$

We note that for logarithmic preferences ($\gamma = 1$), the constants α_o and β_o^2 coincide with \tilde{m} and \tilde{v}^2 , respectively.

We now construct the representative investor in our dynamically complete market economy to infer her implied bias in beliefs. The representative investor solves

$$\begin{aligned} U(D_T; \boldsymbol{\lambda}) = \max & \int_{\Theta} \lambda(\theta) \eta_T(\theta) \frac{W_T(\theta)^{1-\gamma}}{1-\gamma} d\theta, \\ \text{s.t.} & \int_{\Theta} W_T(\theta) d\theta = D_T, \end{aligned} \quad (\text{A.16})$$

for some strictly positive weights $\lambda(\theta)$ for each θ -type investor, where the collection of weights is denoted by $\boldsymbol{\lambda} = \{\lambda(\theta)\}_{\theta \in \Theta}$. The first order conditions of (A.16) yield

$$\frac{W_T(\theta)}{W_T(0)} = \left(\frac{\lambda(0) \eta_T(0)}{\lambda(\theta) \eta_T(\theta)}\right)^{-\frac{1}{\gamma}},$$

where $\lambda(0)$ and $\eta_T(0)$ denote the weight and the Radon-Nikodým derivative of the 0-type investor, respectively. Imposing $\int_{\Theta} W_T(\theta) d\theta = D_T$ yields the equilibrium horizon wealth allocations, which after substituting into (A.16) gives the representative investor's utility function as

$$U(D_T; \boldsymbol{\lambda}) = \frac{D_T^{1-\gamma}}{1-\gamma} \left(\int_{\Theta} [\lambda(\theta) \eta_T(\theta)]^{\frac{1}{\gamma}} d\theta\right)^{\gamma}. \quad (\text{A.17})$$

We next identify, from the second welfare theorem, the weights as $\lambda(\theta) = 1/y(\theta)$ where $y(\theta)$ is the θ -type investor's Lagrange multiplier given by (A.8). Substituting these weights into (A.17) gives the representative investor's utility function as

$$U(D_T; \boldsymbol{\lambda}) = \left(\int_{\Theta} y(\theta)^{-\frac{1}{\gamma}} \eta_T(\theta)^{\frac{1}{\gamma}} d\theta\right)^{\gamma} \frac{D_T^{1-\gamma}}{1-\gamma}. \quad (\text{A.18})$$

We observe that the parenthesis term in (A.18) is equal to M_T in (A.6). Applying Itô's Lemma to (A.9), we obtain the dynamics of M^γ as

$$\frac{dM_t^\gamma}{M_t^\gamma} = -\frac{1}{2} \left(1 - \frac{1}{\gamma}\right) \frac{\beta_t^2}{\sigma^2} dt + \frac{\alpha_t}{\sigma} d\omega_t. \quad (\text{A.19})$$

Since the drift term in (A.19) is deterministic, we may write M_t^γ as $M_t^\gamma = K^\gamma Y_t Z_t$ where Y is

a deterministic process and Z is a martingale process with dynamics

$$\frac{dY_t}{Y_t} = -\frac{1}{2} \left(1 - \frac{1}{\gamma}\right) \frac{\beta_t^2}{\sigma^2} dt, \quad (\text{A.20})$$

$$\frac{dZ_t}{Z_t} = \frac{\alpha_t}{\sigma} d\omega_t = \int_{\Theta} \frac{\theta}{\sigma} \left(\frac{y(\theta)^{-\frac{1}{\gamma}} \eta_t(\theta)^{\frac{1}{\gamma}}}{\int_{\Theta} y(\theta)^{-\frac{1}{\gamma}} \eta_t(\theta)^{\frac{1}{\gamma}} d\theta} \right) d\theta d\omega_t, \quad (\text{A.21})$$

and initial values $Y_0 = Z_0 = 1$. The solution to (A.20) is given by

$$Y_t = \left(\frac{\beta_t}{\beta_o} \right)^{\gamma-1} = \left(\frac{\sigma^2}{\sigma^2 + \frac{1}{\gamma} \beta_o^2 t} \right)^{\frac{1}{2}(\gamma-1)}. \quad (\text{A.22})$$

Since equilibrium is unique up to a constant, we set the constant $K = Y_T^{-\frac{1}{\gamma}}$ in (A.8) without loss of generality and substitute (A.20)–(A.22) to obtain the representative investor's utility function as

$$U(D_T; \boldsymbol{\lambda}) = Z_T \frac{D_T^{1-\gamma}}{1-\gamma},$$

where Z is given by the martingale process (A.21). Hence, we identify Z_T as being the Radon-Nikodým derivative of the representative investor's subjective belief \mathbb{P}^R with respect to the true belief \mathbb{P} , that is $d\mathbb{P}^R/d\mathbb{P} = Z_T$. Moreover, (A.21) implies that α_t is the time- t (stochastic) bias of the representative investor, and so is the time- t average bias in beliefs in our economy, as denoted by m_t ,

$$m_t = \alpha_t = \sigma \frac{\sigma \alpha_o + \frac{1}{\gamma} \beta_o^2 \omega_t}{\sigma^2 + \frac{1}{\gamma} \beta_o^2 t} = \int_{\Theta} \theta \left(\frac{y(\theta)^{-\frac{1}{\gamma}} \eta_t(\theta)^{\frac{1}{\gamma}}}{\int_{\Theta} y(\theta)^{-\frac{1}{\gamma}} \eta_t(\theta)^{\frac{1}{\gamma}} d\theta} \right) d\theta. \quad (\text{A.23})$$

Substituting $\sigma \omega_t$ by $\ln D_t - (\mu - \frac{1}{2}\sigma^2)t$ yields the expression stated in (12).

From the last equality in (A.23) we identify the unique weights $h_t(\theta)$ such that the weighted-average of investors' biases equals to the average bias in beliefs as

$$h_t(\theta) = \frac{y(\theta)^{-\frac{1}{\gamma}} \eta_t(\theta)^{\frac{1}{\gamma}}}{\int_{\Theta} y(\theta)^{-\frac{1}{\gamma}} \eta_t(\theta)^{\frac{1}{\gamma}} d\theta}. \quad (\text{A.24})$$

Substituting (A.7)–(A.8) into (A.24), and rearranging yields

$$h_t(\theta) = \frac{1}{\sqrt{2\pi\beta_t^2}} e^{-\frac{1}{2} \frac{(\theta - \alpha_t)^2}{\beta_t^2}}, \quad (\text{A.25})$$

where α_t and β_t are as in (A.10) and (A.11), respectively.

Finally, to determine the belief dispersion, we use the definition in (11) with the average bias in beliefs (A.23) and weights (A.25) substituted in to obtain

$$v_t^2 = \int_{\Theta} (\theta - m_t)^2 h_t(\theta) d\theta = \int_{\Theta} (\theta - m_t)^2 \frac{1}{\sqrt{2\pi\beta_t^2}} e^{-\frac{1}{2} \frac{(\theta - m_t)^2}{\beta_t^2}} d\theta.$$

This gives the (squared) dispersion in our economy as $v_t^2 = \beta_t^2$. By equating the initial values α_o and β_o^2 to m and v^2 in (A.14) and (A.15), respectively we obtain the (squared) dispersion and the weights in the economy as in (13) and (16). \square

Proof of Proposition 2. By no arbitrage, the stock price in our complete market economy is given by

$$S_t = \frac{1}{\xi_t} \mathbb{E}_t [\xi_T D_T]. \quad (\text{A.26})$$

To determine the stock price, we first compute the equilibrium state price density at time t by using the fact that it is a martingale, $\xi_t = \mathbb{E}_t [\xi_T]$. The equilibrium state price density at time T is as in the proof of Proposition 1, given by (A.5). Hence,

$$\begin{aligned} \xi_t &= \mathbb{E}_t [D_T^{-\gamma} M_T^\gamma] \\ &= D_t^{-\gamma} M_t^\gamma \left(\frac{v_T}{v_t} \right)^{\gamma-1} e^{-\gamma(\mu - \frac{1}{2}\sigma^2)(T-t)} e^{-\gamma m_t(T-t)} e^{\frac{1}{2}\gamma^2 \sigma^2 \frac{v_t^2}{v_T^2}(T-t)}, \end{aligned} \quad (\text{A.27})$$

where the last equality follows from Lemma 1 of the technical Appendix A.3 by taking $a = -\gamma$ and $b = \gamma$ and using the equalities $m_t = \alpha_t$ and $v_t = \beta_t$ to express the equation in terms of model parameters.

Next, we substitute (A.5) into (A.26) and obtain the expectation $\mathbb{E}_t [\xi_T D_T] = \mathbb{E}_t [D_T^{1-\gamma} M_T^\gamma]$. Again employing Lemma 1 of the technical Appendix A.3 with $a = 1 - \gamma$ and $b = \gamma$, we obtain

$$\mathbb{E}_t [D_T^{1-\gamma} M_T^\gamma] = D_t^{1-\gamma} M_t^\gamma \left(\frac{v_T}{v_t} \right)^{\gamma-1} e^{(1-\gamma)(\mu - \frac{1}{2}\sigma^2)(T-t)} e^{(1-\gamma)m_t(T-t)} e^{\frac{1}{2}(1-\gamma)^2 \sigma^2 \frac{v_t^2}{v_T^2}(T-t)}. \quad (\text{A.28})$$

Substituting (A.27) and (A.28) into (A.26) and manipulating yields the stock price expression (19) in Proposition 2. To determine the benchmark economy stock price, we set $\tilde{m} = \tilde{v} = 0$, which yields $m_t = v_t = 0$. Substituting into (19) gives the benchmark stock price (20).

The condition for property (i) that the stock price is higher than in the benchmark economy follows immediately from comparing (19)–(20). Property (ii) that the stock price is convex in cash-flow news follows once we substitute (12) into the stock price equation (19) and differentiate with respect to D_t .

The condition for property (iii) that the stock price is increasing in belief dispersion follows from the partial derivative of (19) with respect to v_t . This property holds when

$$\frac{\partial}{\partial v_t} m_t > \frac{1}{2\gamma} (2\gamma - 1) (T - t) \frac{\partial}{\partial v_t} v_t^2. \quad (\text{A.29})$$

Taking the partial derivative of m_t and v_t^2 using the expression (12) while taking account of the

dependency on v and m , yields

$$\frac{\partial}{\partial v_t} m_t = \frac{2}{v_t} (m_t - \tilde{m}), \quad (\text{A.30})$$

$$\frac{\partial}{\partial v_t} v_t^2 = 2v_t, \quad (\text{A.31})$$

which after substituting into (A.29) and rearranging gives the desired condition.

Finally, property (iv) that the stock price is increasing in investors' risk aversion for relatively bad cash-flow states and low values of γ follows from the partial derivative of (19) with respect to γ . This property holds when

$$\frac{\partial}{\partial \gamma} m_t > \sigma^2 + \left[\frac{1}{2\gamma^2} v_t^2 + \left(1 - \frac{1}{2\gamma} \right) \left(\frac{\partial}{\partial \gamma} v_t^2 \right) \right] (T - t). \quad (\text{A.32})$$

In this regard, using (14)–(15), we first compute $\partial v^2 / \partial \gamma$ and $\partial m / \partial \gamma$, and to simplify notation denote them by C and D , respectively

$$C \equiv \frac{\partial}{\partial \gamma} v^2 = \left(\frac{\tilde{v}^2}{2} - \frac{\gamma}{T} \sigma^2 \right) + \frac{\left(\frac{\gamma}{2} \tilde{v}^2 - \frac{\gamma^2}{2T} \sigma^2 \right) \left(\frac{\tilde{v}^2}{2} - \frac{\gamma}{T} \sigma^2 \right) + \frac{\gamma}{T} \tilde{v}^2 \sigma^2}{\sqrt{\left(\frac{\gamma}{2} \tilde{v}^2 - \frac{\gamma^2}{2T} \sigma^2 \right)^2 + \frac{\gamma^2}{T} \tilde{v}^2 \sigma^2}}, \quad (\text{A.33})$$

$$D \equiv \frac{\partial}{\partial \gamma} m = \frac{1}{\gamma^2} v^2 T + \left(1 - \frac{1}{\gamma} \right) C T. \quad (\text{A.34})$$

Using the expressions (12) and (13), we then obtain the required partial derivatives as

$$\frac{\partial}{\partial \gamma} m_t = \frac{v_t^2}{v^2} \left[D - \left(\frac{1}{\gamma} - \frac{C}{v^2} \right) (m_t - m) \right], \quad (\text{A.35})$$

$$\frac{\partial}{\partial \gamma} v_t^2 = \frac{v_t^4}{v^2} \left(\frac{C}{v^2} + \frac{v^2 t}{\sigma^2 \gamma^2} \right). \quad (\text{A.36})$$

Substituting (A.35) and (A.36) into (A.32) and rearranging yields the condition as

$$m_t < m + \left(\frac{1}{\gamma} - \frac{C}{v^2} \right)^{-1} \left\{ D - \frac{v^2}{v_t^2} \sigma^2 - \left[\frac{v^2}{2\gamma^2} + \left(1 - \frac{1}{2\gamma} \right) \left(\frac{C}{v^2} + \frac{v^2 t}{\sigma^2 \gamma^2} \right) v_t^2 \right] (T - t) \right\}. \quad (\text{A.37})$$

For any time t , the right hand side of (A.37) is constant while its left hand side is the average bias in beliefs, which is a normally distributed random variable, hence for sufficiently low levels of m_t and γ , (A.37) always holds. \square

Proof of Proposition 3. Applying Itô's Lemma to the stock price (19) yields

$$\frac{dS_t}{S_t} = \left[\gamma \sigma + \frac{v_t^2}{\sigma} (T - t) - \frac{m_t}{\sigma} \right] \left[\sigma + \frac{1}{\gamma} \frac{v_t^2}{\sigma} (T - t) \right] dt + \left[\sigma + \frac{1}{\gamma} \frac{v_t^2}{\sigma} (T - t) \right] d\omega_t, \quad (\text{A.38})$$

where its drift term gives the equilibrium mean return as

$$\mu_{St} = \left[\gamma \sigma + \frac{v_t^2}{\sigma} (T - t) - \frac{m_t}{\sigma} \right] \left[\sigma + \frac{1}{\gamma} \frac{v_t^2}{\sigma} (T - t) \right]. \quad (\text{A.39})$$

Substituting the equality

$$\sigma + \frac{1}{\gamma} \frac{v_t^2}{\sigma} (T - t) = \sigma \frac{v_t^2}{v_T^2}, \quad (\text{A.40})$$

into (A.39) and manipulating gives (22). To determine the benchmark economy mean return, we set $\tilde{m} = \tilde{v} = 0$, which yields $m_t = v_t = 0$. Substituting these into (A.39) gives the benchmark mean return (23).

