Where to locate in a Circular City? [1, 2]

by

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[1] *Genesis of the paper*: The problem was independently analyzed by Gupta, Pal and Sarkar (2001), and Yu and Lai (2003; a). Subsequently, both papers were simultaneously submitted for possible publication at the *International Journal of Industrial Organization*. At the suggestion of the journal editor, Simon Anderson, the two papers are combined giving rise to the present article.

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Abstract:

We study location choices by Cournot oligopolists in a circular market, where consumers are located uniformly along the circumference. We analyze the subgame perfect Nash equilibria of a two stage location - quantity game. We demonstrate that the equidistant location pattern is only one of the many equilibrium location patterns that arise in a circular market. Non-equidistant, multiple or a continuum of location equilibria may also arise. Both spatial agglomeration and dispersion, or a combination of agglomeration and dispersion may occur in equilibrium. In the case of infinitely many location equilibria, we establish the welfare equivalence of the equilibrium location patterns.

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1. Introduction

Spatial competition has a rich and diverse literature, with its origin dating back to the seminal work by Hotelling (1929). There are two standard models of spatial competition that are considered in the literature: the linear city model pioneered by Hotelling (1929), and the circular city model pioneered by Vickrey (1964) and made popular by Salop (1979).[4] In the linear city model, the consumers are assumed to be distributed on a line of a finite length, whereas in the circular city model, the consumers are distributed on the circumference of a circle.

Most of work in the spatial competition literature is based on the linear city model. It is well recognized, however, that there are scenarios where the circular city model is more appropriate. For example, in many U.S. cities, suburbs are located along a circular belt-way and travel mostly occurs along the belt-way since the city center is difficult to cross, making it attractive for stores to choose locations along the circular belt-way. Other examples include competing television networks choosing time slots for their shows, and competing airlines choosing the arrival and departure times of their flights. The dial of a clock can be interpreted as a circle, and thus, the firms are essentially choosing locations along a circle. Furthermore, in the context of product differentiation, the circular city model often has advantages over a

[4]The modern version of the circular city model is due to Vickrey (1964). An earlier version of the circle model, however, is due to Lerner and Singer (1937). We thank Simon Anderson for bringing these to our attention.

linear city model. With an uniform distribution of consumers, the location space in a circular city is completely homogenous (no location is *a priori* better than another), which makes the relevant analysis more tractable. (See Tirole (1988) for a discussion on the circular city model).

Work in spatial competition that studies location choices by the firms (as opposed to exogenously fixed firm locations) almost exclusively considers the linear city model. This is because, in a linear city model, locations are *a priori* heterogenous and thus, it is interesting to determine location patterns arising from various market characteristics. For example, for a duopoly competing in (mill) prices, Hotelling (1929) claims that with linear transport costs, the firms locate back to back at the market center (*principle of minimum differentiation*), whereas, under quadratic transport costs, d'Aspremont, Gabszewicz and Thisse (1979) find *maximum differentiation*, where the firms locate at the two ends of the linear market.[5] By considering spatial discriminatory pricing and linear transport costs, Lederer and Hurter (1986) show that the firms locate at the market quartiles. In contrast, Anderson and Neven (1991), and Hamilton, Thisse and Weskamp (1989) reestablish the *principle of minimum differentiation* by considering spatial Cournot (quantity) competition.

[5]With linear transport costs, d'Aspremont, Gabszewicz and Thisse (1979) show that if the firms locate relatively close to each other, a pure strategy price equilibrium may not exist. Thus, they claim that "the so-called *Principle of Minimum Differentiation*, as based on Hotelling's 1929 paper "Stability in Competition" is invalid."

The circular city model, on the other hand, is perfectly symmetric and thus, no location is *a priori* better than another. Therefore, it may seem reasonable to conjecture that firms would locate equidistantly along the circle. It is indeed *assumed* in the literature that firms locate equidistantly around the circle, where the distance between any two neighboring firms is constant. The popular work of Salop (1979), which considers subsequent price competition, assumes an equidistant location pattern. Various other work that later applies the circular city model, makes a similar assumption (see, for example, Novshek (1980), Eaton and Wooders (1985), Economides (1993), Papandrea (1997), Steinmetz (1998)). Furthermore, Kats (1995) establishes the symmetric location pattern as an equilibrium location pattern under price competition.

In this paper, we demonstrate that the analysis of location choice in a circular city is much richer than previously anticipated. It is shown that if the firms compete in quantities, the equidistant location pattern is only one of the many equilibrium location patterns that arise in a circular city model. In fact, we identify non-equidistant, multiple and sometimes a continuum of location equilibria that are typically not anticipated in a perfectly symmetric circular city model.

We analyze a two stage problem of location and quantity choice along a circular city. We deviate from the common assumption of price competition in the second stage, by assuming that after choosing their locations, the firms behave as Cournot oligopolist and compete in quantities.

In the non-spatial context, the assumption of Cournot competition needs no further justification. The Cournot model is probably the most widely used oligopoly model. In a spatial context, the predictions arising from a spatial Cournot model often describe the real world better than those arising from a spatial price competition model. For example, it is Cournot competition, not price competition that successfully explains the commonly observed phenomenon of overlapping geographic markets of competing firms selling a homogeneous product. Anderson and Neven (1991) provide compelling arguments justifying the appropriateness of the Cournot assumption in various spatial models.[6] It is established that spatial Cournot competition is appropriate for industries where quantity is less flexible than price at each market point. Such industries would include, for example, oil, natural gas, cement and ready-mixed concrete. In fact, from an empirical perspective, the Cournot model of spatial competition is employed to analyze international oil and natural gas markets (see, for example, Salant 1982). The predictions of the Cournot model in terms of delivered prices are confirmed by McBride (1983) in the cement industry and by Greenhut, Greenhut and Li (1980) in a representative sample of industries.[7]

[6]Also, see Greenhut, Lee and Mansur (1991), which argues that between the Bertrand and Cournot models, the latter warrants major consideration in modelling spatial competition.

[7]Furthermore, following Eaton and Schmitt (1994), a spatial model with Cournot competition can be interpreted as a non-spatial Cournot model involving firms that enjoy economies of scope. Eaton and Schmitt (1994) model economies of scope by considering

In the context of linear markets, spatial models with Cournot competition are becoming increasingly popular in recent years. Anderson and Neven (1991), and Hamilton, Thisse and Weskamp (1989) pioneer the study of spatial Cournot competition with endogenous location choice.[8] They analyze a two-stage problem of location and quantity choices in Hotelling's linear city model, and establish that Cournot competition gives rise to spatial agglomeration of firms. Gupta, Pal and Sarkar (1997) extend Anderson and Neven (1991)'s analysis by considering non-uniform consumer distributions in the linear city model, and confirm the agglomeration result for a wide variety of consumer distributions. Mayer (2000) contributes by allowing the production costs to differ at various locations, and establishes that the firms may still agglomerate.[9]

firms that may produce few *basic products*, which can be modified to produce any other variant in the attribute space. When the firms compete in prices (a la Bertrand), the model mirrors a spatial model with discriminatory pricing. Following this approach, therefore, if the firms compete in quantities, instead of prices, a non-spatial Cournot model with economies of scope would be equivalent to a spatial Cournot model.

[8]Previous literature on spatial Cournot competition treats locations as exogenously fixed. See Ohta (1988) for related discussions.

