On the Interactions Between Routing and Inventory-Management Policies in a One-Warehouse N-Retailer Distribution System

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Abstract

This paper examines the interactions between routing and inventory-management decisions in a two-level supply chain consisting of a cross-docking warehouse and \( N \) retailers. Retailer demand is stochastic; i.e., normally distributed and independent across retailers and over time. Travel times are fixed between pairs of system sites. Every \( m \) time periods, system inventory is replenished at the warehouse, whereupon an uncapacitated vehicle departs on a route that visits each retailer once and only once, “dynamically allocating” its inventory along the way (i.e., allocating its inventory depending of the status of inventory at the retailers who have not yet received allocations). Our goal is to determine a combined system inventory-replenishment, routing, and inventory-allocation policy that minimizes the total expected cost/period of the system over an infinite time horizon. Our analysis begins by examining the determination of the optimal “static” route; i.e., the best route if the vehicle must travel the same route every replenishment-allocation cycle. If total vehicle travel time or cost is the only cost considered, then, of course, the optimal static route is the TSP (traveling-salesman problem) route. However, as we demonstrate, if inventory-holding and backorder-penalty costs are also considered, then the optimal static route also depends on the variances of customer demand, and, if inventory-holding costs are incurred on the vehicle, also on the means of customer demand. As a consequence, the optimal static route can be quite different from the TSP route. We then examine “dynamic” routing policies; i.e., policies that may change the route from one system-replenishment-allocation cycle to another based on the status of the retailers’ inventories. Here, we argue that in the absence of transportation-related cost, the optimal dynamic routing policy should be viewed as balancing management’s ability to respond to system uncertainties (by changing routes) against system uncertainties that are induced by changing routes. We then examine the performance of a heuristic policy for deciding, in any given cycle, whether to use the optimal static route or to switch to one of the \((N!-1)\) alternative routes. Simulation tests of this “change-revert” heuristic for systems with \( N = 2 \) and \( N = 6 \) indicate that its use can substantially reduce system inventory-related costs even if most of the time the chosen route is the optimal static route.
1. Introduction

The goal of this paper is to examine the interactions between routing and inventory-management policies in a one-warehouse, \(N\)-retailer distribution system facing stochastic customer demand. Since 1990, about 25 published papers have examined the joint determination of transportation (e.g., routing, vehicle/customer assignment) and inventory-management policies in supply chains, some in complex business scenarios. See Schwarz, et al., 2004. We believe that this paper is the first to explicitly examine the interactions between routing and inventory-management policies. Although the model we examine is very stylized, we are able to use it to get analytical expressions that represent the trade-off between routing and inventory-management considerations. Then, using simulation, we are able to substantiate the insights provided by the analytical model under more realistic assumptions.

Why should the manager of a warehouse-retailer distribution system, one with authority to determine policy for both vehicle routing and inventory management, care about the interaction between them? Most important, since vehicles are integral to the supply chain, transportation policy influences supply-chain performance. Hence, in the same way that the basic interactions between buyer and supplier inventory-management policies should be understood in managing system inventories, we believe that the basic interactions between transportation and inventory-management policies should also be understood. Competitiveness, particularly in retailing, depends on providing high availability (e.g., high fill-rates) with low average inventory levels (i.e., high turnover). It is well known that “dynamic allocation” (i.e., postponing the allocation of vehicle inventory to the moment that delivery occurs) improves competitiveness given an arbitrary “static” (i.e. fixed) vehicle route (Kumar, et al., 1995).

Technology is available to facilitate both dynamic allocation and dynamic routing. In particular, retailers already use bar codes and RFID to monitor inventory (e.g., Wal-Mart) and transport companies already employ technology to route or reroute vehicles (e.g., Schneider trucking). Given that the technology is already in place, it behooves supply-chain researchers to learn when and how to take advantage of such technology to implement dynamic allocation and dynamic routing.

One contribution of this paper is an analysis of how the choice of a static route affects inventory-management cost (and vice versa). Another contribution is an examination of how
“dynamic” routing (i.e., changing the route from one replenishment-allocation cycle to another based on the status of the retailers’ inventories) impacts system performance. Finally, we develop and test heuristics for making dynamic-routing decisions.

In what follows we first describe some model specifics and the major assumptions used in our analysis. Then we summarize our results.

2. Description of Model, Major Assumptions, and Summary of Results

The model we examine consists of a single cross-docking warehouse, \( N \) retailers, one uncapacitated vehicle, and a single item (SKU). This supply chain is centrally managed, using a periodic-review system. Each retailer \( i \) experiences normally-distributed customer demand \( \left( \mu_i, \sigma_i \right) \) each time period. Demand realizations are independent across time and across retailers. Excess demand is backordered at cost \( \$p \)/unit-period; and a holding cost of \( \$h / \)unit-period is charged on system (i.e., on-vehicle or at-retailer) inventory. Transportation times between system sites are fixed and known. Specifically, the travel time between the warehouse and retailer \( i \) is \( r_{wi} > 0, i = 1, \ldots, N \); the travel time between retailer \( i \) and \( j \) is \( r_{ij} > 0, i, j = 1, \ldots, N \). Transportation costs between system sites are considered in our analysis of static-routing, but not in our analysis of dynamic-routing. The costs of the management systems associated with fixed versus dynamic routing are ignored, since our goal is to examine interactions and benefits.