The condition for property (i) that the mean return is lower than in the benchmark economy follows from comparing (22)–(23). The condition for property (ii) that the mean return is decreasing in belief dispersion follows from the partial derivative of (22) with respect to v_t . This property holds when

$$\frac{\partial}{\partial v_t} \mu_{St} = 2\gamma\sigma^2 \left(\frac{v_t^2}{v_T^2} \right) \frac{\partial}{\partial v_t} \left(\frac{v_t^2}{v_T^2} \right) - m_t \frac{\partial}{\partial v_t} \left(\frac{v_t^2}{v_T^2} \right) - \frac{v_t^2}{v_T^2} \frac{\partial}{\partial v_t} m_t < 0. \quad (\text{A.41})$$

Substituting the partial derivatives (A.30)–(A.31) into (A.41) and using the equality (A.40) and rearranging gives the desired condition.

Finally, property (iii) that the mean return is decreasing in investors' risk aversion for relatively bad cash-flow news and low levels of risk aversion follows from the partial derivative of (22) with respect to γ . This property holds when

$$\frac{\partial}{\partial \gamma} \mu_{St} = \sigma^2 \left(\frac{v_t^2}{v_T^2} \right)^2 + \left[2\gamma\sigma^2 \left(\frac{v_t^2}{v_T^2} \right) - m_t \right] \left(\frac{\partial}{\partial \gamma} \frac{v_t^2}{v_T^2} \right) - \frac{v_t^2}{v_T^2} \left(\frac{\partial}{\partial \gamma} m_t \right) < 0. \quad (\text{A.42})$$

Substituting the partial derivatives (A.35)–(A.36) into (A.42) and using (A.40) yields

$$\frac{\partial}{\partial \gamma} \mu_{St} = \sigma^2 \left(\frac{v_t^2}{v_T^2} \right)^2 + \left[2\gamma\sigma^2 \left(\frac{v_t^2}{v_T^2} \right) - m_t \right] E - \frac{v_t^2}{v_T^2} \frac{v_t^2}{v_T^2} \left[D - \left(\frac{1}{\gamma} - \frac{C}{v^2} \right) (m_t - m) \right], \quad (\text{A.43})$$

where we have defined E as

$$E \equiv \frac{\partial}{\partial \gamma} \frac{v_t^2}{v_T^2} = -\frac{1}{\gamma^2} \frac{v_t^2}{\sigma^2} (T - t) + \frac{1}{\gamma} \frac{1}{\sigma^2} \frac{v_t^4}{v^2} \left(\frac{C}{v^2} + \frac{v^2 t}{\sigma^2 \gamma^2} \right) (T - t).$$

Rearranging (A.43) yields the condition as

$$m_t < \left[\frac{v_t^4}{v_T^2} \left(\frac{1}{\gamma v^2} - \frac{C}{v^4} \right) - E \right]^{-1} \left\{ \frac{v_t^4}{v_T^2} \left[\frac{D}{v^2} + \left(\frac{1}{\gamma} - \frac{C}{v^2} \right) \frac{m}{v^2} \right] - \sigma^2 \frac{v_t^4}{v_T^4} - 2\gamma\sigma^2 \frac{v_t^2}{v_T^2} E \right\}. \quad (\text{A.44})$$

We note that, for any time t , the right hand side of (A.44) is a constant while its left hand side is the average bias in beliefs, which is a normally distributed random variable, hence for sufficiently low levels of m_t (A.44) always holds. \square

Proof of Proposition 4. The volatility of the stock is given by the diffusion term of the dynamics (A.38). The benchmark stock volatility readily is obtained by setting $v_t = 0$ in the diffusion term of (A.38).

The property (i) that the stock volatility is higher than that in the benchmark economy follows immediately by comparing (24) and (25). The property (ii) that the stock volatility is increasing in belief dispersion is immediate from (24). The property (iii) that the stock volatility is decreasing in investors' risk aversion follows from the negative sign of the partial derivative $\partial\sigma_{St}/\partial\gamma = \partial(v_t^2(T-t)/\gamma\sigma)/\partial\gamma$. \square

Proof of Proposition 5. To compute the trading volume measure V , we proceed by first determining the dynamics of each θ -type investor's equilibrium wealth-share, $W(\theta)/S$ and portfolio, $\phi(\theta)$, the fraction of wealth invested in the stock. Then, applying the product rule to $\psi(\theta) = \phi(\theta)(W(\theta)/S)$, we obtain the dynamics of $\psi(\theta)$. Finally, using the definition in (26) we obtain the trading volume measure V in our economy in closed form.

To compute each θ -type investor's wealth share $W(\theta)/S$, we first consider her time- t wealth that satisfies (A.1)

$$\begin{aligned}\xi_t W_t(\theta) &= \mathbb{E}_t[\xi_T W_T(\theta)] \\ &= \frac{K}{\sqrt{2\pi v^2}} e^{-\frac{1}{2}\frac{(\theta-m)^2}{v^2}} \mathbb{E}_t\left[\eta_T(\theta)^{\frac{1}{\gamma}} D_T^{1-\gamma} M_T^{\gamma-1}\right],\end{aligned}\tag{A.45}$$

where the second equality follows by substituting (A.3), (A.5) and (A.8). We also employed the equalities $\alpha_o = m$, $\beta_o^2 = v^2$ (see, proof of Proposition 1) to express (A.45) in terms of model parameters. Using Lemma 2 in the technical Appendix A.3, we have

$$\mathbb{E}_t\left[\eta_T(\theta)^{\frac{1}{\gamma}} D_T^{1-\gamma} M_T^{\gamma-1}\right] = \xi_t S_t K^{-1} \left(\frac{1}{\sqrt{2\pi v^2}} e^{-\frac{1}{2}\frac{(\theta-m)^2}{v^2}}\right)^{-1} \frac{1}{\sqrt{2\pi v_t^2 A_t^2}} e^{-\frac{1}{2}\frac{(\theta - [m_t - (1-\frac{1}{\gamma})v_t^2(T-t)])^2}{v_t^2 A_t^2}},\tag{A.46}$$

with

$$A_t^2 = \frac{1}{\gamma} + \left(1 - \frac{1}{\gamma}\right) \frac{v_T^2}{v_t^2},$$

where we again substituted $\alpha_t = m_t$, $\beta_t^2 = v_t^2$, (see, proof of Proposition 1), and m_t and v_t are as in (12)–(13). Substituting (A.46) into (A.45) and rearranging gives the wealth-share of each θ -type investor as

$$\frac{W_t(\theta)}{S_t} = \frac{1}{\sqrt{2\pi v_t^2 A_t^2}} e^{-\frac{1}{2}\frac{(\theta - [m_t - (1-\frac{1}{\gamma})v_t^2(T-t)])^2}{v_t^2 A_t^2}}.\tag{A.47}$$

It is easy to verify that these expressions coincide with the initial wealth-share distribution (5) once we substitute (14)–(15). We note that the time- t wealth-share distribution is Gaussian

with mean and variance

$$\begin{aligned}\tilde{m}_t &\equiv m_t - \left(1 - \frac{1}{\gamma}\right) v_t^2 (T - t), \\ \tilde{v}_t^2 &\equiv v_t^2 A_t^2 = \frac{1}{\gamma} v_t^2 + \left(1 - \frac{1}{\gamma}\right) v_T^2,\end{aligned}$$

as reported in Remark 1. We then apply Itô's lemma to (A.47) to obtain the dynamics

$$d\frac{W_t(\theta)}{S_t} = \dots dt + \frac{W_t(\theta)}{S_t} \frac{v_t^2}{\gamma\sigma\tilde{v}_t^2} (\theta - \tilde{m}_t) d\omega_t. \quad (\text{A.48})$$

To obtain each θ -type investor's optimal portfolio $\phi(\theta)$ as a fraction of wealth invested in the stock, we match the volatility term in (A.48) with the corresponding one in

$$d\frac{W_t(\theta)}{S_t} = \dots dt + \frac{W_t(\theta)}{S_t} (\phi_t(\theta) - 1) \sigma_{S_t} d\omega_t,$$

which is obtained by using (1) and (9). This yields investors' equilibrium portfolios as

$$\phi_t(\theta) = 1 + \frac{v_T^2}{\gamma\sigma^2\tilde{v}_t^2} (\theta - \tilde{m}_t). \quad (\text{A.49})$$

Applying Itô's lemma to (A.49) yields the dynamics

$$d\phi_t(\theta) = \dots dt - \frac{v_t^2 v_T^2}{\gamma^2 \sigma^3 \tilde{v}_t^2} d\omega_t.$$

We then apply the product rule to $\psi(\theta) = \phi(\theta) (W(\theta)/S)$ to obtain the portfolio dynamics in terms of number of shares invested in the stock $\psi(\theta)$

$$d\psi_t(\theta) = \dots dt + \frac{W_t(\theta)}{S_t} \frac{v_t^2}{\gamma\sigma\tilde{v}_t^2} \left[(\theta - \tilde{m}_t) \phi_t(\theta) - \frac{v_T^2}{\gamma\sigma^2} \right] d\omega_t,$$

which after substituting (A.49) yields the portfolio volatility of each θ -type investor $\sigma_{\psi}(\theta)$ as

$$\sigma_{\psi t}(\theta) = \frac{W_t(\theta)}{S_t} \frac{v_t^2}{\gamma\sigma\tilde{v}_t^2} \left[\frac{v_T^2}{\gamma\sigma^2} \left(\left(\frac{\theta - \tilde{m}_t}{\tilde{v}_t} \right)^2 - 1 \right) + \tilde{v}_t \left(\frac{\theta - \tilde{m}_t}{\tilde{v}_t} \right) \right]. \quad (\text{A.50})$$

We now compute our trading volume measure (26), obtained by summing the absolute value of investors' portfolio volatilities. To find this absolute value, we need to identify the types for whom the portfolio volatility is negative at time t . From (A.50), this occurs when the square bracket term is negative which is a quadratic in types θ . Therefore at time- t , the types for whom the portfolio volatility $\sigma_{\psi t}(\theta)$ is negative lies between two critical types θ_{c1} and θ_{c2} for which $\sigma_{\psi t}(\theta_{c1}) = \sigma_{\psi t}(\theta_{c2}) = 0$. Equating (A.50) to zero and solving the quadratic equation

yields the critical types as

$$\theta_{c1} = \tilde{m}_t + \tilde{v}_t \left(-\frac{1}{2}X_t - \frac{1}{2}\sqrt{X_t^2 + 4} \right), \quad (\text{A.51})$$

$$\theta_{c2} = \tilde{m}_t + \tilde{v}_t \left(-\frac{1}{2}X_t + \frac{1}{2}\sqrt{X_t^2 + 4} \right), \quad (\text{A.52})$$

where $X_t = \gamma\sigma^2\tilde{v}_t/v_T^2$. From definition (26), the trading volume measure is

$$\begin{aligned} V_t &\equiv \frac{1}{2} \int_{\Theta} |\sigma_{\psi t}(\theta)| d\theta = \frac{1}{2} \left[\int_{-\infty}^{\theta_{c1}} \sigma_{\psi t}(\theta) d\theta - \int_{\theta_{c1}}^{\theta_{c2}} \sigma_{\psi t}(\theta) d\theta + \int_{\theta_{c2}}^{\infty} \sigma_{\psi t}(\theta) d\theta \right] \\ &= - \int_{\theta_{c1}}^{\theta_{c2}} \sigma_{\psi t}(\theta) d\theta, \end{aligned} \quad (\text{A.53})$$

where the last equality follows from the fact $\int_{\Theta} \sigma_{\psi t}(\theta) d\theta = 0$ which implies

$$\int_{-\infty}^{\theta_{c1}} \sigma_{\psi t}(\theta) d\theta + \int_{\theta_{c2}}^{\infty} \sigma_{\psi t}(\theta) d\theta = - \int_{\theta_{c1}}^{\theta_{c2}} \sigma_{\psi t}(\theta) d\theta.$$

Substituting (A.47) and (A.50) into (A.53) yields

$$V_t = -\frac{v_t^2}{\gamma\sigma\tilde{v}_t^2} \int_{\theta_{c1}}^{\theta_{c2}} \left[\frac{v_T^2}{\gamma\sigma^2} \left(\left(\frac{\theta - \tilde{m}_t}{\tilde{v}_t} \right)^2 - 1 \right) + \tilde{v}_t \left(\frac{\theta - \tilde{m}_t}{\tilde{v}_t} \right) \right] \frac{1}{\sqrt{2\pi\tilde{v}_t^2}} e^{-\frac{1}{2} \frac{(\theta - \tilde{m}_t)^2}{\tilde{v}_t^2}} d\theta.$$

Changing the variable of integration to $z = \frac{\theta - \tilde{m}_t}{\tilde{v}_t}$ and using the facts that

$$\begin{aligned} \int (z^2 - 1) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz &= -z\phi(z) + C, \\ \int z \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz &= -\phi(z) + C, \end{aligned}$$

where $\phi(\cdot)$ is the standard normal density function and C is a constant, we obtain the trading volume measure as

$$\begin{aligned} V_t &= \frac{v_t^2}{\gamma\sigma\tilde{v}_t^2} \left[\frac{v_T^2}{\gamma\sigma^2} \left(\frac{\theta_{c2} - \tilde{m}_t}{\tilde{v}_t} \right) + \tilde{v}_t \right] \phi \left(\frac{\theta_{c2} - \tilde{m}_t}{\tilde{v}_t} \right) \\ &\quad - \frac{v_t^2}{\gamma\sigma\tilde{v}_t^2} \left[\frac{v_T^2}{\gamma\sigma^2} \left(\frac{\theta_{c1} - \tilde{m}_t}{\tilde{v}_t} \right) + \tilde{v}_t \right] \phi \left(\frac{\theta_{c1} - \tilde{m}_t}{\tilde{v}_t} \right). \end{aligned} \quad (\text{A.54})$$

Finally, substituting (A.51)–(A.52) into (A.54) and rearranging gives (27).