[9]Related work that considers spatial Cournot competition with endogenous location choice in a linear market include Pal and Sarkar (2002), and Norman and Pepall (2000). Pal and Sarkar (2002) analyze spatial Cournot competition among multi-plant firms, whereas Norman and Pepall (2000) analyze horizontal mergers in the context of spatial Cournot competition.

We contribute to the previous literature by analyzing spatial Cournot competition with endogenous location choice in a circular city model. To the best of our knowledge, the only three papers that have attempted to analyze spatial Cournot competition in a circular city model are Pal (1998), Chamorro-Rivas (2000) and Matsushima (2001).[10] Pal (1998) shows that the equilibrium location pattern in a circular city is different from that obtained in a linear city model. In contrast to the agglomeration result obtained in a linear city, he demonstrates that a duopoly never agglomerate at a single location and establishes the equidistant location pattern as the unique equilibrium location pattern in a circular city. He concludes that in a circular city model, both price and Cournot competition yield dispersed and identical equilibrium locations. Chamorro-Rivas (2000) extends the "equidistance result" for a multi-plant duopoly. It is shown that in equilibrium, the plants are equally spaced on the circle. Matsushima (2001), on the other hand, demonstrates an equilibrium location pattern that is significantly different from the equidistant location pattern. For an even number of firms, he establishes that half of the firms locating at a market point and the remaining half locating at the diametrically opposite point can also be sustained as an equilibrium location pattern.

In this paper we demonstrate that the results in both Pal (1998) and Matsushima (2001) are special cases of a far more general location pattern. In

[10]For a duopoly selling *complementary* products, Yu and Lai (2003; b) and Shimizu (2002) are two additional papers that consider quantity competition in a circular city.

fact, we identify multiple and sometimes infinitely many location equilibria in the circular city model. The demonstration of infinitely many location equilibria is distinct from the previous findings in the literature on spatial Cournot competition, where the location equilibria are always finite and mostly unique.

We also identify various equilibrium location patterns that are counter-intuitive in a symmetric circular city model. For example, we demonstrate "dispersed" location equilibria where firms may choose distinct but not equidistant locations (the distance between two neighboring firms may vary). In general, there are many combinations of spatial agglomeration and dispersion that can be sustained as equilibrium locations. For example, firms agglomerating at various points along the circular city, which are not necessarily equidistant from each other, may be an equilibrium. An equilibrium location pattern may also involve some firms choosing isolated locations while other firms may agglomerate at various market points.

Typical agglomeration results in the literature involve all firms agglomerating at the same location. In this work, perhaps one of the most distinct properties of the agglomeration equilibria is that all firms never agglomerate at the same location; each agglomeration point includes only a subset of firms. This pattern is consistent with actual firm behavior in a metropolis, where all firms most definitely do not agglomerate at one point and each point typically has different numbers of firms. Neither Cournot competition in a linear market, nor price competition in either linear or circular markets give rise to results that are consistent with agglomeration of different number of firms at various market points.

The paper is organized as follows. Section 2 describes the model. Section 3 characterizes various properties of a location equilibrium. Section 4 identifies the equilibrium locations and discusses the results. Section 5 concludes the paper.

2. Model

We consider a spatial Cournot oligopoly serving a circular market with perimeter one. The consumers are distributed uniformly on the circle. The market demand at each point x on the circle is given by $p(x) = a - bQ(x)$, where $a > 0, b > 0$ are constants and independent of x. $Q(x)$ is the aggregate quantity supplied at x and $p(x)$ is the market price at x.

There are n firms who choose their locations on the circle. The points on the circle are identified with numbers in $[0, 1]$, the north most point being $\frac{1}{4}$ and the values increase in a counter clockwise direction. Thus, the east most point is considered both 0 and 1. The vector $\underline{\xi} = (\xi_1, \xi_2, ..., \xi_n)$ denotes the locations of the n firms and the vector $\underline{\xi}^{-i} = (\xi_1, ..., \xi_{i-1}, \xi_{i+1}, ..., \xi_n)$ denotes the location of all other firms except Firm i $(1 \leq i \leq n)$. The firms produce and sell a homogeneous output to the consumers. The firms deliver the product to the consumers and arbitrage among consumers is assumed to be infeasible. Thus, the firms can discriminate across consumers. The firms have identical production and transportation technologies. Each firm produces at a constant marginal and average cost (both normalized to zero) and pays a linear transport cost of $t > 0$, per unit distance. Without loss of generality, we assume $t = 1$. The good can be transported only along the perimeter of the circle. Each firm serves a market point x incurring the lowest possible transport cost. We also assume that $a \geq \frac{n}{2}$. This condition ensures that all firms will always serve the entire market.

We study the subgame perfect Nash equilibria (SPNE) of a two-stage game, where in stage one, the firms choose their locations and in stage two, after observing their competitors' locations, the firms compete in quantities. We proceed by backward induction and first characterize the quantity equilibrium in the second stage for given locations.

Since marginal production costs are constant and arbitrage among the consumers is not feasible, quantities set at different points by the same firm are strategically independent. Therefore, the second stage Cournot equilibrium can be characterized by a set of independent Cournot equilibria, one for each market point x.

At each market point $x \in [0, 1]$, Firm i $(i = 1, 2, .., n)$ chooses $q_i(x)$ to maximize its profit $[p(x) - c_i(x)]\, q_i(x)$, where $c_i(x) = \min\{|\xi_i - x|, 1 - |\xi_i - x|\}$ is Firm i's delivered marginal cost at x. By simultaneously solving the first order conditions for profit maximization of the firms, we obtain the following equilibrium outcomes at each x:

$$p\left(x,\underline{\xi}\right)=\frac{a+\sum_{j=1}^{n}c_j\left(x\right)}{n+1} \tag{2.1}$$

$$q_i\left(x,\xi_i,\underline{\xi}^{-i}\right)=\frac{\left[a-nc_i\left(x\right)+\sum_{j\neq i}c_j\left(x\right)\right]}{\left(n+1\right)b} \tag{2.2}$$

$$\pi_i\left(x,\xi_i,\underline{\xi}^{-i}\right)=\frac{\left[a-nc_i\left(x\right)+\sum_{j\neq i}c_j\left(x\right)\right]^2}{\left(n+1\right)^2b} \tag{2.3}$$

where $i=1,2,...,n$, $q_i\left(x,\xi_i,\underline{\xi}^{-i}\right)$ and $\pi_i\left(x,\xi_i,\underline{\xi}^{-i}\right)$ denote Firm i's equilibrium quantity and equilibrium profit at market point x, given the locations $\underline{\xi}$.

Therefore, given the locations $\underline{\xi}$, Firm i's $(i=1,2,...,n)$ equilibrium aggregate profit is

$$\Pi_i\left(\xi_i,\underline{\xi}^{-i}\right)=\int_0^1\pi_i\left(x,\xi_i,\underline{\xi}^{-i}\right)dx \tag{2.4}$$

We solve for a location vector $\underline{\xi}^*$ such that, given $(\xi_1^*,...,\xi_{i-1}^*,\xi_{i+1}^*,...,\xi_n^*)$, ξ_i^* maximizes Firm i's aggregate profit for all $i=1,2,...,n$. Thus, $\underline{\xi}^*$ is a subgame perfect location equilibrium in which no firm finds it profitable to relocate unilaterally.

3. Properties of a Location Equilibrium

In this section, we characterize several properties of a sub-game perfect location equilibrium. These properties are used in Section 4 to determine the SPNE locations. We first define a firm's *quantity median* and its *competitors'* *aggregate cost median,* then we associate each firm's SPNE locations with its quantity median and its competitors' aggregate cost median. The following notations are used to simplify our presentation.