The system operates as follows: Every \( m \) time periods the system places an order on an outside supplier, and the warehouse receives it instantaneously. Upon receipt, an uncapacitated vehicle immediately departs on a route that visits each retailer once and only once, “dynamically allocating” its inventory along the way (i.e., allocating its inventory depending of the status of inventory at the retailers who have not yet been allocated inventory). Our goal is to determine a combined system inventory-replenishment, routing, and inventory-allocation policy that minimizes the total expected cost/period of the system over an infinite time horizon.

In what follows we provide a few definitions that are helpful in understanding our results.

**Time Periods, Replenishment Cycles and Allocation Cycles:** Without loss of generality, we assume that delivery and allocation to a given retailer occur simultaneously at the start of the corresponding time period, that demands occur during each time period, and that inventory-related costs are charged on end-of-period net inventory. The \( m \) periods between system
replenishments are defined to be the system \textit{replenishment cycle}. The periods between successive allocations of vehicle stock to retailer \(i\) is defined to be retailer \(i\)'s \textit{allocation cycle}. We define the “start” of any given retailer’s \(k\)th allocation cycle to be the time period when it receives its allocation from the \(k\)th system replenishment. Correspondingly, the “end” of the retailer’s \(k\)th allocation cycle is the period immediately prior to receiving its allocation from the \((k+1)\)st system replenishment. The set of \(N\) retailer allocation cycles, together with their associated system replenishment cycle is called the system \textit{replenishment-allocation cycle}.

Figure 1 illustrates a typical replenishment-allocation cycle.

\textbf{Routes and Routing Policy:} A route specifies the sequence of retailer visits/allocations associated with some given replenishment-allocation cycle. A \textit{routing policy} describes the routes to be used over two or more replenishment-allocation cycles. Under a fixed routing policy, the route used in each cycle is known prior to the start of the first cycle. A special case of fixed routing is static routing. Under a \textit{static routing policy}, the route associated with every replenishment-allocation cycle is the same. We examine the determination of the optimal static route in Section 4. Under a \textit{dynamic routing policy} the route may change between replenishment-allocation cycles, depending on the status of the system. For example, a \textit{least-inventory-first (LIF) routing policy} (Park, et al., 2002) examines the vector of net inventories for those retailers not yet visited on the given route, and travels next to the retailer with the smallest net inventory. Given these basics, we can now provide an overview of the interactions between inventory-management and routing. Details are provided in Section 5.

\textit{An Overview of How Routing and Inventory Policy Interact}

Most fundamentally, each retailer’s delivery leadtime (i.e., the number of time periods between a given system replenishment and each retailer’s allocation) depends on which of the \(N!\) possible routes is chosen. Let \([j]\) be the index of the \(j\)th retailer on the route, and let \(B_{[j]}\) be its delivery leadtime. Then \(B_{[1]} = r_{0[1]}\); \(B_{[2]} = r_{0[1]} + r_{1[2]}\) and, in general, \(B_{[j]} = \sum_{k=0}^{j-1} r_{[k][k+1]}\).

Since, inventory-management (e.g., inventory-holding and backorder penalty) costs depend on leadtimes, it follows from a supply-chain perspective, that routing and inventory-management policies should be determined jointly, not sequentially. Routes also determine the first and last periods in each retailer’s allocation cycle. Under \textit{any} fixed routing policy, the number of periods in every retailer’s allocation cycle is known at the time of allocation. The number of
periods in each allocation cycle plays a key role in determining appropriate allocation quantities. Since routes impact leadtimes and allocation quantities, the optimal routing policy is not typically found by solving a traditional traveling-salesman problem (TSP). Indeed, as we will demonstrate, determining the optimal static route is more complicated than solving a TSP.

Under a dynamic routing policy, the interactions between routing and inventory management are even more complex. As with a fixed routing policy, the route determines each retailer’s delivery leadtime. However, under dynamic routing, delivery leadtimes and the number of periods in each retailer’s allocation cycle become uncertain; i.e., subject to the quantities allocated, to future demand realizations, and to the specific routing decision-rule. This makes the allocation decision considerably more complex than in the fixed-route case.

Major Assumptions:

Three major assumptions are made in order to facilitate our analysis. We describe each assumption below. As we report in Section 8, we tested the validity of these assumptions via simulation; and, conclude that for the high service-level systems we examine: (1) these assumptions are violated no more than 5% of the time; and (2) even when they are violated, the insights provided by our analysis appear to be insensitive to these assumptions. (See Section 8.)