The condition for property (i) that the trading volume measure is increasing in belief dispersion follows from the partial derivative of (27) with respect to v_t , or equivalently v^2 . To compute this partial derivative we rewrite (27) compactly as

$$V_t = \frac{1}{2\sigma} \left[Z_t^+ \phi \left(\frac{1}{2} \frac{X_t^2}{\sigma^2} \frac{v_T^2}{v_t^2} Z_t^- \right) + Z_t^- \phi \left(\frac{1}{2} \frac{X_t^2}{\sigma^2} \frac{v_T^2}{v_t^2} Z_t^+ \right) \right], \quad (\text{A.55})$$

where we defined the positive deterministic processes

$$\begin{aligned} Z_t^+ &\equiv \sqrt{\left(\frac{\sigma^2 v_t^2}{X_t^2 v_T^2}\right)^2 (X_t^2 + 4) + \frac{\sigma^2 v_t^2}{X_t v_T^2}}, \\ Z_t^- &\equiv \sqrt{\left(\frac{\sigma^2 v_t^2}{X_t^2 v_T^2}\right)^2 (X_t^2 + 4) - \frac{\sigma^2 v_t^2}{X_t v_T^2}}, \end{aligned}$$

with $0 < Z_t^- < Z_t^+$, and

$$\frac{\partial}{\partial v^2} Z_t^+ - \frac{\partial}{\partial v^2} Z_t^- = 2 \frac{\partial}{\partial v^2} \left(\frac{\sigma^2 v_t^2}{X_t v_T^2} \right). \quad (\text{A.56})$$

The partial derivative of (A.55) with respect to v^2 is given by

$$\begin{aligned} \frac{\partial}{\partial v^2} V_t &= \frac{1}{2\sigma} \left[\left(\frac{\partial Z_t^+}{\partial v^2} \right) \phi \left(\frac{1}{2} \frac{X_t^2 v_T^2}{\sigma^2 v_t^2} Z_t^- \right) + Z_t^+ \frac{\partial}{\partial v^2} \phi \left(\frac{1}{2} \frac{X_t^2 v_T^2}{\sigma^2 v_t^2} Z_t^- \right) \right] \\ &+ \frac{1}{2\sigma} \left[\left(\frac{\partial Z_t^-}{\partial v^2} \right) \phi \left(\frac{1}{2} \frac{X_t^2 v_T^2}{\sigma^2 v_t^2} Z_t^+ \right) + Z_t^- \frac{\partial}{\partial v^2} \phi \left(\frac{1}{2} \frac{X_t^2 v_T^2}{\sigma^2 v_t^2} Z_t^+ \right) \right]. \end{aligned} \quad (\text{A.57})$$

Substituting

$$\begin{aligned} \frac{\partial}{\partial v^2} \phi \left(\frac{1}{2} \frac{X_t^2 v_T^2}{\sigma^2 v_t^2} Z_t^- \right) &= -\frac{1}{2} \frac{X_t^2 v_T^2}{\sigma^2 v_t^2} Z_t^- \phi \left(\frac{1}{2} \frac{X_t^2 v_T^2}{\sigma^2 v_t^2} Z_t^- \right) \frac{\partial}{\partial v^2} \left(\frac{1}{2} \frac{X_t^2 v_T^2}{\sigma^2 v_t^2} Z_t^- \right), \\ \frac{\partial}{\partial v^2} \phi \left(\frac{1}{2} \frac{X_t^2 v_T^2}{\sigma^2 v_t^2} Z_t^+ \right) &= -\frac{1}{2} \frac{X_t^2 v_T^2}{\sigma^2 v_t^2} Z_t^+ \phi \left(\frac{1}{2} \frac{X_t^2 v_T^2}{\sigma^2 v_t^2} Z_t^+ \right) \frac{\partial}{\partial v^2} \left(\frac{1}{2} \frac{X_t^2 v_T^2}{\sigma^2 v_t^2} Z_t^+ \right), \end{aligned}$$

with

$$\begin{aligned} \frac{\partial}{\partial v^2} \left(\frac{1}{2} \frac{X_t^2 v_T^2}{\sigma^2 v_t^2} Z_t^- \right) &= \frac{1}{2} \left[\left(\frac{X_t^2 v_T^2}{\sigma^2 v_t^2} \right) \frac{\partial Z_t^-}{\partial v^2} + Z_t^- \frac{\partial}{\partial v^2} \left(\frac{X_t^2 v_T^2}{\sigma^2 v_t^2} \right) \right], \\ \frac{\partial}{\partial v^2} \left(\frac{1}{2} \frac{X_t^2 v_T^2}{\sigma^2 v_t^2} Z_t^+ \right) &= \frac{1}{2} \left[\left(\frac{X_t^2 v_T^2}{\sigma^2 v_t^2} \right) \frac{\partial Z_t^+}{\partial v^2} + Z_t^+ \frac{\partial}{\partial v^2} \left(\frac{X_t^2 v_T^2}{\sigma^2 v_t^2} \right) \right], \end{aligned}$$

into (A.57), and using (A.56) and the equality

$$Z_t^+ Z_t^- = 4 \left(\frac{\sigma^2 v_t^2}{X_t^2 v_T^2} \right)^2,$$

yields the required partial derivative

$$\begin{aligned} \frac{\partial}{\partial v^2} V_t &= \frac{1}{2\sigma} \left[2 \frac{\partial}{\partial v^2} \left(\frac{\sigma^2 v_t^2}{X_t v_T^2} \right) - Z_t^- \frac{\sigma^2 v_t^2}{X_t^2 v_T^2} \frac{\partial}{\partial v^2} \left(\frac{X_t^2 v_T^2}{\sigma^2 v_t^2} \right) \right] \phi \left(\frac{1}{2} \frac{X_t^2 v_T^2}{\sigma^2 v_t^2} Z_t^- \right) \\ &+ \frac{1}{2\sigma} \left[-2 \frac{\partial}{\partial v^2} \left(\frac{\sigma^2 v_t^2}{X_t v_T^2} \right) - Z_t^+ \frac{\sigma^2 v_t^2}{X_t^2 v_T^2} \frac{\partial}{\partial v^2} \left(\frac{X_t^2 v_T^2}{\sigma^2 v_t^2} \right) \right] \phi \left(\frac{1}{2} \frac{X_t^2 v_T^2}{\sigma^2 v_t^2} Z_t^+ \right). \end{aligned} \quad (\text{A.58})$$

However, (A.58) is always positive because

$$\frac{\partial}{\partial v^2} \left(\frac{X_t^2 v_T^2}{\sigma^2 v_t^2} \right) < 0 \quad \text{and} \quad \frac{\partial}{\partial v^2} \left(\frac{\sigma^2 v_t^2}{X_t v_T^2} \right) > 0,$$

which implies that the first square bracket term in (A.58) is positive, and if the second square bracket terms is also positive then it is easy to see that (A.58) is positive. However, if the second square bracket term is negative then we use the inequality

$$2\frac{\partial}{\partial v^2}\left(\frac{\sigma^2 v_t^2}{X_t v_T^2}\right) - Z_t^- \frac{\sigma^2 v_t^2}{X_t^2 v_T^2} \frac{\partial}{\partial v^2}\left(\frac{X_t^2 v_T^2}{\sigma^2 v_t^2}\right) > \left| -2\frac{\partial}{\partial v^2}\left(\frac{\sigma^2 v_t^2}{X_t v_T^2}\right) - Z_t^+ \frac{\sigma^2 v_t^2}{X_t^2 v_T^2} \frac{\partial}{\partial v^2}\left(\frac{X_t^2 v_T^2}{\sigma^2 v_t^2}\right) \right|,$$

and the fact that

$$0 < \phi\left(\frac{1}{2}\frac{X_t^2 v_T^2}{\sigma^2 v_t^2} Z_t^+\right) < \phi\left(\frac{1}{2}\frac{X_t^2 v_T^2}{\sigma^2 v_t^2} Z_t^-\right), \quad (\text{A.59})$$

to show that the first line in (A.58) dominates the second line, and therefore (A.58) is positive.

Property (ii) that the trading volume measure is positively related to the stock volatility follows from the fact that an increase in belief dispersion leads to both a higher trading volume measure and a stock volatility.

Finally, property (iii) that the trading volume measure is decreasing in investors' risk aversion follows from the partial derivative of (27) with respect to γ . Following the similar steps as above, we obtain

$$\begin{aligned} \frac{\partial}{\partial \gamma} V_t &= \frac{1}{2\sigma} \left[2\frac{\partial}{\partial \gamma}\left(\frac{\sigma^2 v_t^2}{X_t v_T^2}\right) - Z_t^- \frac{\sigma^2 v_t^2}{X_t^2 v_T^2} \frac{\partial}{\partial \gamma}\left(\frac{X_t^2 v_T^2}{\sigma^2 v_t^2}\right) \right] \phi\left(\frac{1}{2}\frac{X_t^2 v_T^2}{\sigma^2 v_t^2} Z_t^-\right) \\ &+ \frac{1}{2\sigma} \left[-2\frac{\partial}{\partial \gamma}\left(\frac{\sigma^2 v_t^2}{X_t v_T^2}\right) - Z_t^+ \frac{\sigma^2 v_t^2}{X_t^2 v_T^2} \frac{\partial}{\partial \gamma}\left(\frac{X_t^2 v_T^2}{\sigma^2 v_t^2}\right) \right] \phi\left(\frac{1}{2}\frac{X_t^2 v_T^2}{\sigma^2 v_t^2} Z_t^+\right). \end{aligned} \quad (\text{A.60})$$

However, (A.60) is always negative because

$$\frac{\partial}{\partial \gamma}\left(\frac{X_t^2 v_T^2}{\sigma^2 v_t^2}\right) > 0 \quad \text{and} \quad \frac{\partial}{\partial \gamma}\left(\frac{\sigma^2 v_t^2}{X_t v_T^2}\right) < 0,$$

which implies the first square bracket term in (A.60) is negative, and if the second square bracket term is also negative then it is easy to see that (A.60) is negative. However, if the second square bracket term is positive then we use the inequality

$$-2\frac{\partial}{\partial \gamma}\left(\frac{\sigma^2 v_t^2}{X_t v_T^2}\right) - Z_t^+ \frac{\sigma^2 v_t^2}{X_t^2 v_T^2} \frac{\partial}{\partial \gamma}\left(\frac{X_t^2 v_T^2}{\sigma^2 v_t^2}\right) < \left| 2\frac{\partial}{\partial \gamma}\left(\frac{\sigma^2 v_t^2}{X_t v_T^2}\right) - Z_t^- \frac{\sigma^2 v_t^2}{X_t^2 v_T^2} \frac{\partial}{\partial \gamma}\left(\frac{X_t^2 v_T^2}{\sigma^2 v_t^2}\right) \right|,$$

and (A.59) to show that the first line in (A.60) dominates the second line, and therefore (A.60) is negative. \square

A.2 Equilibrium in Finitely-Many-Investor Economy

In this Appendix, we consider a variant of our economy in Section 2 in which there are finitely many investors instead of a continuum of them. The other features of our economy remain the same. In particular, the securities market is as in Section 2.1 and the investors' beliefs are

as in Section 2.2, that is, under the θ_n -type investor's beliefs, the cash-flow news process has dynamics $dD_t = (\mu + \theta_n) D_t dt + \sigma D_t d\omega_{nt}$, where ω_n is her perceived Brownian motion with respect to her own probability measure \mathbb{P}^{θ_n} , and is given by $\omega_{nt} = \omega_t - \theta_n t / \sigma$. We again index each θ_n -type investor by her bias θ_n , with the type space now becoming $\Theta = \{\theta_1, \dots, \theta_N\}$ rather than $\Theta = \mathbb{R}$ as in our main model. We assume that each θ_n -type investor is initially endowed with f_n units of stock shares so that $\sum_{n=1}^N f_n = 1$. This gives the initial wealth of each θ_n -type investor as $W_{n0} = S_0 f_n$. For tractability, we here only consider the case when investors have logarithmic preferences.

As in the Proof of Proposition 1, we begin by first solving each θ_n -type investor's optimization problem. Dynamic market completeness implies a unique state price density process ξ under \mathbb{P} . The static budget constraint of each θ_n -type investor under \mathbb{P} is given by

$$\mathbb{E}_t [\xi_T W_{nT}] = \xi_t W_{nt}. \quad (\text{A.61})$$

We write each θ_n -type investor's expected utility under the objective measure \mathbb{P} as

$$\mathbb{E} [\eta_{nT} \ln W_{nT}], \quad (\text{A.62})$$

where η_{nT} is the Radon-Nikodým derivative of the subjective measure \mathbb{P}^{θ_n} with respect to the true measure \mathbb{P} given by

$$\eta_{nT} = \frac{d\mathbb{P}^{\theta_n}}{d\mathbb{P}} = e^{\frac{\theta_n}{\sigma} \omega_T - \frac{1}{2} \frac{\theta_n^2}{\sigma^2} T}.$$

Maximizing each θ_n -type investor's expected objective function (A.62) subject to (A.61) evaluated at time $t = 0$ leads to the optimal horizon wealth of each θ_n -type as

$$W_{nT} = S_0 \frac{f_n \eta_{nT}}{\xi_T}. \quad (\text{A.63})$$

Applying the market clearing condition yields the time- T equilibrium state price density as

$$\xi_T = S_0 D_T^{-1} \sum_{n=1}^N f_n \eta_{nT}. \quad (\text{A.64})$$

The time- t equilibrium state price density is then given by $\xi_t = \mathbb{E}_t [\xi_T]$ which yields

$$\xi_t = S_0 e^{(\sigma^2 - \mu)(T-t)} D_t^{-1} \sum_{n=1}^N e^{-\theta_n(T-t)} f_n \eta_{nt}, \quad (\text{A.65})$$

where $\eta_{nt} = \mathbb{E}_t [\eta_{nT}] = e^{\frac{\theta_n}{\sigma} \omega_t - \frac{1}{2} \frac{\theta_n^2}{\sigma^2} t}$. By no arbitrage, the stock price in this economy is given by $\xi_t S_t = \mathbb{E}_t [\xi_T D_T]$. Substituting (A.64)–(A.65) and computing the expectation yields the stock price. To derive the equilibrium stock mean return and volatility, we apply Itô's Lemma to the stock price. These quantities are presented below.