Notation 1. *Let $\widehat{\xi}$ be the point diametrically opposite ξ. Then $L(\xi)$ denotes the half circle from ξ to $\widehat{\xi}$ (not including $\widehat{\xi}$) in the clockwise direction and $R(\xi)$ denotes the half circle from ξ to $\widehat{\xi}$ (not including ξ) in the counter-clockwise direction.*

Observe that $\xi \in L(\xi)$, $\widehat{\xi} \in R(\xi)$ and $L(\xi) = R\left(\widehat{\xi}\right)$, $R(\xi) = L\left(\widehat{\xi}\right)$. Also note that if Firm i locates at ξ_i, its delivered marginal cost at x, denoted by $c_i(x)$, equals the length of the shorter arc from ξ_i to x (since $t = 1$). Hence, for all $x \neq \xi_i, \widehat{\xi}_i$

$$\frac{\partial c_i(x)}{\partial \xi_i} = \left\{ \begin{matrix} 1 \text{ if } x \in L(\xi_i) \\ -1 \text{ if } x \in R(\xi_i) \end{matrix} \right\} \tag{3.1}$$

Definition 1. *Quantity median: ξ is a quantity median for Firm i if and only if the aggregate quantity supplied by Firm i in $L(\xi)$ equals the aggregate quantity supplied by it in $R(\xi)$. That is, $\int_{x \in L(\xi)} q_i(x)\, dx = \int_{x \in R(\xi)} q_i(x)\, dx$*

Definition 2. *Aggregate cost median: ξ is an aggregate cost median of n firms if and only if the aggregate delivered marginal cost of all n firms in $L(\xi)$ equals the aggregate delivered marginal cost of these firms in $R(\xi)$. That is, $\int_{x \in L(\xi)} \left[\sum_{j=1}^{n} c_j(x)\right] dx = \int_{x \in R(\xi)} \left[\sum_{j=1}^{n} c_j(x)\right] dx$*

Definition 3. *Competitors' aggregate cost median: ξ is a competitors' aggregate cost median for Firm i if and only if the aggregate delivered marginal cost of all other firms in $L(\xi)$ equals the aggregate delivered marginal cost of all other firms in $R(\xi)$. That is, $\int_{x\in L(\xi)}\left[\sum_{j\neq i} c_j(x)\right]dx = \int_{x\in R(\xi)}\left[\sum_{j\neq i} c_j(x)\right]dx$*

Remark 1. *If ξ is a competitors' aggregate cost median for Firm i, then $\int_{x\in L(\xi)}\left[\sum_{j\neq i} c_j(x)\right]dx = \int_{x\in R(\xi)}\left[\sum_{j\neq i} c_j(x)\right]dx = \frac{n-1}{8}$.*

In the context of a linear market, the quantity median property is useful in determining the SPNE locations. In the context of a circular market, however, it turns out that the competitors' aggregate cost median property is more useful in determining the SPNE locations and we make extensive use of this property. Previous work in this area has not made use of the competitors' aggregate cost median property. The following lemma presents a relationship between a firm's quantity median and its competitors' aggregate cost median.

Lemma 1. *In a circular market, ξ is a quantity median for Firm i if and only if it is also Firm i's competitors' aggregate cost median.*

Proof. From Definition 1 and equation (2.2), ξ is a quantity median for Firm i if and only if

$$\int_{x\in L(\xi)}\left[a+\sum_{j\neq i} c_j(x) - nc_i(x)\right]dx = \int_{x\in R(\xi)}\left[a+\sum_{j\neq i} c_j(x) - nc_i(x)\right]dx$$

$$\Leftrightarrow \int_{x\in L(\xi)} [a - nc_i(x)]\,dx + \int_{x\in L(\xi)} \left[\sum_{j\neq i} c_j(x)\right] dx$$

$$= \int_{x\in R(\xi)} [a - nc_i(x)]\,dx + \int_{x\in R(\xi)} \left[\sum_{j\neq i} c_j(x)\right] dx$$

Note that $L(\xi)$ and $R(\xi)$ are of equal length, and $c_i(x)$ is symmetric in x around ξ. Therefore,

$$\int_{x\in L(\xi)} [a - nc_i(x)]\,dx = \int_{x\in R(\xi)} [a - nc_i(x)]\,dx$$

Thus, ξ is a quantity median for Firm i if and only if

$$\int_{x\in L(\xi)} \left[\sum_{j\neq i} c_j(x)\right] dx = \int_{x\in R(\xi)} \left[\sum_{j\neq i} c_j(x)\right] dx$$

That is, ξ is Firm i's competitors' aggregate cost median. □

Observe that Lemma 1 does not hold in the context of a linear market. This is because, in a linear market, a firm's market length on its left may differ from its market length on its right. Thus, in the context of a linear market, only the quantity median property is emphasized in determining the SPNE locations. In a circular market, however, the competitors' aggregate cost median property will be very useful in determining the SPNE locations.

Also, observe that given the locations of all other competitors, there always exists a point ξ_i, which is Firm i's quantity median or its competitors' aggregate cost median. This follows from the intermediate value theorem since $\int_{x\in L(\xi)} \left[\sum_{j\neq i} c_j(x)\right] dx$ is continuous in ξ. The point ξ_i, however, is not unique. For example, if ξ_i is such a point, so is $\widehat{\xi_i}$, which is diametrically opposite to ξ_i.

How does a firm's optimal location relate to its competitors' aggregate cost median? The following proposition establishes a useful relationship between a firm's optimal location and its competitors' aggregate cost median.

Proposition 1. *Given the locations of all other firms, a firm maximizes its profit only if it locates at its competitors' aggregate cost median or equivalently, at its quantity median.*

Proof. Without loss of generality, consider Firm i located at ξ_i. From equations (2.3) and (2.4), observe that Firm i's aggregate profit from the entire circular city is given by

$$\Pi_i(\xi_i) = \frac{1}{b(n+1)^2} \int_0^1 \left[a + \sum_{j \neq i} c_j(x) - nc_i(x) \right]^2 dx$$

Differentiating $\Pi_i(\xi_i)$ with respect to ξ_i and using equation (3.1), we have

$$\left[\frac{b(n+1)^2}{-2n} \right] \frac{\partial \Pi_i(\xi_i)}{\partial \xi_i}$$

$$= \int_{x \in L(\xi_i)} \left[a + \sum_{j \neq i} c_j(x) - nc_i(x) \right] dx - \int_{x \in R(\xi_i)} \left[a + \sum_{j \neq i} c_j(x) - nc_i(x) \right] dx$$

$$= \int_{x \in L(\xi_i)} \left[\sum_{j \neq i} c_j(x) \right] dx - \int_{x \in R(\xi_i)} \left[\sum_{j \neq i} c_j(x) \right] dx$$

The first order condition for profit maximization requires $\frac{\partial \Pi_i(\xi_i)}{\partial \xi_i} = 0$, which completes the proof of the proposition. □

The intuition behind Proposition 1 is as follows. Consider a small movement by a firm located at ξ. It alters its profits in both $L(\xi)$ and $R(\xi)$. At equilibrium, the change in profit in $L(\xi)$ must be equal to the change in profit in $R(\xi)$. For Cournot competition, a firm's profit in a market is proportional to the square of the quantity served by the firm in that market. Hence, the change in profit in a market is proportional to the quantity served in that market. Therefore, the total quantity served in $L(\xi)$ must be equal to the total quantity served in $R(\xi)$, establishing the necessity of the quantity median property at an equilibrium. Furthermore, since a firm's quantity median and its competitors' aggregate cost median are coincident in a circular market, the competitors' cost median property must be satisfied at an equilibrium.