The Allocation Assumption: The allocation assumption was introduced by Eppen and Schrage (1981) and has been widely adopted by others (e.g., Kumar, et al). In the language of our model, it means that as the vehicle travels along its route allocating inventory, it is always possible to make a cost-minimizing allocation (i.e., one that equalizes the marginal expected costs) among the retailers not yet visited on that route. In effect, this assumption allows negative allocations. The role of the allocation assumption is that, for any given system-replenishment and routing policy, the allocation assumption decouples the allocation decisions in any given replenishment-allocation cycle from the allocation decisions made in any other replenishment-allocation cycle.

The Returns-Without-Penalty Assumption: In analyzing multi-period newsvendor-type models it is often assumed that, if desired, system stock can be reduced instantaneously and without penalty. In effect, negative replenishments are allowed. See, for example, Kahn (1987) and Lee, et al. (1997). Although this assumption is seldom valid in practice, it is
typically innocuous when costs and demand parameters are stationary, and demands are non-negative.

In combination, the dynamic allocation and returns-without-penalty assumptions decouple replenishment and allocation decisions made in any given replenishment-allocation cycle from the corresponding decisions in any other replenishment-allocation cycle. More specifically, under a fixed routing policy the optimal replenishment and allocation policy for a single replenishment-allocation cycle yields the optimal replenishment and allocation policy for any arbitrary number of consecutive replenishment allocation cycles. We summarize this single-cycle problem below.

**The Last-Period Backorders Assumption:** This assumption was introduced by Jonsson and Silver (1987). In the language of our model it means that retailer backorders, if they occur, only occur in the last period of that retailer’s allocation cycle.

**The Single Cycle**

The shaded portion of Figure 1 illustrates a single replenishment-allocation cycle — henceforth called a “cycle” — in which \( m = 8 \) periods and \( N = 3 \) retailers. We have arbitrarily identified the first period as period \( t = 0 \). Note that the vehicle first goes to Retailer 2, then Retailer 1, then Retailer 3. Note, further, that in the next cycle the vehicle goes to Retailer 1 first. Hence, Retailer 1’s allocation cycle starts in period 3, because in this cycle \( B_1 = 3 \), and ends in period 9, because in the following cycle \( B_1 = 2 \).

The total cost in each period is given by \( h \) times the sum of all end-of-period on-hand inventories, plus \( p \) times the sum of all end-of-period backorders from all the retailers. Let \( E_t \) be the system net inventory in period \( t \), which equals the sum of the net inventory (backorders count as negative net inventory) at the vehicle and all retailers. Let \( S_t \) denote the backorders in period \( t \) for any retailer whose allocation cycle ends in period \( t \). The corresponding system inventory cost in period \( t \) is given by \( C_t \), where

\[
C_t = hE_t + (h + p)S_t
\]

We uniquely assign each element of (1) to a specific replenishment-allocation cycle, using the same cost assignment as in Park (2002). The holding cost, \( hE_t \), is assigned to the
replenishment cycle containing period \( t \). The backorder-related cost, \((h + p)S_i\), is assigned to the allocation cycle in which the backorders occur. To illustrate, the total costs assigned to the shaded cycle in Figure 1 would be
\[
\sum_{t=0}^{7} E_t + (h + p)(s_{1,9} + s_{2,11} + s_{3,12}),
\]
where \( s_{i,t} \) is the backorders at retailer \( i \) in period \( t \).

Given any fixed route, and under the assumptions described above, the optimal replenishment-and-allocation policy for the single-cycle problem is also the optimal replenishment-and-allocation policy for any arbitrary number of adjacent cycles. Further, the optimal replenishment policy is a base-stock policy; and, an optimal allocation policy is one that minimizes expected backorders. See Appendix A for details.

However, under dynamic routing, the above assumptions are not sufficient to decouple decisions in adjacent cycles. In particular, under dynamic routing, allocation decisions in the current cycle impact the route selection in the next cycle, which define allocation cycle lengths in the current cycle. Section 5 considers the complexity of dynamic routing in more detail.

**Summary of Results:**

Our examination begins in Section 4 where we provide the expected inventory-related cost/cycle associated with any given static route. We also provide necessary and sufficient conditions to select the route that minimizes these costs in the case where inventory-holding costs are charged only on retailer inventory; and, sufficient conditions for the optimal static route in the case where inventory-holding costs are also charged on vehicle inventory. We also describe how transportation (i.e., traveling-salesman) costs can be combined with inventory-related costs to determine the optimal static route.

Section 5 examines the interactions between routing and inventory-management decisions in detail, summarizes the Park, et al.’s (2002) results for “symmetric” retailers, and lays the foundation for the “change-revert” dynamic-routing decision rule. Sections 6 develops the change-revert routing decision-rule. Section 7 describes our analytical model. Section 8 uses simulation to assess the validity of the analytical model and to assess the value of the change-revert heuristic under a variety of parameterizations for \( N = 2 \) retailers. We observed statistically-significant savings of between 0.55% and 10.44% over 27 parameterizations with change-route decisions in fewer than 20% of the replenishment cycles for most parameterizations. A modified, “threshold” change-revert, heuristic is demonstrated to provide
most of the savings of the original, but with significantly fewer change-route decisions. We also demonstrate the effectiveness of the change-revert heuristic for example systems with negative-binomial retailer demand. Section 9 examines the application of the change-route heuristic to systems with $N = 6$ retailers. Here, too, we observe that change-revert provides statistically significant and managerially meaningful savings.