Proposition A.1. *In the economy with finitely many investors, the equilibrium stock price, mean return and volatility are given by*

$$\begin{aligned}
S_t &= e^{(\mu-\sigma^2)(T-t)} D_t \frac{\sum_{n=1}^N f_n \eta_{nt}}{\sum_{n=1}^N e^{-\theta_n(T-t)} f_n \eta_{nt}}, \tag{A.66} \\
\mu_{St} &= \left[\sigma - \frac{1}{\sigma} \frac{\sum_{n=1}^N \theta_n e^{-\theta_n(T-t)} f_n \eta_{nt}}{\sum_{n=1}^N e^{-\theta_n(T-t)} f_n \eta_{nt}} \right] \left[\sigma + \frac{1}{\sigma} \left(\frac{\sum_{n=1}^N \theta_n f_n \eta_{nt}}{\sum_{n=1}^N f_n \eta_{nt}} - \frac{\sum_{n=1}^N \theta_n e^{-\theta_n(T-t)} f_n \eta_{nt}}{\sum_{n=1}^N e^{-\theta_n(T-t)} f_n \eta_{nt}} \right) \right], \\
\sigma_{St} &= \sigma + \frac{1}{\sigma} \left(\frac{\sum_{n=1}^N \theta_n f_n \eta_{nt}}{\sum_{n=1}^N f_n \eta_{nt}} - \frac{\sum_{n=1}^N \theta_n e^{-\theta_n(T-t)} f_n \eta_{nt}}{\sum_{n=1}^N e^{-\theta_n(T-t)} f_n \eta_{nt}} \right).
\end{aligned}$$

Finally to determine the trading volume measure, we first determine the each θ_n -type investor's time- t wealth by (A.61). Substituting (A.63)–(A.64) and taking the expectation yields

$$W_{nt} = S_0 \frac{f_n \eta_{nt}}{\xi_t}. \tag{A.67}$$

We then determine the investors' equilibrium portfolios ϕ_n , the fraction of wealth invested in the stock, by applying Itô's Lemma to (A.67), and obtain

$$\phi_{nt} = \frac{1}{\sigma_{St}} \left(\frac{\mu_{St}}{\sigma_{St}} + \frac{\theta_n}{\sigma} \right), \tag{A.68}$$

where μ_{St} and σ_{St} as in Proposition A.1. We also compute each θ_n -type investor's wealth-share W_n/S by dividing (A.67) by (A.66) and substituting (A.65) which yields

$$\frac{W_{nt}}{S_t} = \frac{f_n \eta_{nt}}{\sum_{n=1}^N f_n \eta_{nt}}. \tag{A.69}$$

Hence the investors' equilibrium portfolios ψ_n in terms of the number of shares invested in the stock becomes

$$\begin{aligned}
\psi_{nt} &= \phi_{nt} \frac{W_{nt}}{S_t} \\
&= \frac{1}{\sigma_{St}} \left(\frac{\mu_{St}}{\sigma_{St}} + \frac{\theta_n}{\sigma} \right) \frac{f_n \eta_{nt}}{\sum_{n=1}^N f_n \eta_{nt}}, \tag{A.70}
\end{aligned}$$

where the last equality follows by substituting (A.68)–(A.69). Denoting the portfolio dynamics as $d\psi_{nt} = \mu_{n\psi t} dt + \sigma_{n\psi t} d\omega_t$, we compute the volatility term $\sigma_{n\psi}$ by applying Itô's Lemma to (A.70). After straightforward but lengthy computations we obtain the portfolio volatility and the trading volume measure as the following.

Proposition A.2. *In the economy with finitely many investors, the equilibrium portfolio volatil-*

ity of each investor is given by

$$\begin{aligned}
\sigma_{n\psi t} &= \frac{\phi_{nt}}{\sigma} \left(\theta_n - \frac{\sum_{n=1}^N \theta_n f_n \eta_{nt}}{\sum_{n=1}^N f_n \eta_{nt}} \right) \frac{W_{nt}}{S_t} \\
&\quad - \frac{1}{\sigma^3 \sigma_{St}^2} \left(\theta_n - \frac{\sum_{n=1}^N \theta_n f_n \eta_{nt}}{\sum_{n=1}^N f_n \eta_{nt}} \right) \frac{W_{nt}}{S_t} \left[\frac{\sum_{n=1}^N \theta_n^2 f_n \eta_{nt}}{\sum_{n=1}^N f_n \eta_{nt}} - \frac{\sum_{n=1}^N \theta_n^2 e^{-\theta_n(T-t)} f_n \eta_{nt}}{\sum_{n=1}^N e^{-\theta_n(T-t)} f_n \eta_{nt}} \right] \\
&\quad - \frac{1}{\sigma^3 \sigma_{St}^2} \left(\theta_n - \frac{\sum_{n=1}^N \theta_n f_n \eta_{nt}}{\sum_{n=1}^N f_n \eta_{nt}} \right) \frac{W_{nt}}{S_t} \left[2 \left(\frac{\sum_{n=1}^N \theta_n f_n \eta_{nt}}{\sum_{n=1}^N f_n \eta_{nt}} \right) (\sigma^2 - \sigma \sigma_{St}) + (\sigma^2 - \sigma \sigma_{St})^2 \right] \\
&\quad - \frac{1}{\sigma^2 \sigma_{St}} \frac{W_{nt}}{S_t} \left[\frac{\sum_{n=1}^N \theta_n^2 f_n \eta_{nt}}{\sum_{n=1}^N f_n \eta_{nt}} - \left(\frac{\sum_{n=1}^N \theta_n f_n \eta_{nt}}{\sum_{n=1}^N f_n \eta_{nt}} \right)^2 \right], \tag{A.71}
\end{aligned}$$

and the trading volume measure is given by

$$V_t = \frac{1}{2} \sum_{n=1}^N |\sigma_{n\psi t}|.$$

where $\sigma_{n\psi t}$ is as in (A.71).

A.3 Technical Lemmas

Lemma 1. *Let the processes M , α and β be defined as in (A.9), (A.10) and (A.11), respectively. Then for all numbers a and b we have*

$$\begin{aligned}
\mathbb{E}_t [D_T^a M_T^b] &= D_t^a M_t^b \left(\frac{\beta_T}{\beta_t} \right)^b e^{a(\mu - \frac{1}{2}\sigma^2)(T-t)} e^{-\frac{b}{2} \frac{\alpha_t^2}{\beta_t^2} \left(1 - \frac{\beta_T^2}{\beta_t^2} \left[1 - \frac{b}{\gamma} \left(1 - \frac{\beta_T^2}{\beta_t^2} \right) \right]^{-1} \right)} \\
&\quad \times \left[1 - \frac{b}{\gamma} \left(1 - \frac{\beta_T^2}{\beta_t^2} \right) \right]^{-\frac{1}{2}} e^{\frac{1}{2} \left[1 - \frac{b}{\gamma} \left(1 - \frac{\beta_T^2}{\beta_t^2} \right) \right]^{-1} \left(\frac{2ab\alpha_t}{\gamma} \frac{\beta_T^2}{\beta_t^2} + a^2 \sigma^2 \right) (T-t)}, \tag{A.72}
\end{aligned}$$

provided $1 - \frac{b}{\gamma} \left(1 - \frac{\beta_T^2}{\beta_t^2} \right) > 0$.

Proof of Lemma 1. By (A.9), we have

$$M_T = M_t \left(\frac{\beta_T}{\beta_t} \right) e^{-\frac{1}{2} \frac{\alpha_t^2}{\beta_t^2} + \frac{1}{2} \frac{\alpha_T^2}{\beta_T^2}}, \tag{A.73}$$

and (A.10)–(A.11) give

$$\begin{aligned}
\frac{\alpha_T^2}{\beta_T^2} &= \beta_T^2 \left(\frac{m^2}{v^4} + 2 \frac{m \omega_T}{v^2 \gamma \sigma} + \frac{\omega_T^2}{\gamma^2 \sigma^2} \right) \\
&= \frac{\alpha_t^2}{\beta_t^2} \frac{\beta_T^2}{\beta_t^2} + 2 \frac{\alpha_t}{\gamma \sigma} \frac{\beta_T^2}{\beta_t^2} (\omega_T - \omega_t) + \frac{\beta_T^2}{\gamma^2 \sigma^2} (\omega_T - \omega_t)^2. \tag{A.74}
\end{aligned}$$

Substituting (A.74) into (A.73) and using the lognormality of D_T , we obtain

$$\mathbb{E}_t [D_T^a M_T^b] = D_t^a e^{a(\mu - \frac{1}{2}\sigma^2)(T-t)} M_t^b \left(\frac{\beta_T}{\beta_t} \right)^b e^{-\frac{b}{2} \frac{\alpha_t^2}{\beta_t^2} \left(1 - \frac{\beta_T^2}{\beta_t^2}\right)} \mathbb{E}_t \left[e^{\left(\frac{b\alpha_t}{\gamma\sigma} \frac{\beta_T^2}{\beta_t^2} + a\sigma \right) (\omega_T - \omega_t) + \frac{1}{2} \frac{b\beta_T^2}{\gamma^2\sigma^2} (\omega_T - \omega_t)^2} \right]. \quad (\text{A.75})$$

Let $Z \sim \mathcal{N}(0, 1)$. Then the independence property of conditional expectations yields

$$\begin{aligned} \mathbb{E}_t \left[e^{\left(\frac{b\alpha_t}{\gamma\sigma} \frac{\beta_T^2}{\beta_t^2} + a\sigma \right) (\omega_T - \omega_t) + \frac{1}{2} \frac{b\beta_T^2}{\gamma^2\sigma^2} (\omega_T - \omega_t)^2} \right] &= \mathbb{E} \left[e^{\left(\frac{b\alpha_t}{\gamma\sigma} \frac{\beta_T^2}{\beta_t^2} + a\sigma \right) \sqrt{T-t}Z + \frac{1}{2} \frac{b\beta_T^2}{\gamma^2\sigma^2} (T-t)Z^2} \right] \\ &= \int_{-\infty}^{\infty} e^{\left(\frac{b\alpha_t}{\gamma\sigma} \frac{\beta_T^2}{\beta_t^2} + a\sigma \right) \sqrt{T-t}z + \frac{1}{2} \frac{b\beta_T^2}{\gamma^2\sigma^2} (T-t)z^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\ &= \left[1 - \frac{b}{\gamma} \left(1 - \frac{\beta_T^2}{\beta_t^2} \right) \right]^{-\frac{1}{2}} e^{\frac{1}{2} \left[1 - \frac{b}{\gamma} \left(1 - \frac{\beta_T^2}{\beta_t^2} \right) \right]^{-1} \left(\frac{b\alpha_t}{\gamma\sigma} \frac{\beta_T^2}{\beta_t^2} + a\sigma \right)^2 (T-t)}, \end{aligned}$$

which after substituting into (A.75) and rearranging gives (A.72). \square

Lemma 2. *Let the processes $\eta(\theta)$ and M be as in (A.7) and (A.9), respectively. Then*

$$\mathbb{E}_t \left[\eta_T(\theta)^{\frac{1}{\gamma}} D_T^{1-\gamma} M_T^{\gamma-1} \right] = \xi_t S_t K^{-1} \left(\frac{1}{\sqrt{2\pi\beta_o^2}} e^{-\frac{1}{2} \frac{(\theta - \alpha_o)^2}{\beta_o^2}} \right)^{-1} \frac{1}{\sqrt{2\pi\beta_t^2 A_t^2}} e^{-\frac{1}{2} \frac{(\theta - [\alpha_t - (1 - \frac{1}{\gamma})\beta_t^2(T-t)])^2}{\beta_t^2 A_t^2}}, \quad (\text{A.76})$$

where the positive deterministic process A^2 is given by

$$A_t^2 = \frac{1}{\gamma} + \left(1 - \frac{1}{\gamma} \right) \frac{\beta_T^2}{\beta_t^2}.$$

Proof of Lemma 2. We first express $\eta_T(\theta)^{\frac{1}{\gamma}}$ in terms of D_T . Using (A.7) and the lognormality of D_T , we have

$$\eta_T(\theta)^{\frac{1}{\gamma}} = e^{-\frac{\theta}{\gamma\sigma^2}(\mu - \frac{1}{2}\sigma^2)T - \frac{1}{2\gamma} \frac{\theta^2}{\sigma^2} T} D_T^{\frac{\theta}{\gamma\sigma^2}}.$$

Therefore, the required expectation becomes

$$\mathbb{E}_t \left[\eta_T(\theta)^{\frac{1}{\gamma}} D_T^{1-\gamma} M_T^{\gamma-1} \right] = e^{-\frac{\theta}{\gamma\sigma^2}(\mu - \frac{1}{2}\sigma^2)T - \frac{1}{2\gamma} \frac{\theta^2}{\sigma^2} T} \mathbb{E}_t \left[D_T^{1-\gamma + \frac{\theta}{\gamma\sigma^2}} M_T^{\gamma-1} \right]. \quad (\text{A.77})$$

Letting $a = 1 - \gamma + \frac{\theta}{\gamma\sigma^2}$ and $b = \gamma - 1$ in Lemma 1 of the technical Appendix A.3 yields

$$\begin{aligned} \mathbb{E}_t \left[D_T^{1-\gamma + \frac{\theta}{\gamma\sigma^2}} M_T^{\gamma-1} \right] &= D_t^{1-\gamma + \frac{\theta}{\gamma\sigma^2}} M_t^{\gamma-1} \frac{1}{A_t} \left(\frac{\beta_T}{\beta_t} \right)^{(\gamma-1)} e^{(1-\gamma + \frac{\theta}{\gamma\sigma^2})(\mu - \frac{1}{2}\sigma^2)(T-t)} e^{-\frac{\gamma-1}{2} \frac{\alpha_t^2}{\beta_t^2} \left(1 - \frac{\beta_T^2}{\beta_t^2} \frac{1}{A_t^2}\right)} \\ &\quad \times e^{\frac{\alpha_t}{A_t^2} (1 - \frac{1}{\gamma}) \left(1 - \gamma + \frac{\theta}{\gamma\sigma^2}\right) \frac{\beta_T^2}{\beta_t^2} (T-t)} e^{\frac{1}{2} \frac{1}{A_t^2} \left(1 - \gamma + \frac{\theta}{\gamma\sigma^2}\right)^2 \sigma^2 (T-t)}, \end{aligned} \quad (\text{A.78})$$

where, for notational simplicity, we have defined the deterministic positive process

$$A_t^2 \equiv \frac{1}{\gamma} + \left(1 - \frac{1}{\gamma} \right) \frac{\beta_T^2}{\beta_t^2}.$$