From Proposition 1, it follows that the search for a firm's profit maximizing location can be limited to the points that satisfy the competitors' aggregate cost median property. Recall, however, that a firm's competitors' aggregate cost median is not unique. In fact, if ξ_i is such a point, so is $\widehat{\xi_i}$, which is diametrically opposite to ξ_i. In this scenario, the second order condition can be used to eliminate points that satisfy competitors' aggregate cost median property, but are not profit maximizing locations.

Observe that

$$\frac{\partial^2 \Pi_i(\xi_i)}{\partial \xi_i^2} = \frac{-4n}{b(n+1)^2}\left[\sum_{j\neq i} c_j(\xi_i) - \sum_{j\neq i} c_j\left(\widehat{\xi_i}\right)\right]$$

Therefore, the second order condition for profit maximization does not hold at ξ_i if $\sum_{j\neq i} c_j(\xi_i) < \sum_{j\neq i} c_j\left(\widehat{\xi_i}\right)$, or equivalently, the aggregate delivered marginal cost of the competitors is lower at ξ_i than at $\widehat{\xi_i}$. On the other hand, if a location ξ_i satisfies the competitors' aggregate cost median property and $\sum_{j\neq i} c_j(\xi_i) \geq \sum_{j\neq i} c_j\left(\widehat{\xi_i}\right)$, then the search for global profit maximizing locations for Firm i can be limited to such points.

The next corollary follows as an immediate implication of Proposition 1.

Corollary 1. *At SPNE locations, each firm locates at its competitors' cost median, or equivalently, at its quantity median.*

The search for SPNE locations, therefore, can be confined to a set of locations such that each firm locates at its competitors' aggregate cost median and $\forall i = 1, ..., n$, $\sum_{j\neq i} c_j(\xi_i) \geq \sum_{j\neq i} c_j\left(\widehat{\xi_i}\right)$. The following section uses this property to identify the SPNE locations.

4. Determination of the SPNE Locations

In this section, we first derive and characterize the SPNE locations for an even number of firms. The analysis for an odd number of firms follows in Subsection 4.2.

4.1. **SPNE locations for an even number of firms.**

Proposition 2. *For $n = 2m$ $(m \geq 1)$, if the firms locate at opposite ends of any set of m diameters of the circle, then the firm locations are sustainable as SPNE locations.*

Proof. Consider any set of m diameters of the circle. Without loss of generality, suppose that first $n-2$ firms locate at opposite ends of $m-1$ diameters. Now, consider the m th diameter. Let the $(n-1)$ st firm locate at one end of the m th diameter. We need to show that the n th firm maximizes its profit by locating at the other end of the m th diameter.

Note that the aggregate delivered marginal cost of any two firms that are located at opposite ends of a diameter, is exactly $\frac{1}{2}$ at each point on the circle. Therefore, the aggregate delivered marginal cost of the first $n-2$ firms, located at opposite ends of $m-1$ diameters, is constant (and equals $\frac{m-1}{2}$) at each point on the circle. Hence, with respect to these first $n-2$ firms (without considering the $(n-1)$ st firm), any point on the circle will satisfy the competitors' aggregate cost median property for the n th firm.

Hence, if the $(n-1)$ st firm locates at one end of the m th diameter, then with respect to the $n-1$ competitors, there are only two points that satisfy the market median property of the n th firm. These two points are precisely the two ends of the m th diameter. The competitors' aggregate delivered marginal cost, however, is lower at the end where the $(n-1)$ st firm is located. Therefore, the n th firm maximizes its profit by locating at the other end of the m th diameter. $\square$

We first illustrate Proposition 2 using an example. The intuition behind Proposition 2 follows the example.

Example 1. *SPNE locations for $n = 8$*

Diagram 1 illustrates various location patterns that are sustainable as SPNE locations.

[Insert Diagram 1 here.]

An intuition behind Proposition 2 is as follows. Any two firms, located at the opposite ends of a diameter, give rise to constant aggregate delivered marginal cost (of these two firms) at each point on the circle. Therefore, with respect to firms $1, 2, ..., n-2$, located at the opposite ends of $m-1$ diameters, the aggregate delivered marginal cost is constant at each point on the circle. Consequently, the total quantity supplied by these $n-2$ firms at each point on the circle is constant (this follows from the fact that total quantity is a function of aggregate marginal cost. See equation (2.2)). So, when the $n-1$ th firm considers where to locate, all locations are equally attractive to it, since firms $1, 2, ..., n-2$ supply the same total amount at every location. For the n th firm, however, the aggregate quantity supplied by its competitors (including the $n-1$ th firm) varies periodically, with the minimum being attained at the location diametrically opposite to firm $n-1$ and the maximum being attained at the point where firm $n-1$ is located.

Consequently, firm n maximizes its profit by locating diametrically opposite to firm $n-1$. Interchanging firms $n-1$ and n, the same argument applies, giving rise to Proposition 2. In other words, firms $1, 2, ..., n-2$, located at the opposite ends of $m-1$ diameters, are strategically irrelevant for firms $n-1$ and n, who form an equilibrium by locating at the opposite ends of the m th diagonal.[11]

In the context of a circular market with quantity competition, Pal (1998) and Matsushima (2001) establish two different types of equilibria and thus, they indicate the possibility of multiple equilibria. Pal (1998) establishes that if the firms locate equidistantly from each other, then the firms are in equilibrium, whereas Matsushima (2001) shows that if half of the firms agglomerate at a point and the rest locate at the diametrically opposite end, then the firms are in equilibrium. Proposition 2 establishes that the number of equilibria are actually infinitely many and the two SPNE locations that are characterized in the earlier literature, are in fact special cases.

Case 1. *Pal (1998) establishes that if the firms locate equidistantly from each other, then the locations are sustainable as SPNE locations. This is a special case of Proposition 2, when the diameters are distinct and divide the circle evenly.*

[11]In Yu and Lai (2003; a), the vector of locations of firms $1, 2, ..., n-2$ is said to present the "neutral property", which is one of the sufficient conditions to implement a combination of Nash equilibria (called "Nash combination") shown in Proposition 2.

Case 2. *Matsushima (2001) establishes that for n even, if $\frac{n}{2}$ firms locate at any point ξ on the circle and the remaining $\frac{n}{2}$ firms locate at diametrically opposite end $\widehat{\xi}$, then the locations are sustainable as SPNE locations. This is a special case of Proposition 2, when all diameters overlap on each other.*

Observe that Proposition 2 demonstrates the possibility of a continuum of SPNE locations in the context of Cournot competition in a circular market. In contrast, the number of equilibria obtained in the prior literature on spatial competition has mostly been finite and unique.[12] It also identifies various equilibrium location patterns that are quite counter-intuitive in a symmetric circular city model. For example, it demonstrates "dispersed" equilibria that are not equidistant. In general, there are many combinations of spatial agglomeration and dispersion which can be sustained as equilibrium locations. For example, firms agglomerating at various points along the circular city, which are not necessarily equidistant from each other, may be an equilibrium. An equilibrium location pattern may also involve some firms choosing isolated locations while other firms may agglomerate at various market points. Note that all firms never agglomerate at the same location and each agglomeration point includes only a subset of the firms. This is in contrast to the typical agglomeration results in a linear market

[12]For an exception, see Kats (1995), which demonstrates the possibility of a continuum of equilibria in a spatial market. Also, free entry in to market (as opposed to exogenously fixed number of firms in the market) usually yields multiple equilibria (see Anderson and Engers (2001) for an in-depth investigation).

involving quantity competition, where all firms agglomerate at the same market point.[13]

Proposition 2 demonstrates a continuum of SPNE locations. Since there are infinitely many SPNE locations, two natural questions arise. First, are there properties common to all SPNE locations? Second, is it possible to rank the SPNE locations? The following proposition sheds light in this regard.