3. Literature Review

The literature of supply-chain models that involve both transportation and inventory-management considerations is large and growing. A recent survey (Schwarz, et al, 2004) cites approximately 50 articles published in Management Science, Operations Research, Transportation Science, Transportation Research, the European Journal of Operations Research, the Journal of Business Logistics, etc. since 1990. Roughly half of these models either choose a transportation policy (e.g., vehicle and/or customer assignment, routing) for a prespecified inventory policy or choose an inventory policy for a prespecified transportation policy. The other half consider transportation and inventory as joint decision variables. In what follows we cite the most closely-related work; i.e., models with stochastic customer demand that consider both transportation and inventory management as joint decision variables.

To the best of our knowledge, Federgruen and Zipkin (1984), was the first attempt to integrate inventory-management and routing decisions in a single model. They use a nonlinear-programming formulation to make vehicle-assignment, routing, and inventory-replenishment decisions for a single day (i.e., single replenishment-allocation cycle). Vehicle-travel cost between locations is considered, but transportation times are ignored. Their heuristic solution is based on decomposing the joint decision into an inventory-allocation and routing decisions, but coordinating them appropriately. Our model can be viewed as extending Federgruen and Zipkin to account for transportation times between sites and to employ both dynamic routing and dynamic allocation.

In Golden, et al. (1984), the retailers use tanks with limited capacity to store inventory. The business scenario involves the distribution of liquid propane. The system they develop makes three kinds of decisions: (1) customer selection; i.e., which customers to deliver to on a given day or week; (2) customer/vehicle assignment; and (3) vehicle routing. Customers are selected on the basis of their “eligibility” (i.e., the ratio of remaining tank capacity to tank
capacity) and the vehicle travel time available. Inventory-management costs are not explicitly modeled. Instead, the focus is on maintaining customer inventories above some specified level (e.g. 30% of capacity) and avoiding “premature” deliveries (i.e., deliveries when customer inventories are above 50%). Although demand on tank inventory is stochastic, it is modeled deterministically using forecasted demand. Based on simulation tests, the proposed system compared favorably to existing company policy.

Kleywegt, et al. (2002a, 2002) examine a business scenario similar to that of Golden, et al., with \( M \) vehicles, each with capacity \( C_v \), and each customer with storage capacity \( C_n \). Periodically the supplier makes decisions about which customers to replenish and how much (using static allocation), how to combine customers into vehicle routes, and which routes to assign to which vehicles. Their model includes travel costs (but ignores travel times) between sites, inventory-holding costs, and lost-sale penalty costs. Their objective is to maximize expected discounted profit over an infinite time horizon. The authors formulate this problem as a Markov decision process and propose approximate methods (i.e., customer decomposition) to find “good solutions with reasonable computational effort”.

Adelman (2001) examines a multi-item routing/inventory-management model with joint replenishment costs. His model is similar to the inventory-routing problem (IRP) of Kelywegt, et al., except that in the IRP vehicle capacity is limited and only a single product is considered, whereas in Adelman’s model the focus is on managing a set of items, which might represent a product, a location, or a product-location pair. In the specific scenario Adelman describes, a dispatcher decides which items and quantities to replenish in the current period, and then combines them into disjoint subsets, \( I \), incurring a fixed cost \( C_I \) for each. Each item has a storage limit and each subset has a limit on the total replenishment quantity across items. The dispatcher’s problem is to manage replenishments each period in order to minimize the expected total discounted cost over an infinite horizon. Adelman also formulates the problem as a Markov decision process, but uses a price-directed heuristic control policy that approximates the future using dual-prices from linear-programming relaxations.

Cetinkaya and Lee (2000) formulate a renewal-theoretic model of a vendor-managed inventory business scenario in order to jointly determine, in the language of our model, system replenishment times (i.e., “agreeable dispatch times”) and replenishment quantities. Qu, et al.
(1999) examine a multi-item joint-replenishment problem for a warehouse that is replenishing its inventory by routing a vehicle to collect shipments from its suppliers. They develop an (R,T)-based heuristic to minimize dispatching, stopover, routing, and inventory-management (ordering, holding, backordering) costs.

**Dynamic Allocation:** Kumar, et al (1995) examine the same supply-chain model we examine here, operating under a static (i.e., fixed) routing policy and using static allocation, and assess the reduction in system variance and cost/time provided by dynamic allocation. Reiman, et al. (1999) incorporate dynamic allocation in their heavy-traffic analysis of a similar supply-chain model, but with a single capacitated vehicle. Berman and Larson (2001) incorporate dynamic allocation in their model of industrial-gas distribution. Their analysis does not incorporate inventory costs, per se, but, instead, linear costs associated with making deliveries either earlier or later than desired. Bassok and Ernst (1995) incorporate dynamic allocation in a multi-product distribution setting. Their model also allocates space on the vehicle to different products.