Substituting the equality

$$D_t^{\frac{\theta}{\gamma\sigma^2}} = e^{\frac{\theta}{\gamma\sigma^2}(\mu - \frac{1}{2}\sigma^2)t + \theta\left(\frac{\alpha_t}{\beta_t^2} - \frac{\alpha_o}{\beta_o^2}\right)},$$

and (A.78) into (A.77) and rearranging leads to

$$\begin{aligned} \mathbb{E}_t \left[\eta_T (\theta)^{\frac{1}{\gamma}} D_T^{1-\gamma} M_T^{\gamma-1} \right] &= D_t^{1-\gamma} M_t^{\gamma-1} \frac{1}{A_t} \left(\frac{\beta_T}{\beta_t} \right)^{(\gamma-1)} e^{(1-\gamma)(\mu - \frac{1}{2}\sigma^2)(T-t)} \\ &\times e^{-\frac{1}{2}\frac{\theta^2}{\gamma\sigma^2}T} e^{\theta\left(\frac{\alpha_t}{\beta_t^2} - \frac{\alpha_o}{\beta_o^2}\right)} e^{-\frac{\gamma-1}{2}\frac{\alpha_t^2}{\beta_t^2}\left(1 - \frac{\beta_T^2}{\beta_t^2} \frac{1}{A_t^2}\right)} \\ &\times e^{\frac{\alpha_t}{A_t^2}\left(1 - \frac{1}{\gamma}\right)\left(1 - \gamma + \frac{\theta}{\gamma\sigma^2}\right)\frac{\beta_T^2}{\beta_t^2}(T-t)} e^{\frac{1}{2}\frac{1}{A_t^2}\left(1 - \gamma + \frac{\theta}{\gamma\sigma^2}\right)^2\sigma^2(T-t)}. \end{aligned} \quad (\text{A.79})$$

We then substitute

$$\xi_t S_t = \mathbb{E}_t \left[D_T^{1-\gamma} M_T^\gamma \right] = D_t^{1-\gamma} M_t^\gamma \left(\frac{\beta_T}{\beta_t} \right)^{\gamma-1} e^{(1-\gamma)(\mu - \frac{1}{2}\sigma^2)(T-t)} e^{(1-\gamma)\alpha_t(T-t)} e^{\frac{1}{2}(1-\gamma)^2\sigma^2\frac{\beta_t^2}{\beta_T^2}(T-t)},$$

into (A.79) to obtain

$$\begin{aligned} \mathbb{E}_t \left[\eta_T (\theta)^{\frac{1}{\gamma}} D_T^{1-\gamma} M_T^{\gamma-1} \right] &= M_t^{-1} \frac{\xi_t S_t}{A_t} e^{(\gamma-1)\alpha_t(T-t)} e^{-\frac{1}{2}(1-\gamma)^2\sigma^2\frac{\beta_t^2}{\beta_T^2}(T-t)} \\ &\times e^{-\frac{1}{2}\frac{\theta^2}{\gamma\sigma^2}T} e^{\theta\left(\frac{\alpha_t}{\beta_t^2} - \frac{\alpha_o}{\beta_o^2}\right)} e^{-\frac{\gamma-1}{2}\frac{\alpha_t^2}{\beta_t^2}\left(1 - \frac{\beta_T^2}{\beta_t^2} \frac{1}{A_t^2}\right)} \\ &\times e^{\frac{\alpha_t}{A_t^2}\left(1 - \frac{1}{\gamma}\right)\left(1 - \gamma + \frac{\theta}{\gamma\sigma^2}\right)\frac{\beta_T^2}{\beta_t^2}(T-t)} e^{\frac{1}{2}\frac{1}{A_t^2}\left(1 - \gamma + \frac{\theta}{\gamma\sigma^2}\right)^2\sigma^2(T-t)}. \end{aligned} \quad (\text{A.80})$$

Moreover, substituting the equalities

$$1 - \frac{\beta_T^2}{\beta_t^2} \frac{1}{A_t^2} = \frac{1}{A_t^2} \frac{\beta_T^2}{\gamma^2\sigma^2} (T-t),$$

$$\left(1 - \frac{1}{\gamma}\right) \frac{\beta_T^2}{\gamma\sigma^2} (T-t) = 1 - A_t^2,$$

and (A.9) into (A.80) yields

$$\begin{aligned} \mathbb{E}_t \left[\eta_T (\theta)^{\frac{1}{\gamma}} D_T^{1-\gamma} M_T^{\gamma-1} \right] &= \xi_t S_t K^{-1} \left(\frac{1}{\sqrt{2\pi\beta_o^2}} e^{-\frac{1}{2}\frac{(\theta - \alpha_o)^2}{\beta_o^2}} \right)^{-1} \frac{1}{\sqrt{2\pi\beta_t^2 A_t^2}} \\ &\times e^{-\frac{1}{2}\frac{(\theta - \alpha_o)^2}{\beta_o^2}} e^{\frac{1}{2}\frac{\alpha_o^2}{\beta_o^2} - \frac{1}{2}\frac{\alpha_t^2}{\beta_t^2}} e^{(\gamma-1)\alpha_t(T-t)} e^{-\frac{1}{2}(1-\gamma)^2\sigma^2\frac{\beta_t^2}{\beta_T^2}(T-t)} \\ &\times e^{-\frac{1}{2}\frac{\theta^2}{\gamma\sigma^2}T} e^{\theta\left(\frac{\alpha_t}{\beta_t^2} - \frac{\alpha_o}{\beta_o^2}\right)} e^{-\frac{\gamma-1}{2}\frac{\alpha_t^2}{\beta_t^2} \frac{1}{A_t^2} \frac{\beta_T^2}{\gamma^2\sigma^2}(T-t)} \\ &\times e^{\frac{\alpha_t}{A_t^2}\left(1 - \gamma + \frac{\theta}{\gamma\sigma^2}\right)\frac{\gamma\sigma^2}{\beta_t^2}(1 - A_t^2)} e^{\frac{1}{2}\frac{1}{A_t^2}\left(1 - \gamma + \frac{\theta}{\gamma\sigma^2}\right)^2\sigma^2(T-t)}. \end{aligned}$$

We note that the last three rows in the above equation is equal to

$$e^{-\frac{1}{2} \frac{(\theta - [\alpha_t - (1 - \frac{1}{\gamma}) \beta_t^2 (T-t)])^2}{\beta_t^2 A_t^2}},$$

and so we obtain (A.76). \square

Lemma 3. *Let the processes M , α and β be as in (B.7), (B.8) and (B.9), respectively. Then for all numbers a and b we have*

$$\begin{aligned} \mathbb{E}_t [D_T^a M_T^b] &= D_t^a M_t^b \left(\frac{\beta_T}{\beta_t} \right)^b \left(\frac{s_T}{s_t} \right)^{\frac{b}{\gamma}} e^{a(\mu - \frac{1}{2}\sigma^2)(T-t)} e^{-\frac{1}{2} \frac{b}{\gamma} \left(\frac{\hat{\alpha}_t}{\sigma} \right)^2 \frac{\beta_T^2}{\beta_t^2} \frac{s_T^2}{s_t^2} (T-t)} \\ &\quad \times \left[1 - \frac{b}{\gamma} \left(1 - \frac{\beta_T^2}{\beta_t^2} \frac{s_T^2}{s_t^2} \right) \right]^{-\frac{1}{2}} e^{\frac{1}{2} \left[1 - \frac{b}{\gamma} \left(1 - \frac{\beta_T^2}{\beta_t^2} \frac{s_T^2}{s_t^2} \right) \right]^{-1} \left(\frac{b}{\gamma} \frac{\hat{\alpha}_t}{\sigma} \frac{\beta_T^2}{\beta_t^2} \frac{s_T^2}{s_t^2} + a \right)^2 \sigma^2 (T-t)} \end{aligned} \quad (\text{A.81})$$

provided $1 - \frac{b}{\gamma} \left(1 - \frac{\beta_T^2}{\beta_t^2} \frac{s_T^2}{s_t^2} \right) > 0$.

Proof of Lemma 3. By (B.7), we have

$$M_T = M_t \left(\frac{\beta_T}{\beta_t} \right) \left(\frac{s_T}{s_t} \right)^{\frac{1}{\gamma}} e^{-\frac{1}{2} \frac{\alpha_t^2}{\beta_t^2} + \frac{1}{2} \frac{\alpha_T^2}{\beta_T^2} + \frac{1}{2} \gamma s^2 \left[\frac{s_T^2}{s_t^2} \left(\frac{\alpha_T}{\beta_T} - \frac{\alpha_t}{\beta_t} \right)^2 - \frac{s_T^2}{s_t^2} \left(\frac{\alpha_t}{\beta_t} - \frac{\alpha_0}{\beta_0} \right)^2 \right]},$$

which after rearranging this and using (B.8)–(B.9) lead to

$$M_T = M_t \left(\frac{\beta_T}{\beta_t} \right) \left(\frac{s_T}{s_t} \right)^{\frac{1}{\gamma}} e^{G_{0t}(T-t) + G_{1t} \frac{(\omega_T - \omega_t)}{\sigma} + \frac{1}{2} G_{2t} \frac{(\omega_T - \omega_t)^2}{\sigma^2}},$$

where the time- t measurable processes G_0 , G_1 , and G_2 are defined as

$$G_{0t} \equiv -\frac{1}{2} \frac{1}{\gamma} \left(\frac{\hat{\alpha}_t}{\sigma} \right)^2 \frac{\beta_T^2}{\beta_t^2} \frac{s_T^2}{s_t^2}, \quad G_{1t} \equiv \frac{\hat{\alpha}_t}{\gamma} \frac{\beta_T^2}{\beta_t^2} \frac{s_T^2}{s_t^2}, \quad G_{2t} \equiv \frac{\beta_T^2}{\gamma^2} \frac{s_T^4}{s_t^4} + \frac{s_T^2}{\gamma},$$

and $\hat{\alpha}_t$ is given by (B.12). Therefore, using the lognormality of D_T we obtain the expectation

$$\mathbb{E}_t [D_T^a M_T^b] = D_t^a e^{a(\mu - \frac{1}{2}\sigma^2)(T-t)} M_t^b \left(\frac{\beta_T}{\beta_t} \right)^b \left(\frac{s_T}{s_t} \right)^{\frac{b}{\gamma}} e^{bG_{0t}(T-t)} \mathbb{E}_t \left[e^{(b \frac{G_{1t}}{\sigma} + a\sigma)(\omega_T - \omega_t) + \frac{1}{2} b \frac{G_{2t}}{\sigma^2} (\omega_T - \omega_t)^2} \right]. \quad (\text{A.82})$$

To compute the expectation in (A.82), we let $Z \sim \mathcal{N}(0, 1)$ and use the independence property of conditional expectations to obtain

$$\begin{aligned} \mathbb{E}_t \left[e^{(b \frac{G_{1t}}{\sigma} + a\sigma)(\omega_T - \omega_t) + \frac{1}{2} b \frac{G_{2t}}{\sigma^2} (\omega_T - \omega_t)^2} \right] &= \mathbb{E} \left[e^{(b \frac{G_{1t}}{\sigma} + a\sigma) \sqrt{T-t} Z + \frac{1}{2} b \frac{G_{2t}}{\sigma^2} (T-t) Z^2} \right] \\ &= \left[1 - b \frac{G_{2t}}{\sigma^2} (T-t) \right]^{-\frac{1}{2}} e^{\frac{1}{2} [1 - b \frac{G_{2t}}{\sigma^2} (T-t)]^{-1} (b \frac{G_{1t}}{\sigma} + a\sigma)^2 (T-t)}. \end{aligned}$$

Substituting the last line into (A.82) and rearranging gives (A.81). \square

B Internet Appendix

B.1 Proofs of Bayesian Economy

Proof of Proposition 6. We proceed by first finding the optimal estimate of the mean return of the expected payoff μ for each θ -type investor and elicit their respective beliefs. We then proceed as in the proof of Proposition 1 in Appendix A.1.

Following the standard filtering theory (see, Liptser and Shiryaev (2001)), for each θ -type investor, the time- t posterior distribution of μ , conditional on the information set $\mathcal{F}_t = \{D_u : 0 \leq u \leq t\}$, is normally distributed, $\mathcal{N}(\mu + \hat{\theta}_t, s_t^2)$, where the time- t bias of θ -type investor $\hat{\theta}_t$ and the type-independent mean squared error s_t^2 are given by

$$d\hat{\theta}_t = \frac{s_t^2}{\sigma} d\omega_t(\theta), \quad (\text{B.1})$$

$$s_t^2 = \frac{s^2 \sigma^2}{\sigma^2 + s^2 t}, \quad (\text{B.2})$$

and her perceived Brownian motion process $\omega(\theta)$ is given by

$$d\omega_t(\theta) = \frac{1}{\sigma} \left(\frac{dD_t}{D_t} - (\mu + \hat{\theta}_t) dt \right) = d\omega_t - \frac{\hat{\theta}_t}{\sigma} dt,$$

which implies the likelihood process $\eta(\theta)$ of

$$\begin{aligned} d\eta_t(\theta) &= \eta_t(\theta) \frac{\hat{\theta}_t}{\sigma} d\omega_t, \\ \eta_t(\theta) &= e^{\int_0^t \frac{\hat{\theta}_u}{\sigma} d\omega_u - \frac{1}{2} \int_0^t \left(\frac{\hat{\theta}_u}{\sigma} \right)^2 du}. \end{aligned} \quad (\text{B.3})$$

Solution to (B.1) (θ -type investor's bias at time t) is given by

$$\hat{\theta}_t = \sigma \frac{\sigma \theta + s^2 \omega_t}{\sigma^2 + s^2 t} = \frac{s_t^2}{s^2} \theta + \frac{s_t^2}{\sigma} \omega_t.$$

Substituting this into (B.3) and solving yields the likelihood process as

$$\eta_t(\theta) = \frac{s_t}{s} e^{-\frac{1}{2} \frac{\theta^2}{s^2} + \frac{1}{2} \frac{\hat{\theta}_t^2}{s_t^2}}. \quad (\text{B.4})$$

Each θ -investor's maximization problem

$$\mathbb{E}^\theta \left[\frac{W_T^{1-\gamma}}{1-\gamma} \right],$$

subject to the budget constraint

$$\begin{aligned} dW_t(\theta) &= \phi_t(\theta) W_t(\theta) (\mu_{St}(\theta) dt + \sigma_{St} d\omega_t(\theta)), \\ \mu_{St}(\theta) &= \mu_{St} + \sigma_{St} \frac{\widehat{\theta}_t}{\sigma}, \end{aligned}$$

where the expectation is taken such that $\omega(\theta)$ is a standard Brownian motion, can be solved using standard martingale techniques.