Proposition 3. *Consider the SPNE locations where $n = 2m$ firms locate at the opposite ends of any set of m diameters. The SPNE locations satisfy the following. (1) The total quantity supplied and the market price at each point on the circle are constants and identical for all SPNE locations. Therefore, all SPNE locations give rise to identical consumer surplus for all consumers. (2) The total profit earned by each firm is identical for all SPNE locations. Therefore, the subgame perfect Nash equilibria are equivalent with respect to consumer surplus, profits and welfare.*

Proof. First observe that the aggregate delivered marginal cost for any two firms located at opposite ends of a diameter is exactly $\frac{1}{2}$ at each point on the circle. It can then be checked that for any SPNE location, the total quantity

[13]Firms competing in prices may agglomerate in the spatial dimension if their products are differentiated by attribute(s) other than location (see, for example, dePalma et al. 1985, Anderson and dePalma 1988, Irmen and Thisse 1998).

supplied at each point on the circle is $\frac{n\left(a-\frac{1}{4}\right)}{b(n+1)}$ and the market price at any point on the circle is $\frac{a+\frac{n}{4}}{(n+1)}$. Since these expressions are independent of the locations of the firms and of the consumers, the first part of the proposition follows.

The second part of the proposition follows from the fact that the market price at each point on the circle is constant and identical. Consider Firm i located at ξ_i. In equilibrium, Firm i's profit at a market point x is $\pi_i\left(x|\xi_i\right) = \frac{(p^*(x)-c_i(x))^2}{b}$, where $p^*(x)$ is the equilibrium market price at x. Since $p^*(x) = \frac{a+\frac{n}{4}}{(n+1)}$ is independent of the locations of the firms, Firm i's total profit is identical at all SPNE locations. □

Proposition 3 describes a surprising result. Given the geographic differences among the SPNE locations, one would hardly anticipate that all SPNE locations give rise to identical consumer surplus for all consumers and identical total profit for each firm. For example, for $n = 4$, firms located equidistantly at $\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$, and two firms located at 0 and two firms located at $\frac{1}{2}$ are both sustainable as SPNE locations. Despite the obvious geographic differences between the two SPNE location patterns, both SPNE locations are equivalent with respect to consumer surplus and profits for *all* consumers and firms.

4.2. **SPNE locations for an odd number of firms.** Our analysis can be extended to determine SPNE locations for an odd number of firms. Similar to the outcomes for an even number of firms, any odd number of firms

($n \geq 3$) gives rise to multiple equilibria. Also, non-equidistant location patterns are possible in equilibrium.

To illustrate the possibility of multiple equilibria and non-equidistant location pattern, we first consider an example. More general results follow the example.

Example 2. *SPNE locations for n=3*

The first equilibrium is the one where the firms are equidistant from each other. Firms located at 0, $\frac{1}{3}$ and $\frac{2}{3}$ are in equilibrium. The second equilibrium is the one where one firm locates at ξ and the other two locate at $\widehat{\xi}$, which is diametrically opposite of ξ. To see how this location pattern can be sustained as a equilibrium, first note that if two firms locate at $\widehat{\xi}$, then the third firm maximizes its profit by locating at ξ. Now consider two firms located at ξ and $\widehat{\xi}$, respectively. Since the firms are located at two ends of a diameter, their aggregate delivered marginal cost is constant over the circle. In fact, at each point on the circle, the aggregate delivered marginal cost of these two firms is $\frac{1}{2}$. Therefore, any point on the circle satisfies the competitors' aggregate cost median property for the remaining firm. Also, since the aggregate delivered marginal cost of its competitors is constant over the circle, any location on the circle gives rise to identical profit for the remaining firm. Therefore, it cannot increase its profit by moving away from $\widehat{\xi}$ and consequently, one firm locating at ξ and the other two locating at $\widehat{\xi}$ can be sustained as an equilibrium.

Observe that for $n = 3$, the two SPNE location patterns described above are the only equilibria. When $n > 3$, however, more SPNE locations emerge. The following proposition provides a more general result regarding equilibrium location patterns for an odd number of firms. The motivation behind the proposition is as follows. The addition of a diametrically opposite pair of firms increases the aggregate delivered marginal cost by a constant $\frac{1}{2}$ throughout the entire circle, and therefore, the aggregate cost medians remain unchanged. Thus, it may be possible to construct additional SPNE locations by considering an initial SPNE location and then suitably adding a diametrically opposite pair of firms.

We first highlight the main argument used in the proof of Proposition 4 below. When $2k+1$ firms locate equidistantly on the circle, their aggregate delivered marginal cost (and hence the market price function) is periodic and piecewise linear, with the minimum attained at the firms' locations and the maximum attained at the midpoints between successive firms. The same property of the aggregate delivered marginal cost holds when $2(m-k)$ additional firms locate at the opposite ends of any of the $2k+1$ diameters passing through the locations of the first $2k+1$ firms. These properties of the aggregate delivered marginal cost help to sustain the locations stipulated in Proposition 4 as SPNE locations.

Proposition 4. *For $n = 2m+1$, suppose that $2k+1$ firms ($k \leq m$) locate equidistantly on the circle, say at 0, $\frac{1}{2k+1}$, ..., $\frac{2k}{2k+1}$ and the remaining*

$2(m-k)$ firms locate at the opposite ends of any of the $2k+1$ diameters passing through 0, $\frac{1}{2k+1}$, ..., $\frac{2k}{2k+1}$. Then the firm locations are sustainable as SPNE locations.

Proof. Suppose Firms $1,2,...,2k+1$ are located equidistantly on the circle, say at 0, $\frac{1}{2k+1}$, ..., $\frac{2k}{2k+1}$. Note that in the absence of any additional firm, Firms $1,2,...,2k+1$ are in equilibrium. Now, consider Firm $2k+2$. For Firm $2k+2$, the aggregate delivered marginal cost of firms $1,2,...,2k+1$ is periodic (in fact, piecewise linear), with the minimum at the firms' locations and the maximum at $\frac{1}{2(2k+1)}$, $\frac{3}{2(2k+1)}$,..., $\frac{4k+1}{2(2k+1)}$. Hence, the optimal location for Firm $2k+2$ is any of the $2k+1$ points $\frac{1}{2(2k+1)}$, $\frac{3}{2(2k+1)}$,..., $\frac{4k+1}{2(2k+1)}$.

If Firm $2k+2$ locates at one of these points ξ, then the aggregate delivered marginal cost of Firms $1,2,...,2k+2$ is minimized at ξ with value $\left[\frac{k^2}{2k+1}+\frac{1}{2}\right]$ and maximized at $\widehat{\xi}$ (which is diametrically opposite to ξ) with value $\left[\frac{k(k+1)}{(2k+1)}+\frac{1}{2}\right]$. Hence Firm $2k+3$ maximizes its profit by locating at $\widehat{\xi}$. Note that ξ and $\widehat{\xi}$ are at the opposite ends of a diameter and $\widehat{\xi} \in \left\{0, \frac{1}{2k+1}, ..., \frac{2k}{2k+1}\right\}$. Now, given the locations of $1,2,...,2k+1$ and Firm $2k+3$'s location at $\widehat{\xi}$, Firm $2k+2$ indeed maximizes its profit by locating at ξ. Next, note that since Firms $2k+2$ and $2k+3$ are located at the opposite ends of a diameter, their the aggregate delivered marginal cost is constant over the circle, implying that Firms $1,2,...,2k+1$ are still in equilibrium given the locations of $2k+2$ and $2k+3$ at ξ and $\widehat{\xi}$.