**Dynamic Routing:** Of course, any model that examines only a single system replenishment and/or allocation cycle can be viewed as incorporating dynamic routing, since, from cycle to cycle, different routes will be chosen. However, such models are, in effect, selecting dynamic routes myopically. Only a few published models explicitly address dynamic routing policy from both a long-term (i.e., multi-cycle) perspective and in conjunction with the corresponding inventory-management policy. Trudeau and Dror (1992), examine what they call the “stochastic vehicle routing problem”, which is the traditional inventory-routing problem, but with customer demands not revealed until the vehicle arrives at a customer to make a delivery. Inventory costs are not modeled, per se, but “route failures” (i.e., what happens when a vehicle is unable to satisfy all the demand on a given route) are examined. They compare several different solution procedures for this problem. Reiman, et al. (1999) incorporate a limited form of dynamic routing in their model (described above). Savelsbergh and Goetschalckx (1995) compare the efficiency of fixed (i.e., static) versus variable (i.e., dynamic routes) in a stochastic inventory-routing problem. Their objective is to minimize the total cost of travel for servicing all customers. As in Trudeau and Dror (1992), inventory-management costs are not modeled. Instead, it is assumed that every customer has its tank filled, even if this requires having the vehicle return to the depot for resupply. Finally, Park, et al. (2002) examine both dynamic
allocation and dynamic routing in the same model as we examine here, but under the limiting assumption that all between-retailer travel times are the same and all warehouse-retailer travel times are the same. Their results are described more completely in Section 5.

4. Choosing the Optimal Static Route

We begin examining the choice of the optimal static route using the model of Kumar, et al. (1995), henceforth labeled “Kumar”, because its results are well known, and because its analysis sheds light on the impact of on-vehicle inventory-holding costs on routing decisions. Kumar’s static-routing model is identical to ours except, in Kumar, inventory-holding cost is charged only on retailer inventory. Initially, our analysis ignores transportation-related costs.

Given any static route, Kumar describes how to make optimal dynamic allocations (Theorem 5.1) and establishes the optimality of a system base-stock inventory-replenishment policy (Theorem 5.2). Both results are based on showing that these decisions are equivalent to the corresponding decisions for a hypothetical “composite” retailer. For example, Kumar establishes that each allocation equalizes the runout probabilities of two retailers: the one currently receiving an allocation and the other, a composite retailer that represents the set of all retailers that have not yet received an allocation. Appendix A describes its construction. Note that the construction of the composite retailer in Appendix A differs slightly from Kumar. However, the resulting allocations and replenishments are the same. Similarly, the system-replenishment decision is shown to be equivalent to the replenishment of the single composite retailer representing all \( N \) retailers. In particular, the optimal base-stock is given by

\[
Y^* = \mu_i^C + \sigma_i^C K^*
\]  

where \( K^* \) is such that

\[
\Phi \left( K^* \right) = \left( p - h(m-1) \right) / \left( p + h \right)
\]  

where \( \Phi \) is the standard normal cdf and

\[
\mu_i^C = \sum_{i=1}^{N} \mu_i (m + B_i)
\]

\( \sigma_i^C \) is determined recursively as follows
\[(\sigma_N^C)^2 = (m + h_{(N)}) (\sigma_{(N)})^2\]
\[(\sigma_j^C)^2 = h_{(j)} \sum_{r=1}^{N} (\sigma_{r(j)})^2 + (\sigma_{r(j)} \sqrt{m + \sigma_{j,1}^C})^2,\]

for \( 1 \leq j \leq N-1 \) \hfill (5)

In the above, \( \mu_j^c \) and \( \sigma_j^c \) are the mean and standard deviation of the composite retailer representing all \( N \) retailers, and \( h_{(j)} \) is the incremental leadtime to retailer \([j]\) (i.e. \( h_{(j)} = B_{(j)} - B_{(j-1)} \)). Note that (2)-(3) can be interpreted to be the optimal base stock for the standard newsvendor model with normally-distributed demand (with mean \( \mu_j^C \) and standard deviation \( \sigma_j^C \)), leftover cost \( hm \) and backorder cost \( p - h(m-1) \).

Although Kumar (1995) based (2) through (5) on the assumption that inventory-holding cost (\$/unit-period) is charged only at the retailers (i.e., “retailer-only holding” cost), we show in Appendix A, that they remain valid under our cost structure, in which inventory-holding cost is also charged on vehicle inventory (i.e., “system-based holding” cost). While \( Y^*, K^*, \mu_j^C, \) and \( \sigma_j^C \) are identical under the two cost structures, \( Z^s \), the optimal expected per cycle cost differ. Specifically,

(retailer-only holding costs) \hfill \( Z^s = H^0 + (p + h)\phi(K^*)\sigma_j^C \) \hfill (6a)

(system-based holding costs) \hfill \( Z^s = H^0 + H^F + (p + h)\phi(K^*)\sigma_j^C \) \hfill (6b)

where \( H^0 = h[m(m-1)(\sum_{i=1}^{N} \mu_i) / 2] \) is the same for all routes, and \( H^F = hm \sum_{i=1}^{N} \mu_i B_i \) depends upon the route. (6a) is from Theorem 5.2b of Kumar, and (6b) is (A6) from Appendix A, specialized for a static route. \( H^0 \) is the expected holding cost of the inventory required to satisfy average demand over an \( m \)-period replenishment cycle. Under system-based holding
costs (6b), $Z^S$ has an additional term not present in (6a), $H^F$, equal to the average on-vehicle holding cost per replenishment cycle.