Following similar steps as in the proof of Proposition 1 in Appendix A.1, we obtain the optimal horizon wealth of each θ -type as

$$W_T(\theta) = \left(\frac{\eta_T(\theta)}{y(\theta) \xi_T} \right)^{\frac{1}{\gamma}}, \quad (\text{B.5})$$

where $\eta_T(\theta)$ is (B.4) evaluated at time T and the Lagrange multiplier $y(\theta)$ solves

$$y(\theta)^{-\frac{1}{\gamma}} = \mathbb{E} \left[\eta_T(\theta)^{\frac{1}{\gamma}} \xi_T^{1-\frac{1}{\gamma}} \right]^{-1} \frac{\xi_0 S_0}{\sqrt{2\pi \tilde{v}^2}} e^{-\frac{1}{2} \frac{(\theta - \tilde{m})^2}{\tilde{v}^2}}.$$

The time- T equilibrium state price density ξ_T is again given by $\xi_T = D_T^{-\gamma} M_T^\gamma$, where the auxiliary process M is defined as $M_t \equiv \int_{\Theta} y(\theta)^{-\frac{1}{\gamma}} \eta_t(\theta)^{\frac{1}{\gamma}} d\theta$. As we show below, $y(\theta)^{-\frac{1}{\gamma}}$ is (scaled) Gaussian over the type space Θ for some mean α_o , variance β_o^2 and a constant K :

$$y(\theta)^{-\frac{1}{\gamma}} = K \frac{1}{\sqrt{2\pi \beta_o^2}} e^{-\frac{1}{2} \frac{(\theta - \alpha_o)^2}{\beta_o^2}}. \quad (\text{B.6})$$

Substituting (B.4) and (B.6) into the definition of M yields

$$M_t = K \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi \beta_o^2}} e^{-\frac{1}{2} \frac{(\theta - \alpha_o)^2}{\beta_o^2}} \left(\frac{s_t}{s} \right)^{\frac{1}{\gamma}} e^{-\frac{1}{2} \frac{1}{\gamma} \frac{\theta^2}{s^2} + \frac{1}{2} \frac{1}{\gamma} \frac{\tilde{\theta}_t^2}{s_t^2}} d\theta = K \frac{\beta_t}{\beta_o} e^{-\frac{1}{2} \frac{\alpha_o^2}{\beta_o^2} + \frac{1}{2} \frac{\alpha_t^2}{\beta_t^2}} \left(\frac{s_t}{s} \right)^{\frac{1}{\gamma}} e^{\frac{1}{2} \frac{s_t^2}{\gamma} \frac{\omega_t^2}{\sigma^2}}, \quad (\text{B.7})$$

where the last equality follows by completing the square and integrating, and the processes α and β are

$$\alpha_t = \frac{(\sigma^2 + s^2 t) \alpha_o + \sigma \frac{1}{\gamma} \beta_o^2 \omega_t}{(\sigma^2 + s^2 t) + \frac{1}{\gamma} \beta_o^2 t}, \quad (\text{B.8})$$

$$\beta_t^2 = \frac{\beta_o^2 (\sigma^2 + s^2 t)}{(\sigma^2 + s^2 t) + \frac{1}{\gamma} \beta_o^2 t}, \quad (\text{B.9})$$

with their initial values given by $\alpha_0 = \alpha_o$ and $\beta_0 = \beta_o$, respectively.

Following similar steps as in the proof of Proposition 1 in Appendix A.1, we verify $y(\theta)^{-\frac{1}{\gamma}}$

is as in (B.6) with

$$\alpha_o = \tilde{m} + \left(1 - \frac{1}{\gamma}\right) \beta_o^2 T, \quad (\text{B.10})$$

$$\beta_o^2 = \left(\frac{\gamma}{2} \tilde{v}^2 - \frac{\gamma^2}{2T} (\sigma^2 + s^2 T)\right) + \sqrt{\left(\frac{\gamma}{2} \tilde{v}^2 - \frac{\gamma^2}{2T} (\sigma^2 + s^2 T)\right)^2 + \frac{\gamma^2}{T} \tilde{v}^2 (\sigma^2 + s^2 T)}. \quad (\text{B.11})$$

The construction of the representative investor is as in the dogmatic beliefs case in the proof of Proposition 1 in Appendix A.1, which in this case yields the time- t average bias in beliefs m_t as

$$m_t = \hat{\alpha}_t = \sigma \frac{\sigma \alpha_o + \left(\frac{1}{\gamma} \beta_o^2 + s^2\right) \omega_t}{\sigma^2 + \left(\frac{1}{\gamma} \beta_o^2 + s^2\right) t} = \int_{\Theta} \hat{\theta}_t \left(\frac{y(\theta)^{-\frac{1}{\gamma}} \eta_t(\theta)^{\frac{1}{\gamma}}}{\int_{\Theta} y(\theta)^{-\frac{1}{\gamma}} \eta_t(\theta)^{\frac{1}{\gamma}} d\theta} \right) d\theta. \quad (\text{B.12})$$

Substituting $\sigma \omega_t = \ln D_t - \left(\mu - \frac{1}{2} \sigma^2\right) t$ yields the expression stated in (30).

From the last equality in (B.12) we identify the unique weights $h_t(\theta)$ such that the weighted-average of investors' biases equals to the average bias in beliefs as

$$h_t(\theta) = \frac{y(\theta)^{-\frac{1}{\gamma}} \eta_t(\theta)^{\frac{1}{\gamma}}}{\int_{\Theta} y(\theta)^{-\frac{1}{\gamma}} \eta_t(\theta)^{\frac{1}{\gamma}} d\theta}. \quad (\text{B.13})$$

Substituting (B.4) and (B.6) into (B.13), and rearranging yields

$$h_t(\theta) = \frac{1}{\sqrt{2\pi} \beta_t^2} e^{-\frac{1}{2} \frac{(\theta - \alpha_t)^2}{\beta_t^2}} = \frac{1}{\sqrt{2\pi} \beta_t^2 (s_t^4/s^4)} e^{-\frac{1}{2} \frac{(\hat{\theta}_t - \hat{\alpha}_t)^2}{\beta_t^2 (s_t^4/s^4)}} \frac{s_t^2}{s^2}, \quad (\text{B.14})$$

where α_t and β_t are as in (B.8) and (B.9), respectively.

Finally, to determine the belief dispersion, we use definition (29) with the average bias in beliefs (B.12) and weights (B.14) substituted in to obtain

$$v_t^2 = \int_{\Theta} \left(\hat{\theta}_t - m_t\right)^2 h_t(\theta) d\theta = \int_{-\infty}^{\infty} \left(\hat{\theta}_t - \hat{\alpha}_t\right)^2 \frac{1}{\sqrt{2\pi} \beta_t^2 (s_t^4/s^4)} e^{-\frac{1}{2} \frac{(\hat{\theta}_t - \hat{\alpha}_t)^2}{\beta_t^2 (s_t^4/s^4)}} \frac{s_t^2}{s^2} \left(\frac{s_t^2}{s^2}\right)^{-1} d\hat{\theta}_t.$$

This gives the (squared) dispersion in our economy as

$$v_t^2 = \beta_t^2 \frac{s_t^4}{s^4}.$$

By equating the initial values α_o and β_o^2 to m and v^2 in (B.10)–(B.11), respectively we obtain the (squared) dispersion and the weights in the economy as in (31) and (35).

The condition for property (i) that the average bias in beliefs is increasing in parameter uncertainty follows from the partial derivative of (30) with respect to s_t (or, equivalently with

respect to s^2 , since $\partial s^2/\partial s_t > 0$). This property holds when $D_t > \exp\left(\left(m + \mu - \frac{1}{2}\sigma^2\right)t\right)$, since

$$\frac{\partial m_t}{\partial s^2} = \left(\ln D_t - \left(m + \mu - \frac{1}{2}\sigma^2\right)t\right) \frac{\sigma^2}{\left(\sigma^2 + \left(\frac{1}{\gamma}v^2 + s^2\right)t\right)^2}. \quad (\text{B.15})$$

The property (ii) that the dispersion in beliefs is decreasing in parameter uncertainty follows from the partial derivative of (31) with respect to s^2 , since

$$\frac{\partial v_t^2}{\partial s^2} = -v^2\sigma^2 \frac{2(\sigma^2 + s^2t) + \frac{1}{\gamma}v^2t}{(\sigma^2 + s^2t)^2} \frac{\sigma^2}{\left(\sigma^2 + \left(\frac{1}{\gamma}v^2 + s^2\right)t\right)^2} t < 0.$$

□

Proof of Proposition 7. To determine the stock price, we first compute the equilibrium state price density at time t by using the fact that it is a martingale, $\xi_t = \mathbb{E}_t[\xi_T]$. The equilibrium state price density at time T is as in the proof of Proposition 6 in Appendix B.1, given by $\xi_T = D_T^{-\gamma} M_T^\gamma$. Hence,

$$\begin{aligned} \xi_t &= \mathbb{E}_t[D_T^{-\gamma} M_T^\gamma] \\ &= D_t^{-\gamma} M_t^\gamma \left(\frac{v_T s_t^2}{v_t s_T^2}\right)^{\gamma-1} e^{-\left[\gamma\left(\mu - \frac{1}{2}\sigma^2\right) + \frac{1}{2}\left(\frac{m_t}{\sigma}\right)^2 \frac{v_T^2 s_t^2}{v_t^2 s_T^2} - \frac{1}{2}\sigma^2 \frac{v_T^2 s_T^2}{v_t^2 s_t^2} \left(\frac{m_t}{\sigma^2} \frac{v_T^2 s_t^2}{v_t^2 s_T^2} - \gamma\right)\right](T-t)}, \end{aligned} \quad (\text{B.16})$$

where the last equality follows from Lemma 3 of the technical Appendix A.3 by taking $a = -\gamma$ and $b = \gamma$ and using the equalities $m_t = \hat{\alpha}_t$ and $v_t = \beta_t(s_t^2/s^2)$ to express the equation in terms of model parameters.

Next, we compute the expectation $\mathbb{E}_t[\xi_T D_T] = \mathbb{E}_t[D_T^{1-\gamma} M_T^\gamma]$. Again, employing Lemma 3 of the technical Appendix A.3 with $a = 1 - \gamma$ and $b = \gamma$, we obtain

$$\mathbb{E}_t[D_T^{1-\gamma} M_T^\gamma] = D_t^{1-\gamma} M_t^\gamma \left(\frac{v_T s_t^2}{v_t s_T^2}\right)^{\gamma-1} e^{(1-\gamma)\left[\left(\mu - \frac{1}{2}\sigma^2\right) + m_t + \frac{1}{2}(1-\gamma)\sigma^2 \frac{v_T^2 s_T^2}{v_t^2 s_t^2}\right](T-t)}. \quad (\text{B.17})$$

By no arbitrage, the stock price in our complete market economy is again given by (A.26). Substituting (B.16) and (B.17) into (A.26) and manipulating yields (37).

To determine the mean return, we apply Itô's Lemma to the stock price (37) and obtain

$$\frac{dS_t}{S_t} = \left[\left(\gamma\sigma \frac{v_t^2 s_T^2}{v_T^2 s_t^2} - \frac{m_t}{\sigma}\right) \sigma \frac{v_t^2 s_T^2}{v_T^2 s_t^2}\right] dt + \left[\sigma + \frac{1}{\sigma} \left(\frac{1}{\gamma}v^2 + s^2\right) \frac{v_t^2 s_T^2}{v^2 s_t^2} (T-t)\right] d\omega_t. \quad (\text{B.18})$$

The drift term in (B.18) gives the equilibrium mean return.

The condition for property (i) that the stock price is increasing in parameter uncertainty follows from the partial derivative of (37) with respect to s_t^2 (or, equivalently with respect to s^2 , since $\partial s^2/\partial s_t > 0$). This property holds if and only if $\partial S_t/\partial s^2 > 0$, which is given by the

condition

$$\frac{\partial}{\partial s^2} m_t > \frac{1}{2} (2\gamma - 1) (T - t) \frac{\partial}{\partial s^2} \left(\left(\frac{1}{\gamma} v^2 + s^2 \right) \frac{v_t^2 s^2}{v^2 s_t^2} \right). \quad (\text{B.19})$$

Substituting (B.15) and

$$\frac{\partial}{\partial s^2} \left(\left(\frac{1}{\gamma} v^2 + s^2 \right) \frac{v_t^2 s^2}{v^2 s_t^2} \right) = \frac{\sigma^4}{\left(\sigma^2 + \left(\frac{1}{\gamma} v^2 + s^2 \right) t \right)^2},$$

into (B.19) and rearranging gives the required condition.

The condition for property (ii) that the mean return is decreasing in parameter uncertainty follows from the partial derivative of (38) with respect to s^2 . This property holds if and only if $\partial \mu_{St} / \partial s^2 < 0$, which is given by the condition

$$2\gamma\sigma^2 \frac{v_t^2 s_T^2}{v_T^2 s_t^2} \frac{\partial}{\partial s^2} \left(\frac{v_t^2 s_T^2}{v_T^2 s_t^2} \right) < \frac{v_t^2 s_T^2}{v_T^2 s_t^2} \frac{\partial}{\partial s^2} m_t + m_t \frac{\partial}{\partial s^2} \left(\frac{v_t^2 s_T^2}{v_T^2 s_t^2} \right),$$

substituting (B.15) along with

$$\frac{\partial}{\partial s^2} \left(\frac{v_t^2 s_T^2}{v_T^2 s_t^2} \right) = \frac{\sigma^2 (T - t)}{\left(\sigma^2 + \left(\frac{1}{\gamma} v^2 + s^2 \right) t \right)^2},$$

and rearranging gives the required condition.