This argument can be repeated for any number of additional pairs of firms. Thus, the locations proposed in the proposition are sustainable as SPNE locations. □

Remark 2. *In Proposition 4, when $k = 0$, a location pattern with $(m+1)$ firms located at any one point on the circle and remaining m firms located at the diametrically opposite point form a SPNE location pattern. On the other hand, when $k = m$, firms located equidistantly on the circle form a SPNE location pattern.*

The following example illustrates the various SPNE locations patterns that can be established by Proposition 4.

Example 3. *SPNE location for $n = 5$.*

When $n = 5$, $m = 2$. So, k can be 0, 1 or 2. Different values of k give rise to different SPNE locations.

Equilibrium 1 ($k = 2$): All five firms locate equidistantly on the circle.

Equilibrium 2 ($k = 0$): Three firms locate at 0 (without loss of generality) and two firms locate at $\frac{1}{2}$. Here the first firm locates at 0 and the next pair of firms locate at 0 and $\frac{1}{2}$, as does the last pair.

Equilibrium 3 ($k = 1$): Two firms locate at 0 (without loss of generality), one locates at $\frac{1}{3}$, one locates at $\frac{1}{2}$ and the remaining one locates at $\frac{2}{3}$. Here the first three firms locate at 0, $\frac{1}{3}$ and $\frac{2}{3}$, and the remaining two firms locate at the opposite ends of the diameter passing through 0.

Diagram 2 illustrates various location patterns for $n = 5$ that are sustainable as SPNE locations.

[Insert Diagram 2 here.]

Observe that Proposition 4 characterizes both equidistant and non - equidistant SPNE locations. In Example 3, Equilibrium 1 illustrates a equidistant location pattern, whereas the other two illustrate non-equidistant location patterns. Also, note that for all the SPNE locations characterized in Proposition 4, the equilibrium price varies along the circle, whereas, all equilibrium location patterns characterized for an even number of firms in Proposition 2 give rise to constant and identical market price at each point on the circle (Proposition 3).

Also, in contrast to the SPNE locations identified in Proposition 2, the SPNE locations characterized in Proposition 4 are not equivalent with respect to consumer surplus, profits and welfare. For instance, consider the three SPNE described in Example 3. In Equilibrium 1, all firms earn the same profit from the entire circle. In Equilibrium 2, the three firms located at 0 earn the same profit each, but it is less than the profit earned by each of the two firms located at $\frac{1}{2}$. In Equilibrium 3, firms located at 0, $\frac{1}{3}$ and $\frac{2}{3}$ earn the same profit each, but less than the profit earned by the firm located at $\frac{1}{2}$. It is interesting, however, that for each of the three equilibria, the total quantity supplied by each firm to the entire circle is a constant. This is a consequence of Corollary 1.

Similar to Proposition 2, Proposition 4 also indicates the possibility of both complete spatial dispersion, and many combinations of spatial agglomeration and dispersion. Unlike the SPNE locations characterized by Proposition 2, however, in case of complete spatial dispersion, firms must be equidistant from each other.

4.3. **Additional SPNE locations for both even and odd numbers of firms.** The SPNE locations we have identified so far are not exhaustive. In fact, there are more equilibrium location patterns for both even and odd numbers of firms. The following two propositions may be used to characterize other equilibrium location patterns.

The motivation behind Proposition 5 below is similar to that of Proposition 4. The removal or addition of a diametrically opposite pair of firms change the aggregate delivered marginal cost by a constant $\frac{1}{2}$ throughout the entire circle, and therefore, the aggregate cost medians remain unchanged. Thus, it may be possible to construct additional SPNE locations by considering an initial SPNE location and then suitably removing or adding a diametrically opposite pair of firms. This intuition turns out to be correct, although now the second order condition for profit maximization requires some restrictions that are stated in the proposition below.

The following notation is used to simplify the presentation.

Notation 2. *$T_p(x)$ denotes the aggregate delivered marginal cost of p firms at point x. That is, $T_p(x) = \sum_{j=1}^{p} c_j(x)$.*

Proposition 5. (A) *If p firms are in equilibrium, and two among them are located at ξ and $\widehat{\xi}$, then their removal leaves the remaining $p-2$ firms still in equilibrium.* (B) *If p firms are in equilibrium and ξ is any aggregate cost median of these p firms satisfying $T_p\left(\widehat{\xi}\right) \leq T_p(\xi) < T_p\left(\widehat{\xi}\right) + \frac{1}{2}$, then with two more firms locating at ξ and $\widehat{\xi}$, all $p+2$ firms are in equilibrium.*

Proof. The removal (or addition) of a diametrically opposite pair of firms decreases (increases) the aggregate delivered marginal cost by a constant $\frac{1}{2}$ throughout the entire circle. Hence, the aggregate cost medians and the maxima of aggregate cost medians remain unchanged. This proves part (A) of the proposition.

To establish part (B), note that ξ and $\widehat{\xi}$ are aggregate cost medians of the original p firms. If $T_p\left(\widehat{\xi}\right) < T_p(\xi)$, the $(p+1)$ st firm will locate at ξ. Next, note that ξ and $\widehat{\xi}$ are still aggregate cost medians of these $p+1$ firms, and $T_{p+1}(\xi) = T_p(\xi) < T_p\left(\widehat{\xi}\right) + \frac{1}{2} = T_{p+1}\left(\widehat{\xi}\right)$. Therefore, the $(p+2)$ nd firm will locate at $\widehat{\xi}$. On the other hand, if the $(p+1)$ st firm locates at $\widehat{\xi}$, we have $T_{p+1}(\xi) = T_p(\xi) + \frac{1}{2} > T_p(\xi) \geq T_p\left(\widehat{\xi}\right) = T_{p+1}\left(\widehat{\xi}\right)$. Hence, the $(p+2)$ nd firm will locate at ξ. With the locating of $(p+1)$ st and $(p+2)$ nd firms at ξ and $\widehat{\xi}$, each of the original p firms are still in equilibrium, since the $(p+1)$ st and $(p+2)$ nd firms increase the aggregate delivered marginal cost by a constant $\frac{1}{2}$ throughout the entire circle. This completes the proof. □

Corollary 2. *Suppose p firms are in equilibrium and two of them are located at ξ and $\widehat{\xi}$. If two additional firms are placed at ξ and $\widehat{\xi}$, then all $p+2$ firms are in equilibrium.*

Proof. With the arrival of the two additional firms at ξ and $\widehat{\xi}$, the previous p firms are still in equilibrium. Now for the two new firms, the locations of the existing firms at ξ and $\widehat{\xi}$ are strategically irrelevant. Therefore, they satisfy exactly the same set of conditions as did the existing pair at ξ and $\widehat{\xi}$. Hence, all $(p+2)$ firms are in equilibrium. □

Remark 3. *The results of Proposition 5 and Corollary 2 extend to any number of pairs of firms located diametrically opposite each other.*

Observe that the SPNE locations characterized in Propositions 2 and 4, may also be obtained by Proposition 5. For example, it is well known that two firms located at diametrically opposite ends are in equilibrium. Starting from this initial equilibrium, we may now apply Corollary 2 to obtain the SPNE locations characterized by Matsushima (2001). Furthermore, if the initial equilibrium configuration involves an odd number of equidistant firms, Proposition 5(B) may be used to obtain the SPNE locations characterized in Proposition 4.