Under either cost structure, the optimal static route minimizes (6a) or (6b) over all possible $N!$ routes. With retailer-only holding costs, observe that the optimal static route minimizes $\sigma_1^C$ in (5), and that $\sigma_1^C$ depends only on the retailer delivery leadtimes and their per-period demand variances.

For the special case of $N = 2$ retailers, (6a) reduces to:

$$H^0 = (\mu_1 + \mu_2)hm(m-1)/2$$  \hspace{1cm} (7)

$$\sigma_1^C = \sqrt{r_{0[1]} + m(\sigma_1^2 + \sigma_2^2) + r_{1[2]}\sigma_2^2 + \sigma_1\sigma_2\sqrt{m(m + r_{1[2]})}}$$  \hspace{1cm} (8)

Using (8), it is straightforward to show that if travel times are such that $r_{01} < r_{02}$ and $r_{12} = r_{21}$, then retailer 1 is first on the optimal static route if and only if:

$$(\sigma_2^2 - \sigma_1^2)/(\sigma_2^2 + \sigma_1^2) < (r_{02} - r_{01})/r_{12}$$  \hspace{1cm} (9)

More generally, a sufficient condition to visit the closest retailer first is for the closest retailer to also have the largest variance. Otherwise, the choice of the optimal static route depends on a trade-off between travel times (i.e., a TSP-oriented parameter) and per-period demand variance (i.e., an inventory-management parameter).

Now, consider the choice of the optimal static route under system-based holding costs; i.e., the minimization of (6b) for the same special case of $N = 2$ retailers. Equation (6b) becomes:

$$Z^* = H^0 + (p + h)\phi(K^*)\sigma_1^C + hm(B_1\mu_1 + B_2\mu_2)$$  \hspace{1cm} (10)

From (10), one can see the impact that retailer mean demand has on the choice of the optimal static route: everything else being equal, the pipeline inventory-holding cost component of (10) favors visiting the retailer with the largest mean demand first.

The role of retailer mean demand — which has no impact on the choice of the optimal static route given retailer-only holding costs — makes the determination of the optimal static route more complicated, even in the case of $N = 2$ retailers, since this choice no longer depends
only on $\sigma_i^C$, but on a trade-off between $\sigma_i^C$ and the mean per-period demand at each retailer. (For $N = 2$ and $r_{12} = r_{21}$, it is trivial to show that if the retailers are labeled so that retailer 1 is closest to the warehouse (i.e., $r_1 \leq r_2$), then a sufficient condition for retailer 1 to be first on the optimal static route is for it to have the largest variance ($\sigma_1 \geq \sigma_2$) and the largest mean demand ($\mu_1 \geq \mu_2$).) More generally, if $\sigma_i^C$ is viewed as a route-dependent measure of system variance, and $H^F$ is viewed as a route-dependent measure of system pipeline mean inventory, then the choice of the static optimal route involves a trade-off between these system mean and variance values.

Since, in general, under either cost structure the optimal static route must be determined by enumeration of the $N!$ alternatives, it is straightforward to incorporate direct transportation costs (e.g., total mileage, total travel time) into the analysis. In particular, if $T_k$ is the total transportation cost of route $k$, and $Z_k^S$ is (6a) or (6b) under route $k$, then the optimal static route $k^*$ solves:

$$k^* = \text{Minimize}_{\text{all possible routes }, k} \{T_k + Z_k^S\}$$

We now turn to our analysis of dynamic routing. Note that the model being analyzed in all the subsequent sections ignores direct transportation cost and assumes a system-based holding-cost structure.

5. Dynamic-Routing Policies

Under a “dynamic-routing policy” the vehicle route is permitted to change from one replenishment-allocation cycle to another, based on the status of system inventories. In the business scenario we examine, dynamic routing provides the opportunity to expedite the allocation of vehicle inventory to retailer/s whose inventory is “running low”, thereby avoiding backorders that otherwise might occur. However, such expediting necessarily postpones deliveries to other retailers, thereby possibly causing backorders that otherwise would not have occurred. Nonetheless, under certain circumstances, it may be desirable to change routes in order to minimize total expected costs in some given single replenishment-allocation cycle. We will illustrate how such a change is evaluated in Section 6.
Of course, regardless of the inventory-management benefit of a “change-route” decision in some given replenishment-allocation cycle, a dynamic-routing policy can increase transportation-related costs. Further, because of its associated instabilities in routing and delivery schedules, dynamic routing introduces two types of uncertainties, neither of which is present under a static routing policy: (1) “\(U_{al}\) uncertainty”; i.e., uncertainty in each retailer’s delivery leadtime; and (2) “\(U_{pd}\) uncertainty”; i.e., uncertainty in the number of periods of customer demand each retailer’s allocation is to supply.