We now prove that properties in Propositions 2–3 also hold in this more general economy. The condition for property that the stock price is higher than in the benchmark economy follows immediately by comparing (20) and (37). The property that the stock price is convex in cash-flow news follows once we substitute (30) into the stock price equation (37). The condition for property that the stock price is increasing in belief dispersion follows from the partial derivative of (37) with respect to v_t . This property holds when

$$m_t > \tilde{m} + \frac{1}{2} (2\gamma - 1) \left(\frac{1}{\gamma} v^2 + s^2 \right) \frac{v_t^2 s^2}{v^2 s_t^2} (T - t) - (\gamma - 1) s^2 T.$$

The property that the stock price is increasing in investors' risk aversion for relatively bad cash-flow states and low values of γ follows from the partial derivative of (37) with respect to γ . This property holds when

$$m_t < m \frac{\sigma^2}{\sigma^2 + \left(\frac{1}{\gamma} v^2 + s^2 \right) t} + \frac{K_{1t} - K_{2t}}{\frac{\sigma^2}{\left(\sigma^2 + \left(\frac{1}{\gamma} v^2 + s^2 \right) t \right)^3} \gamma^2 \left(\frac{1}{\gamma} v^2 + s^2 \right)} \left(1 - \frac{\gamma}{v^2} \frac{\partial v^2}{\partial \gamma} \right),$$

where the deterministic processes K_1, K_2 are defined as

$$K_{1t} \equiv \sigma^2 \frac{\left(\sigma^2 + \left(\frac{1}{\gamma}v^2 + s^2\right)t\right) \left(\frac{1}{\gamma}v^2 + 1 - \gamma\right) T + m \frac{v^2}{\gamma^2} \left(1 - \frac{\gamma}{v^2} \frac{\partial v^2}{\partial \gamma}\right) t}{\left(\sigma^2 + \left(\frac{1}{\gamma}v^2 + s^2\right)t\right)^2}, \quad (\text{B.20})$$

$$K_{2t} \equiv \sigma^2 \frac{v_t^2 s_T^2}{v_T^2 s_t^2} - \frac{1}{2} (2\gamma - 1) \frac{v^2}{\gamma^2} \frac{\sigma^8}{\left(\sigma^2 + \left(\frac{1}{\gamma}v^2 + s^2\right)t\right)^4} \left(1 - \frac{\gamma}{v^2} \frac{\partial v^2}{\partial \gamma}\right) (T - t),$$

and

$$1 - \frac{\gamma}{v^2} \frac{\partial v^2}{\partial \gamma} = \frac{1}{2} \frac{\gamma^2 (\sigma^2 + s^2 T)}{T v^2} \left[1 + \frac{\left(\frac{\gamma}{2} \tilde{v}^2 - \frac{\gamma^2}{2T} (\sigma^2 + s^2 T)\right)}{\sqrt{\left(\frac{\gamma}{2} \tilde{v}^2 - \frac{\gamma^2}{2T} (\sigma^2 + s^2 T)\right)^2 + \frac{\gamma^2}{T} \tilde{v}^2 (\sigma^2 + s^2 T)}} \right] > 0. \quad (\text{B.21})$$

Similarly, the condition for property that the mean return is lower than in the benchmark economy follows from comparing (23) and (38). The condition for property that the mean return is decreasing in belief dispersion follows from the partial derivative of (38) with respect to v_t . This property holds when

$$m_t > \frac{\tilde{m} + 2\gamma\sigma^2 \left(\frac{v_t^2 s_T^2}{v_T^2 s_t^2} - 1\right) - (\gamma - 1) s^2 T}{2 - \frac{v_T^2 s_T^2}{v_t^2 s_t^2}}.$$

The property that the mean return is decreasing in investors' risk aversion for relatively bad cash-flow news and low levels of risk aversion follows from the partial derivative of (38) with respect to γ . This property holds when

$$m_t < \frac{\sigma \frac{v_t^2 s_T^2}{v_T^2 s_t^2} \left(K_{1t} \frac{(\sigma^2 + (\frac{1}{\gamma}v^2 + s^2)t)^2}{\sigma^2} + K_{3t}\right) - K_{4t}}{\frac{v_t^4 \sigma}{v^2 \gamma^2} \frac{s^4}{s_t^4} \left(2(T - t) + \frac{\sigma^2 + (\frac{1}{\gamma}v^2 + s^2)t}{\frac{1}{\gamma}v^2 + s^2}\right) \left(1 - \frac{\gamma}{v^2} \frac{\partial v^2}{\partial \gamma}\right)},$$

where K_1 is as in (B.20) and the deterministic processes K_3, K_4 are defined as

$$K_{3t} \equiv m \frac{\sigma^6}{\left(\sigma^2 + \left(\frac{1}{\gamma}v^2 + s^2\right)t\right)^2} \frac{v^2}{\gamma^2 \left(\frac{1}{\gamma}v^2 + s^2\right)} \left(1 - \frac{\gamma}{v^2} \frac{\partial v^2}{\partial \gamma}\right),$$

$$K_{4t} \equiv \sigma \left(\sigma^2 + \left(\frac{1}{\gamma}v^2 + s^2\right)T\right)^2 + 2 \frac{v_t^6 s_T^2 s^4 \sigma^3}{v_T^2 s_t^6 v^2 \gamma} \left(1 - \frac{\gamma}{v^2} \frac{\partial v^2}{\partial \gamma}\right) (T - t).$$

□

Proof of Proposition 8. The volatility of the stock is given readily by the diffusion term in the dynamics (B.18).

To compute the trading volume measure V , we proceed as in the proof of Proposition 5 in

Appendix A.1 by first determining the dynamics of each θ -type investor's equilibrium wealth-share $W(\theta)/S$ and portfolio $\phi(\theta)$ as the fraction of wealth invested in the stock. Then, we apply the product rule to $\psi(\theta) = \phi(\theta)(W(\theta)/S)$ to obtain the dynamics of the number of shares invested in the stock $\psi(\theta)$. Finally, using the definition (26) we obtain the trading volume measure V in our economy.

We begin by first computing each θ -type investor's wealth share $W(\theta)/S$. The time- t wealth of θ -type investor is given by

$$\begin{aligned}\xi_t W_t(\theta) &= \mathbb{E}_t[\xi_T W_T(\theta)] \\ &= K \frac{1}{\sqrt{2\pi v^2}} e^{-\frac{1}{2} \frac{(\theta-m)^2}{v^2}} \mathbb{E}_t \left[\eta_T(\theta)^{\frac{1}{\gamma}} D_T^{1-\gamma} M_T^{\gamma-1} \right],\end{aligned}\quad (\text{B.22})$$

where the second equality follows by substituting (B.5), (B.6) and $\xi_T = D_T^{-\gamma} M_T^\gamma$. We have also employed the equalities $\alpha_o = m$, $\beta_o^2 = v^2$ (see, proof of Proposition 6 in Appendix B.1), to express (B.22) in terms of the model parameters. Using similar steps as in the proof of Lemma 2 in the technical Appendix A.3, the expectation in (B.22) can be shown to

$$\mathbb{E}_t \left[\eta_T(\theta)^{\frac{1}{\gamma}} D_T^{1-\gamma} M_T^{\gamma-1} \right] = \xi_t S_t K^{-1} \left(\frac{1}{\sqrt{2\pi v^2}} e^{-\frac{1}{2} \frac{(\theta-m)^2}{v^2}} \right)^{-1} \frac{1}{\sqrt{2\pi v_t^2 A_t^2}} e^{-\frac{1}{2} \frac{(\widehat{\theta}_t - [m_t - (\frac{1-\frac{1}{\gamma}}{v_t^2 A_t^2}] v_t^2 (T-t))]^2}{v_t^2 A_t^2}} \frac{s_t^2}{s^2},\quad (\text{B.23})$$

where the positive deterministic process A is given by

$$A_t^2 = \frac{1}{\gamma} + \left(1 - \frac{1}{\gamma}\right) \frac{v_T^2 s_t^4}{v_t^2 s_T^4},$$

and m_t and v_t are as in (30)–(31). Substituting (B.23) into (B.22) and rearranging gives the wealth-share of each θ -type investor as

$$\frac{W_t(\theta)}{S_t} = \frac{1}{\sqrt{2\pi v_t^2 A_t^2}} e^{-\frac{1}{2} \frac{(\widehat{\theta}_t - [m_t - (\frac{1-\frac{1}{\gamma}}{v_t^2 A_t^2}] v_t^2 (T-t))]^2}{v_t^2 A_t^2}} \frac{s_t^2}{s^2}.\quad (\text{B.24})$$

Hence the time- t wealth-share weighted average bias and (squared) dispersion are given by

$$\begin{aligned}\tilde{m}_t &\equiv \int_{\Theta} \widehat{\theta}_t \frac{W_t(\theta)}{S_t} d\theta = m_t - \left(1 - \frac{1}{\gamma}\right) v_t^2 (T-t), \\ \tilde{v}_t^2 &\equiv \int_{\Theta} \left(\widehat{\theta}_t - \tilde{m}_t\right)^2 \frac{W_t(\theta)}{S_t} d\theta = v_t^2 A_t^2 = \frac{1}{\gamma} v_t^2 + \left(1 - \frac{1}{\gamma}\right) v_T^2 \frac{s_t^4}{s_T^4},\end{aligned}$$

and applying Itô's lemma to (B.24) yields the dynamics for the wealth-share as

$$d \frac{W_t(\theta)}{S_t} = \dots dt + \frac{W_t(\theta)}{S_t} \frac{v_t^2}{\gamma \sigma \tilde{v}_t^2} \left(\widehat{\theta}_t - \tilde{m}_t\right) d\omega_t.\quad (\text{B.25})$$

To obtain each θ -type investor's optimal portfolio $\phi(\theta)$ as a fraction of wealth invested in

the stock, we match the volatility term in (B.25) with the corresponding one in

$$d\frac{W_t(\theta)}{S_t} = \dots dt + \frac{W_t(\theta)}{S_t} (\phi_t(\theta) - 1) \sigma_{St} d\omega_t,$$

which yields investors' equilibrium portfolios as

$$\phi_t(\theta) = 1 + \frac{v_T^2 s_t^2}{\gamma \sigma^2 \tilde{v}_t^2 s_T^2} (\hat{\theta}_t - \tilde{m}_t). \quad (\text{B.26})$$

Applying Itô's lemma to (B.26) yields the dynamics

$$d\phi_t(\theta) = \dots dt - \frac{v_t^2 v_T^2 s_t^2}{\gamma^2 \sigma^3 \tilde{v}_t^2 s_T^2} d\omega_t.$$

Finally, applying the product rule to $\psi(\theta) = \phi(\theta) (W(\theta)/S)$ yields

$$d\psi_t(\theta) = \dots dt + \frac{W_t(\theta)}{S_t} \frac{v_t^2}{\gamma \sigma \tilde{v}_t^2} \left[(\hat{\theta}_t - \tilde{m}_t) \phi_t(\theta) - \frac{v_T^2 s_t^2}{\gamma \sigma^2 s_T^2} \right] d\omega_t,$$

which after substituting (B.26) gives the portfolio volatility of each θ -type investor $\sigma_\psi(\theta)$ as

$$\sigma_{\psi t}(\theta) = \frac{W_t(\theta)}{S_t} \frac{v_t^2}{\gamma \sigma \tilde{v}_t^2} \left[\frac{v_T^2 s_t^2}{\gamma \sigma^2 s_T^2} \left(\left(\frac{\hat{\theta}_t - \tilde{m}_t}{\tilde{v}_t} \right)^2 - 1 \right) + \tilde{v}_t \left(\frac{\hat{\theta}_t - \tilde{m}_t}{\tilde{v}_t} \right) \right].$$

Following similar steps as in the proof of Proposition 5 in Appendix A.1, we obtain the trading volume measure as

$$\begin{aligned} V_t &= \frac{\sigma}{X_t^2} \frac{v_t^2 s_T^2}{v_T^2 s_t^2} \left(\frac{1}{2} X_t + \frac{1}{2} \sqrt{X_t^2 + 4} \right) \phi \left(\frac{1}{2} X_t - \frac{1}{2} \sqrt{X_t^2 + 4} \right) \\ &\quad - \frac{\sigma}{X_t^2} \frac{v_t^2 s_T^2}{v_T^2 s_t^2} \left(\frac{1}{2} X_t - \frac{1}{2} \sqrt{X_t^2 + 4} \right) \phi \left(\frac{1}{2} X_t + \frac{1}{2} \sqrt{X_t^2 + 4} \right), \end{aligned}$$

where the positive deterministic process X is defined as

$$X_t^2 \equiv \frac{\gamma^2 \sigma^4}{v_T^4} \left[\frac{1}{\gamma} v_t^2 \frac{s_T^4}{s_t^4} + \left(1 - \frac{1}{\gamma} \right) v_T^2 \right].$$

The property (i) that the stock volatility is increasing in parameter uncertainty follows from the partial derivative of (39) with respect to s_t^2 (or, equivalently with respect to s^2 , since $\partial s^2 / \partial s_t > 0$). This property holds since

$$\frac{\partial \sigma_{St}}{\partial s^2} = \frac{\sigma^3 (T-t)}{\left(\sigma^2 + \left(\frac{1}{\gamma} v^2 + s^2 \right) t \right)^2} > 0.$$

Property (ii) that the effect of parameter uncertainty on stock volatility is decreasing as the dispersion increases follows from the cross partial derivative $\partial^2 \sigma_{St} / \partial v^2 \partial s^2$. This property holds since

$$\frac{\partial}{\partial v^2} \left(\frac{\partial \sigma_{St}}{\partial s^2} \right) = -\frac{2}{\gamma} \frac{\sigma^3 t (T-t)}{\left(\sigma^2 + \left(\frac{1}{\gamma} v^2 + s^2 \right) t \right)^3} < 0.$$

The condition for property (iii) that the trading volume is decreasing in parameter uncertainty for $\gamma \geq 1$ follows from the partial derivative of (40) with respect to s^2 . To compute this partial derivative we rewrite (40) as

$$V_t = \frac{1}{2\sigma} \left[Z_t^+ \phi \left(\frac{Z_t^- X_t^2 v_T^2 s_t^2}{2 \sigma^2 v_t^2 s_T^2} \right) + Z_t^- \phi \left(\frac{Z_t^+ X_t^2 v_T^2 s_t^2}{2 \sigma^2 v_t^2 s_T^2} \right) \right], \quad (\text{B.27})$$

where we have defined the positive deterministic processes

$$\begin{aligned} Z_t^+ &\equiv \sqrt{\left(\frac{\sigma^2 v_t^2 s_T^2}{X_t^2 v_T^2 s_t^2} \right)^2 (X_t^2 + 4)} + \frac{\sigma^2 v_t^2 s_T^2}{X_t v_T^2 s_t^2}, \\ Z_t^- &\equiv \sqrt{\left(\frac{\sigma^2 v_t^2 s_T^2}{X_t^2 v_T^2 s_t^2} \right)^2 (X_t^2 + 4)} - \frac{\sigma^2 v_t^2 s_T^2}{X_t v_T^2 s_t^2}, \end{aligned}$$

with $0 < Z_t^- < Z_t^+$, and

$$\frac{\partial}{\partial s^2} Z_t^+ - \frac{\partial}{\partial s^2} Z_t^- = 2 \frac{\partial}{\partial s^2} \left(\frac{\sigma^2 v_t^2 s_T^2}{X_t v_T^2 s_t^2} \right). \quad (\text{B.28})$$