The following examples illustrate the additional SPNE that can be obtained using Proposition 5.

Example 4. *SPNE locations for $n = 7$*

First note that for $n = 3$, firms located at $0, \frac{1}{3}$ and $\frac{2}{3}$ are in equilibrium. Hence, by Corollary 2, if a pair of firms locate at 0 and $\frac{1}{2}$, then all five firms remain in equilibrium. Thus, $(2, 1, 1, 1)$ firms located at $\left(0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right)$ are in equilibrium for $n = 5$.[14] Then, by Corollary 2, $(3, 1, 2, 1)$ firms located at $\left(0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right)$ are in equilibrium for $n = 7$. Also, by Proposition 5(B), $(2, 1, 1, 1, 2)$ firms located at $\left(0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right)$are in equilibrium for $n = 7$.

Proposition 6 below describes an alternative method to construct SPNE locations from an existing location equilibrium. The motivation behind Proposition 6 is as follows. Superimposing a location equilibrium on top of itself does not alter the aggregate cost medians, thus, if $n = pq$ ($p, q > 1$ are integers), then additional equilibria may be constructed by first determining SPNE locations for p firms and then letting q firms agglomerate at each of these p locations. This intuition turns out to be correct, although the second order condition for profit maximization requires some restrictions that are stated in the proposition below.

[14] The notation $(n_1, n_2, ..., n_k)$ firms locates at $(\eta_1, \eta_2, ..., \eta_k)$ implies that n_i number of firms locate at market point η_i and $\sum_{i=1}^{k} n_i = n$.

Proposition 6. *Suppose that p firms are in equilibrium at* $\left(\xi_1, \xi_2, ..., \xi_p\right)$ *and* $T_p\left(\xi_i\right) \leq T_p\left(\widehat{\xi}_i\right)$ *for* $i = 1, 2, ..., p$. *Let* $q^* = \min\left[2\left(T_p\left(\widehat{\xi}_i\right) - T_p\left(\xi_i\right)\right)\right]^{-1}$ *where* $i = 1, 2, ..., p$. *Then q firms locating at each* ξ_i *form an equilibrium if and only if* $q \leq q^*$.

Proof. Among the original p firms, each firm's location ξ_i is the aggregate cost median of the remaining $p-1$ firms. This implies that among all pq firms, each firm's location ξ_i is still the aggregate cost median of the remaining $pq-1$ firms, since it is the total cost median of all $q-1$ other firms located at ξ_i and also the total cost median of the remaining $q\left(p-1\right)$ firms located at $\underline{\xi}^{-i}$. It remains to show that $T_{pq-1}\left(\xi_i\right) \geq T_{pq-1}\left(\widehat{\xi}_i\right)$.

Note that $T_{pq-1}\left(\xi_i\right) = qT_{p-1}\left(\xi_i\right) = qT_p\left(\xi_i\right)$ and $T_{pq-1}\left(\widehat{\xi}_i\right) = qT_{p-1}\left(\widehat{\xi}_i\right) + \frac{(q-1)}{2} = qT_p\left(\widehat{\xi}_i\right) - \frac{1}{2}$. Hence, $T_{pq-1}\left(\xi_i\right) \geq T_{pq-1}\left(\widehat{\xi}_i\right)$ if and only if $qT_p\left(\xi_i\right) \geq qT_p\left(\widehat{\xi}_i\right) - \frac{1}{2}$ if and only if $q \leq q^*$. This completes the proof. □

Remark 4. *In Proposition 6, if* $p = 2m$ *firms are at opposite ends of any set of* m *diagonals, then* $T_p\left(\xi_i\right) = m = T_p\left(\widehat{\xi}_i\right)$ *and* $q^* = \infty$. *Therefore, q firms locating at each end point of any m diagonals, are in equilibrium for* $1 \leq q \leq \infty$. *This result agrees with Proposition 2.*

Remark 5. *In Proposition 6, if* $p = 2m+1$ *firms are at equidistant equilibrium* $\left(0, \frac{1}{p}, \frac{2}{p}, ..., 1 - \frac{1}{p}\right)$, *then* $T_p\left(\xi_i\right) = \frac{m(m+1)}{(2m+1)}$, $T_p\left(\widehat{\xi}_i\right) = \frac{m^2}{(2m+1)} + \frac{1}{2}$ *and thus,* $q^* = 2m+1$. *Therefore, q firms locating at each point* $\left(0, \frac{1}{p}, \frac{2}{p}, ..., 1 - \frac{1}{p}\right)$ *are in equilibrium if and only if* $1 \leq q \leq 2m+1$.

The following example illustrates the SPNE locations that can be obtained using Propositions 4, 5 and 6. Observe that many of these SPNE locations are asymmetric.

Example 5. *SPNE locations for* $n = 11$

Equilibrium 1: Firms located equidistantly at $\left(0, \frac{1}{11}, \frac{2}{11}, ..., \frac{10}{11}\right)$ are in equilibrium (by Proposition 4).

Equilibrium 2: $(1, 1, 1)$ firm locating at $\left(0, \frac{1}{3}, \frac{2}{3}\right)$ are in equilibrium. Therefore, $(3, 3, 3)$ firm locating at $\left(0, \frac{1}{3}, \frac{2}{3}\right)$ are in equilibrium (by Proposition 6). Now, by Proposition 5, $(4, 3, 1, 3)$ firms at $\left(0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right)$ are in equilibrium.

Propositions 4 and 6 may be combined, giving rise to the result below.

Corollary 3. *Let* $n = (2k + 1)q + 2l$*. Suppose that* $(q, q, ..., q)$ *firms are located at* $(0, \frac{1}{2k+1}, ..., \frac{2k}{2k+1})$*, and the remaining* $2l$ *firms are located at the opposite ends of any of the* $2k+1$ *diameters passing through* $0, \frac{1}{2k+1}, ..., \frac{2k}{2k+1}$*, then the firm locations are sustainable as SPNE locations if* $0 \leq q \leq 2k + 1$*.*

Observe that the SPNE locations characterized in this subsection exhibit characteristics similar to those obtained in the previous two subsections. First, there are multiple equilibria. Second, both equidistant and non-equidistant SPNE locations are possible. Third, the equilibrium location patterns exhibit partial agglomeration, where all firms never agglomerate at the same location and each agglomeration point includes only a subset of the firms.

The theorem below summarizes various simple SPNE location patterns that may arise in a circular city. The previous propositions, however, may give rise to few more (relatively complex) equilibrium location patterns, which are not listed in Theorem 1. Diagram 3 that follows the theorem illustrates alternate SPNE location patterns for $n = 1, ..., 9$, that are obtained using Theorem 1.

Theorem 1. *Consider a two-stage quantity setting oligopoly involving* n *firms, with each firm choosing its location in the first stage. The following location patterns can be sustained as SPNE location:*

(A) n *firms locate equidistantly on the circle.*

(B) $n = 2m$, *the firms locate (pair-wise) at the opposite ends of any sets of diameters.*

(C) $n = 2m + 1$, $(m + 1)$ *firms agglomerate at any one point on the circle and the remaining* m *firms agglomerate at the diametrically opposite point.*

(D) Some of the firms locate equidistantly, such that an even number of firms are remaining, which locate (pair-wise) at the opposite ends of any diameter passing through one of the initial equidistant firm location.