Hence, a management decision to adopt a dynamic-routing policy should be viewed as a trade-off between reducing expected inventory-related costs (compared to the corresponding expected costs if a static route were used), in exchange for increased logistics-related (e.g., transportation) cost. Hence, the obvious questions are: (1) how to determine the optimal dynamic-routing policy; and (2) are the cost savings of dynamic routing versus static routing large enough to justify the increased costs associated with them (i.e., administrative costs and possibly increased direct transportation costs)?

Unfortunately, the optimal dynamic-routing policy is known only for the very special case of “symmetric” retailers. See Park, et al. (2002). We summarize some of their results below. For more general cases, the optimal dynamic-routing policy is unknown, due to complexities that we will also illustrate. Therefore, after reviewing the results of Park, et al., we will turn to an examination of “change-revert” policies.

A “change-revert” policy is a heuristic dynamic-routing policy that, under normal circumstances, uses a predetermined “default route”; but, under exceptional circumstances (e.g., very low inventory at the last retailer on the default route), changes the route to a different one (e.g., visit this retailer sooner). However, regardless of which non-default route is chosen, under a change-revert policy, the inventory-allocation policy in that cycle is based upon the assumption that the route will revert to the default route in the next replenishment-allocation cycle. The intuition behind the change-revert heuristic is as follows: First, there are circumstances when randomness in demand justifies changing the route from the default route. However, given the potential for dynamic routing to decrease system stability, it may not be desirable to change routes frequently. Hence, when a change-route decision is made, the corresponding allocations should be designed to re-establish the default route.
The next section summarizes some of Park, et al.’s results. Our goal in doing so is to
describe its optimal dynamic-routing policy, examine the interactions between routing decisions
and inventory-allocation decisions, and to provide additional motivation for the change-revert
heuristic described above.

Dynamic-Routing: The Symmetric Retailer Case

Park, et al. (2002), henceforth labeled “Park”, examined the same model as ours, but for
the special case of \( N \) “symmetric” retailers. Symmetric retailers have identically-distributed
customer demand, identical travel time to/from the warehouse (i.e., \( r_{01} = r_{02} = \ldots = r_{0N} = a > 0 \)),
and identical between-retailer travel times for all retailer pairs (i.e., \( r_{ij} = b > 0 \) for all \( i \neq j \)).

For a symmetric system, Park proves that for any given inventory-allocation policy, “least-
inventory first” (LIF) is an optimal dynamic-routing policy. Under LIF, at the time of departure
from the warehouse, the vehicle travels first to the retailer with the smallest inventory position,
makes an allocation, then travels to the unvisited retailer that, at that instant, has the smallest
inventory position, etc..

Our interest in Park’s symmetric-retailer model is that it provides a unique framework for
examining the complex interactions between routing decisions and inventory-allocation
decisions. In particular, since, in the Park model, LIF is the optimal dynamic-routing decision-
rule regardless of the allocation policy being used, the amount of \( U_{AL} \) uncertainty (in each
retailer’s delivery leadtime) and \( U_{PD} \) uncertainty (in the number of periods of customer demand
each retailer’s allocation is to supply) depends entirely on the inventory-allocation policy. For
simplicity of exposition, consider \( N = 2 \) symmetric retailers.

Note, first, that, regardless of the route used in any given cycle, if the first retailer on this
route is allocated none (all) of the vehicle inventory, then, under LIF, this retailer is very likely to
be the first (last) retailer on the route chosen for the next cycle. Now consider two possible
heuristic allocation policies: equal allocation and fixed-route allocation. Under an equal-
allocation policy, both retailers are brought to the same inventory position. Equal allocation
makes the next route a 50-50 coin toss, and, hence, induces high levels of \( U_{AL} \) and \( U_{PD} \)
uncertainty. The fixed-route allocation policy allocates inventory to minimize total expected
inventory-holding and backorder cost/period under the assumption that the routes in all
subsequent replenishment-allocation cycles will be the same as the current route (although, in fact, the LIF routing decision-rule will be used to choose the route in all subsequent replenishment-allocation cycles). In summary, fixed-route allocation is designed to reduce the frequency of change-route decisions, and, hence, the amount of $U_{AL}$ and $U_{PD}$ uncertainty; whereas equal-allocation results in frequent (50%) route changes, and, hence, high levels of $U_{AL}$ and $U_{PD}$ uncertainty.

Although neither fixed-route nor equal-allocation (combined with LIF routing) is designed to be the optimal routing-allocation policy, the proximity of either of these allocation policies to the optimal allocation policy (which also uses LIF) is an indicator of how frequently route changes occur in the optimal policy.