The partial derivative of (B.27) with respect to s^2 is given by

$$\begin{aligned} \frac{\partial}{\partial s^2} V_t &= \frac{1}{2\sigma} \left[\left(\frac{\partial}{\partial s^2} Z_t^+ \right) \phi \left(\frac{Z_t^- X_t^2 v_T^2 s_t^2}{2 \sigma^2 v_t^2 s_T^2} \right) + Z_t^+ \frac{\partial}{\partial s^2} \phi \left(\frac{Z_t^- X_t^2 v_T^2 s_t^2}{2 \sigma^2 v_t^2 s_T^2} \right) \right] \\ &+ \frac{1}{2\sigma} \left[\left(\frac{\partial}{\partial s^2} Z_t^- \right) \phi \left(\frac{Z_t^+ X_t^2 v_T^2 s_t^2}{2 \sigma^2 v_t^2 s_T^2} \right) + Z_t^- \frac{\partial}{\partial s^2} \phi \left(\frac{Z_t^+ X_t^2 v_T^2 s_t^2}{2 \sigma^2 v_t^2 s_T^2} \right) \right]. \quad (\text{B.29}) \end{aligned}$$

Substituting

$$\begin{aligned} \frac{\partial}{\partial s^2} \phi \left(\frac{Z_t^- X_t^2 v_T^2 s_t^2}{2 \sigma^2 v_t^2 s_T^2} \right) &= -\frac{Z_t^- X_t^2 v_T^2 s_t^2}{2 \sigma^2 v_t^2 s_T^2} \phi \left(\frac{Z_t^- X_t^2 v_T^2 s_t^2}{2 \sigma^2 v_t^2 s_T^2} \right) \frac{\partial}{\partial s^2} \left(\frac{Z_t^- X_t^2 v_T^2 s_t^2}{2 \sigma^2 v_t^2 s_T^2} \right), \\ \frac{\partial}{\partial s^2} \phi \left(\frac{Z_t^+ X_t^2 v_T^2 s_t^2}{2 \sigma^2 v_t^2 s_T^2} \right) &= -\frac{Z_t^+ X_t^2 v_T^2 s_t^2}{2 \sigma^2 v_t^2 s_T^2} \phi \left(\frac{Z_t^+ X_t^2 v_T^2 s_t^2}{2 \sigma^2 v_t^2 s_T^2} \right) \frac{\partial}{\partial s^2} \left(\frac{Z_t^+ X_t^2 v_T^2 s_t^2}{2 \sigma^2 v_t^2 s_T^2} \right), \end{aligned}$$

with

$$\begin{aligned} \frac{\partial}{\partial s^2} \left(\frac{Z_t^- X_t^2 v_T^2 s_t^2}{2 \sigma^2 v_t^2 s_T^2} \right) &= \frac{1}{2} \left[\left(\frac{X_t^2 v_T^2 s_t^2}{\sigma^2 v_t^2 s_T^2} \right) \frac{\partial}{\partial s^2} Z_t^- + Z_t^- \frac{\partial}{\partial s^2} \left(\frac{X_t^2 v_T^2 s_t^2}{\sigma^2 v_t^2 s_T^2} \right) \right], \\ \frac{\partial}{\partial s^2} \left(\frac{Z_t^+ X_t^2 v_T^2 s_t^2}{2 \sigma^2 v_t^2 s_T^2} \right) &= \frac{1}{2} \left[\left(\frac{X_t^2 v_T^2 s_t^2}{\sigma^2 v_t^2 s_T^2} \right) \frac{\partial}{\partial s^2} Z_t^+ + Z_t^+ \frac{\partial}{\partial s^2} \left(\frac{X_t^2 v_T^2 s_t^2}{\sigma^2 v_t^2 s_T^2} \right) \right], \end{aligned}$$

into (B.29), and using the equalities (B.28) and

$$Z_t^+ Z_t^- = 4 \left(\frac{\sigma^2 v_t^2 s_T^2}{X_t^2 v_T^2 s_t^2} \right)^2,$$

yields the required partial derivative

$$\begin{aligned} \frac{\partial}{\partial s^2} V_t &= \frac{1}{2\sigma} \left[2 \frac{\partial}{\partial s^2} \left(\frac{\sigma^2 v_t^2 s_T^2}{X_t v_T^2 s_t^2} \right) - Z_t^- \frac{\sigma^2 v_t^2 s_T^2}{X_t v_T^2 s_t^2} \frac{\partial}{\partial s^2} \left(\frac{X_t^2 v_T^2 s_t^2}{\sigma^2 v_t^2 s_T^2} \right) \right] \phi \left(\frac{Z_t^- X_t^2 v_T^2 s_t^2}{2 \sigma^2 v_t^2 s_T^2} \right) \\ &+ \frac{1}{2\sigma} \left[-2 \frac{\partial}{\partial s^2} \left(\frac{\sigma^2 v_t^2 s_T^2}{X_t v_T^2 s_t^2} \right) - Z_t^+ \frac{\sigma^2 v_t^2 s_T^2}{X_t v_T^2 s_t^2} \frac{\partial}{\partial s^2} \left(\frac{X_t^2 v_T^2 s_t^2}{\sigma^2 v_t^2 s_T^2} \right) \right] \phi \left(\frac{Z_t^+ X_t^2 v_T^2 s_t^2}{2 \sigma^2 v_t^2 s_T^2} \right). \end{aligned} \quad (\text{B.30})$$

However, (B.30) is always negative for $\gamma \geq 1$ because

$$\frac{\partial}{\partial s^2} \left(\frac{X_t^2 v_T^2 s_t^2}{\sigma^2 v_t^2 s_T^2} \right) > 0 \quad \text{and} \quad \frac{\partial}{\partial s^2} \left(\frac{\sigma^2 v_t^2 s_T^2}{X_t v_T^2 s_t^2} \right) < 0, \quad \text{for} \quad \gamma \geq 1,$$

which implies that for $\gamma \geq 1$, the first square bracket term in (B.30) is negative, and if the second square bracket term is also negative then it is easy to see that (B.30) is negative. However, if the second square bracket term is positive then we use the inequality

$$-2 \frac{\partial}{\partial s^2} \left(\frac{\sigma^2 v_t^2 s_T^2}{X_t v_T^2 s_t^2} \right) - Z_t^+ \frac{\sigma^2 v_t^2 s_T^2}{X_t v_T^2 s_t^2} \frac{\partial}{\partial s^2} \left(\frac{X_t^2 v_T^2 s_t^2}{\sigma^2 v_t^2 s_T^2} \right) < \left| 2 \frac{\partial}{\partial s^2} \left(\frac{\sigma^2 v_t^2 s_T^2}{X_t v_T^2 s_t^2} \right) - Z_t^- \frac{\sigma^2 v_t^2 s_T^2}{X_t v_T^2 s_t^2} \frac{\partial}{\partial s^2} \left(\frac{X_t^2 v_T^2 s_t^2}{\sigma^2 v_t^2 s_T^2} \right) \right|,$$

and the fact that

$$0 < \phi \left(\frac{Z_t^+ X_t^2 v_T^2 s_t^2}{2 \sigma^2 v_t^2 s_T^2} \right) < \phi \left(\frac{Z_t^- X_t^2 v_T^2 s_t^2}{2 \sigma^2 v_t^2 s_T^2} \right), \quad (\text{B.31})$$

to show that the first line in (B.30) dominates the second line, and therefore (B.30) is negative when $\gamma \geq 1$.

We now prove that properties in Propositions 4–5 also hold in this more general economy. The property that the stock volatility is higher than that in the benchmark economy follows immediately by comparing (25) and (39). The property that the stock volatility is increasing in belief dispersion is also immediate from (39). The property that the stock volatility is decreasing in investors' risk aversion follows from the negative sign of the partial derivative of (39) with respect to γ

$$\frac{\partial \sigma_{St}}{\partial \gamma} = - \frac{\sigma^7}{\left(\sigma^2 + \left(\frac{1}{\gamma} v^2 + s^2 \right) t \right)^4 \gamma^2} \frac{v^2}{\gamma^2} \left(1 - \frac{\gamma}{v^2} \frac{\partial v^2}{\partial \gamma} \right) (T - t) < 0,$$

where $1 - (\gamma/v^2)(\partial v^2/\partial \gamma)$ is positive and as in (B.21).

Similarly, the condition for property that the trading volume measure is increasing in belief dispersion follows from the partial derivative of (40) with respect to v_t , or equivalently v^2 . Following similar steps as in the case with respect to s^2 above, we obtain

$$\begin{aligned} \frac{\partial}{\partial v^2} V_t &= \frac{1}{2\sigma} \left[2 \frac{\partial}{\partial v^2} \left(\frac{\sigma^2 v_t^2 s_T^2}{X_t v_T^2 s_t^2} \right) - Z_t^- \frac{\sigma^2 v_t^2 s_T^2}{X_t v_T^2 s_t^2} \frac{\partial}{\partial v^2} \left(\frac{X_t^2 v_T^2 s_t^2}{\sigma^2 v_t^2 s_T^2} \right) \right] \phi \left(\frac{Z_t^- X_t^2 v_T^2 s_t^2}{2 \sigma^2 v_t^2 s_T^2} \right) \\ &+ \frac{1}{2\sigma} \left[-2 \frac{\partial}{\partial v^2} \left(\frac{\sigma^2 v_t^2 s_T^2}{X_t v_T^2 s_t^2} \right) - Z_t^+ \frac{\sigma^2 v_t^2 s_T^2}{X_t v_T^2 s_t^2} \frac{\partial}{\partial v^2} \left(\frac{X_t^2 v_T^2 s_t^2}{\sigma^2 v_t^2 s_T^2} \right) \right] \phi \left(\frac{Z_t^+ X_t^2 v_T^2 s_t^2}{2 \sigma^2 v_t^2 s_T^2} \right). \end{aligned} \quad (\text{B.32})$$

However, (B.32) is always positive because

$$\frac{\partial}{\partial v^2} \left(\frac{X_t^2 v_T^2 s_t^2}{\sigma^2 v_t^2 s_T^2} \right) < 0 \quad \text{and} \quad \frac{\partial}{\partial v^2} \left(\frac{\sigma^2 v_t^2 s_T^2}{X_t v_T^2 s_t^2} \right) > 0,$$

which implies that the first square bracket term in (B.32) is positive, and if the second square bracket term is also positive then it is easy to see that (B.32) is positive. However, if the second square bracket term is negative then we use the inequality

$$2 \frac{\partial}{\partial v^2} \left(\frac{\sigma^2 v_t^2 s_T^2}{X_t v_T^2 s_t^2} \right) - Z_t^- \frac{\sigma^2 v_t^2 s_T^2}{X_t v_T^2 s_t^2} \frac{\partial}{\partial v^2} \left(\frac{X_t^2 v_T^2 s_t^2}{\sigma^2 v_t^2 s_T^2} \right) > \left| -2 \frac{\partial}{\partial v^2} \left(\frac{\sigma^2 v_t^2 s_T^2}{X_t v_T^2 s_t^2} \right) - Z_t^+ \frac{\sigma^2 v_t^2 s_T^2}{X_t v_T^2 s_t^2} \frac{\partial}{\partial v^2} \left(\frac{X_t^2 v_T^2 s_t^2}{\sigma^2 v_t^2 s_T^2} \right) \right|,$$

and (B.31) to show that the first line in (B.32) dominates the second line, and therefore (B.32) is positive. The property that the trading volume measure is positively related to the stock volatility follows from the fact that an increase in belief dispersion leads to both a higher trading volume measure and a stock volatility as before. Finally, the property that the trading volume measure is decreasing in investors' risk aversion follows from the partial derivative of (40) with respect to γ . Following similar steps as in the case with respect to s^2 again, we obtain

$$\begin{aligned} \frac{\partial}{\partial \gamma} V_t &= \frac{1}{2\sigma} \left[2 \frac{\partial}{\partial \gamma} \left(\frac{\sigma^2 v_t^2 s_T^2}{X_t v_T^2 s_t^2} \right) - Z_t^- \frac{\sigma^2 v_t^2 s_T^2}{X_t v_T^2 s_t^2} \frac{\partial}{\partial \gamma} \left(\frac{X_t^2 v_T^2 s_t^2}{\sigma^2 v_t^2 s_T^2} \right) \right] \phi \left(\frac{Z_t^- X_t^2 v_T^2 s_t^2}{2 \sigma^2 v_t^2 s_T^2} \right) \\ &+ \frac{1}{2\sigma} \left[-2 \frac{\partial}{\partial \gamma} \left(\frac{\sigma^2 v_t^2 s_T^2}{X_t v_T^2 s_t^2} \right) - Z_t^+ \frac{\sigma^2 v_t^2 s_T^2}{X_t v_T^2 s_t^2} \frac{\partial}{\partial \gamma} \left(\frac{X_t^2 v_T^2 s_t^2}{\sigma^2 v_t^2 s_T^2} \right) \right] \phi \left(\frac{Z_t^+ X_t^2 v_T^2 s_t^2}{2 \sigma^2 v_t^2 s_T^2} \right). \end{aligned} \quad (\text{B.33})$$

However, (B.33) is always negative because

$$\frac{\partial}{\partial \gamma} \left(\frac{X_t^2 v_T^2 s_t^2}{\sigma^2 v_t^2 s_T^2} \right) > 0 \quad \text{and} \quad \frac{\partial}{\partial \gamma} \left(\frac{\sigma^2 v_t^2 s_T^2}{X_t v_T^2 s_t^2} \right) < 0,$$

which implies that the first square bracket term in (B.33) is negative, and if the second square bracket term is also negative then it is easy to see that (B.33) is negative. However, if the second square bracket term is positive then we use the inequality

$$-2 \frac{\partial}{\partial \gamma} \left(\frac{\sigma^2 v_t^2 s_T^2}{X_t v_T^2 s_t^2} \right) - Z_t^+ \frac{\sigma^2 v_t^2 s_T^2}{X_t v_T^2 s_t^2} \frac{\partial}{\partial \gamma} \left(\frac{X_t^2 v_T^2 s_t^2}{\sigma^2 v_t^2 s_T^2} \right) < \left| 2 \frac{\partial}{\partial \gamma} \left(\frac{\sigma^2 v_t^2 s_T^2}{X_t v_T^2 s_t^2} \right) - Z_t^- \frac{\sigma^2 v_t^2 s_T^2}{X_t v_T^2 s_t^2} \frac{\partial}{\partial \gamma} \left(\frac{X_t^2 v_T^2 s_t^2}{\sigma^2 v_t^2 s_T^2} \right) \right|,$$

and (B.31) to show that the first line in (B.33) dominates the second line, and therefore (B.33) is negative. \square