(E) $n = pq$, p *firms are in equilibrium at* $(\xi_1, \xi_2, ..., \xi_p)$ *and* q *firms agglomerate at each of* ξ_i $\forall i = 1, ..., p$ *(subject to a possible upper bound on* q*, see Proposition 6).*

(F) $n = (2k + 1)q + 2l$, q *firms agglomerate at each of the* $(2k + 1)$ *equidistant points and remaining* $2l$ *firms locate at the opposite ends of any*

of the $2k + 1$ *diameters passing through one of the initial equidistant firm location (subject to* $0 \leq q \leq 2k + 1$*).*

[Insert Diagram 3 here.]

5. Conclusion

In this paper, we have demonstrated that the analysis of location choice in the circular city model is much richer than previously anticipated. Since the circular city model is perfectly symmetric and no location is *a priori* better than another, the literature on spatial competition typically assumes that firms would locate equidistantly around the circle. We have established that under quantity competition, the equidistant location pattern is only one of the many location equilibria that may arise in a circular city model. In fact, we have identified non-equidistant, multiple and sometimes a continuum of location equilibria in the perfectly symmetric circular city model.[15] The demonstration of non-equidistant equilibria in a perfectly symmetric circular city model is counter-intuitive. The demonstration of infinitely many location equilibria is distinct from the previous findings in the literature on spatial Cournot competition, where the location equilibria are always finite and mostly unique.

[15]Note that not all equilibria of the model have been characterized in the paper. We do not anticipate, however, to find an equilibrium that cannot be obtained by using the techniques documented in the paper, although we lack a formal proof to support the claim.

Our work is useful in understanding the contrasting location patterns that arise under alternative market configurations (linear or circular markets), and the nature of the competition (price or quantity competitions). For both linear and circular markets, and for a variety of alternative pricing strategies, price competition yields spatial dispersion (d'Aspremont, Gabszewicz and Thisse (1979), Lederer and Hurter (1986), Kats (1995)). On the other hand, quantity (Cournot) competition in a linear market gives rise to a unique equilibrium, where all firms agglomerate at the market center (Hamilton, Thisse and Weskamp (1989), Anderson and Neven (1991), Gupta, Pal and Sarkar (1997)). In contrast, we show that quantity competition in a circular city may yield various combinations of spatial agglomeration and dispersion. Thus, unfortunately, spatial agglomeration or dispersion cannot simply be categorized either by the market structure or by the instruments of competition.

The nature of the agglomeration equilibria we have demonstrated is distinct from those characterized in the existing literature of spatial competition. The agglomeration results in the literature typically involve all firms agglomerating at the same location. In contrast, we have shown that in a circular city model, all firms never agglomerate at the same location; each agglomeration point includes only a subset of firms. This pattern is consistent with actual firm locations in major cities, where all firms most definitely do not agglomerate at a single market point and each agglomeration point typically involves different numbers of firms. Neither Cournot competition in a linear market, nor price competition in either linear or circular markets yield results that are consistent with the observation.

The results we have demonstrated have significant implications. For example, the circular city model is often used to analyze various economic phenomena ranging from entry in a market with differentiated products to competition among airlines and television networks. The analyses are always carried out under the assumption that the firms would locate equidistantly around the circle. We have demonstrated multiple and sometimes infinitely many location equilibria that are distinct from the equidistant locations. Obviously, it raises questions about the appropriateness of policy analysis based on the assumption of symmetric and equidistant locations. It also calls for a reasonable mechanism to select one of the many location equilibria.[16]

Variations in transport costs may resolve the multiplicity problem by giving rise to a unique equilibrium. In this paper, we assume linear transport cost that is low enough (relative to demand at each market point) such that each firm serves the entire market. A sufficiently non-linear transport cost or a transport cost that is large enough such that no firm serves the entire

[16] In this paper, multiple location equilibria arise with an exogenously fixed number of firms. In contrast, for spatial models with free entry, multiple location equilibria arise due to different number of firms at various equilibria. In this context, the "zero profit equilibrium" is typically assumed in literature. Recently, Anderson and Engers (2001) resolve the multiplicity by making the firms compete for entry positions. Remarkably, the resulting unique equilibrium is distinct from the "zero profit equilibrium".

market, each is likely to eliminate most of the location equilibria involving agglomeration.

Observe that in this paper, multiple equilibria arise primarily due to the fact that two firms located diametrically opposite to each other are "strategically irrelevant" to the remaining firms. This is because the aggregate delivered marginal cost of these two firms is constant along the circumference of the circle. The variations in the transport cost proposed above would give rise to variable (non-constant) aggregate delivered marginal cost along the perimeter and may destroy the "strategic irrelevance" of the two diametrically opposite firms, eliminating several of the equilibria demonstrated in this paper.

For example, consider the scenario where the transport cost is linear, but large enough so that no firm serves the entire market. Let the transport cost function be symmetric and linear on either side of the firm's location but only up to an arc distance, equal to one-fourth of the entire circumference. Thereafter, the transport cost becomes prohibitively large (relative to demand) so that market points any farther than one-fourth of the circumference are not served by the firm. Under this set up, the aggregate transport cost function of the two firms located at 0 and $\frac{1}{2}$ is no longer constant along the perimeter. It is piece-wise linear on arcs $\left(0, \frac{1}{4}\right)$, $\left(\frac{1}{4}, \frac{1}{2}\right)$, $\left(\frac{1}{4}, \frac{3}{4}\right)$, and $\left(\frac{3}{4}, 1\right)$ with maximum at $\frac{1}{4}$ and $\frac{3}{4}$, and minimum at 0 and $\frac{1}{2}$. It can be verified that with $n = 3$, two firms locating at 0 and one locating at $\frac{1}{2}$ is no longer sustainable as an SPNE. One of the firms at 0 finds it profitable to relocate at

$\frac{1}{4}$ or at $\frac{3}{4}$, where it's competitors' aggregate delivered marginal cost reaches its maximum. Here, the unique SPNE location is the one where the firms locate equidistantly from one another.

Similarly, non-linear transport cost may destroy the "strategic irrelevance" of two diametrically opposite firms and may eliminate several equilibria obtained using the logic. For example, consider a concave transport cost, where the transport cost is proportional to the square root of the length of the shorter arc between the location of the firm and the market point. It can be checked that with $n = 3$, the only SPNE is the one where the firms locate equidistantly from one another.[17] The case for a convex transport cost, however, is more intricate, since a convex transport cost may preserve the peaks of the aggregate transport cost that are obtained under a linear transport cost. Although we anticipate most of the SPNE patterns identified in this paper, other than the one where all firms locate equidistantly, would fail to be supported as SPNE under sufficiently non-linear transport cost, we, however, make no claim that other new patterns of SPNE cannot be constructed for some specific transport cost functions. Further investigation in this direction should be worthwhile.

[17] A piece-wise linear transport cost may also destroy the "strategic irrelevance" of two diametrically opposite firms and eliminate several equilibria involving agglomeration. For example, if $t(x-\xi_i) = \left\{ \begin{array}{c} |x-\xi_i| \;\; if \;\;\; |x-\xi_i| \leq \frac{1}{4} \\ \frac{1}{4} \; if \;\; |x-\xi_i| > \frac{1}{4} \end{array} \right\}$ then with $n = 3$, the only SPNE is the one where the firms locate equidistantly from one another.

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