Park compared the simulated performance of the equal-allocation and fixed-route allocation heuristics with the optimal allocation policy (all using LIF routing) for 128 parameter sets. Park reported that the fixed-route heuristic increased the average cost/cycle compared with the optimal allocation policy, only 0.48%; the maximum increase was only 4.12%. On the other hand, the equal-allocation heuristic “performed poorly, increasing cost/cycle by an average of 17% and a maximum of 95%”. Park further reported that more than two-thirds of reduction in cost/cycle provided by optimal dynamic routing and allocation (compared to optimal static routing and allocation) is provided by dynamic routing and fixed-route allocation.

Park’s results suggest that the optimal dynamic routing and allocation policy allocates inventory to favor the long term over the short term in the trade-off described above (i.e., to support relatively few change-route decisions) and then, when a route change occurs, allocates inventory to re-establish a stable (i.e., near-static) route. This is the motivation behind the change-revert heuristic described above and examined in detail below.

6. The Change-Revert Heuristic

As already noted (Section 2), the routing decision in any given replenishment cycle determines for each retailer both: (1) the first period in the corresponding allocation cycle for each of the retailers; and (2) the last period in the allocation cycle corresponding to the immediately preceding (i.e., “last”) replenishment cycle. In other words, the routing decision in the current replenishment cycle “ends” the allocation cycles associated with the last
replenishment cycle, and “starts” the allocation cycles associated with the current replenishment cycle. Therefore, in order to determine the optimal dynamic routing policy, one must evaluate the consequences of each alternative route on the backorder costs incurred in the previous cycle, the holding and backorder costs of the current cycle. The latter depends, impart, on the end-periods of the allocation cycles, which are determined by the route chosen in the next cycle. Hence, the optimal route in this cycle depends on, among other things, the optimal route in the next cycle. Although this is conceptually straightforward, it is analytically impossible. This is the complexity that limits what is known about optimal dynamic-routing decisions to the symmetric case (since, in the symmetric case, routing decisions in all subsequent cycles are independent of the routing decision in the current cycle). Hence, in order to examine dynamic-routing decisions, one must examine heuristics.

In this section, we develop the “change-revert” heuristic introduced above. As already described, this heuristic is based on the premise that there is a preferred, “default” route (e.g., the optimal static route), and that there are long-term advantages in using this route most of the time. The heuristic’s decision-rule is applied before the vehicle leaves the warehouse, and yields one of two decisions: (a) use the default route in the current replenishment cycle; or (b) choose another route. In either case, the decision-rule assumes that the route in the next replenishment cycle will be the default route. This assumption fixes the last periods of all the retailers’ allocation cycles for the current replenishment cycle. It is important to note, however, that although it is assumed that the route chosen in the next replenishment cycle will be the default route, this choice will, in fact, be made at the beginning of the next replenishment cycle, using the same decision-rule.

In choosing a route for the current replenishment cycle, the decision-rule recognizes that, at the time it is being made, this decision affects total expected cost in two ways; (1) by determining the end of the retailers’ allocation cycles associated with the last replenishment cycle, and, hence, their backorders-related cost; and (2) by determining the start of the allocation cycles associated with the current replenishment cycle. Since, by assumption, the route to be chosen in the next replenishment cycle is the default route, (2), in effect, determines the expected inventory costs in the current replenishment-allocation cycle. More specifically, (2) determines the number of periods, denoted \( m_i \), in the allocation cycle for retailer \( i, i=1,\ldots,n \), in the current...
cycle. Appendix A provides the optimal replenishment-and-allocation decisions associated with any choice of allocation cycles (i.e., starting and ending periods) for all $N$ retailers, and their expected cost.

Consider the routing decision in any given replenishment cycle. Each possible route has an associated set of travel times to each retailer $i$, $B_i$, and an associated set of starting and ending periods for each retailer’s respective allocation cycle. From Appendix A, equation (A6) and (A7), the single cycle cost is

$$Z^* = H^0 + H^F + (p + h)\sigma_i^F \phi(K^*)$$

(12)

where both $H^0 + H^F$ are “unmanageable” (i.e., non-incremental) pipeline inventory-holding cost associated with any $m$-period replenishment cycle. Note that $H^F$ depends on the route used in the next cycle. In the static routing case considered in Section 4, $H^F$ depends on the choice of the static route. Under the change-revert heuristic, the next route is assumed to be the default route, so $H^F$ is independent of the route used in the current cycle. In either case $\sigma_i^C$ depends on the route used in the current cycle.

In addition, the expected shortages of retailer $i$ in its last allocation cycle, as viewed from the start of the current replenishment cycle, are given by

$$s_i^p = \sigma_i \sqrt{B_i} R \left( \frac{\mu - B_i}{\sigma_i \sqrt{B_i}} \right),$$

where $x_i$ is retailer $i$’s net inventory at the start of the replenishment cycle, and $R$ is the unit-normal loss function. Define $S^p = \sum_{i=1}^{N} s_i^p$.

Given the expected costs of (1) and (2), above, the decision-rule then chooses the route with the smallest value of $C^*$, where

$$C^* = Z^* + (p + h)S^p.$$

(13)

For purposes of illustration, consider the special case of two retailers having the same per period mean, $\mu$, and standard deviation, $\sigma$, of demand. Label the two retailers R1 and R2, assume that $r_{12} = r_{21}$, and, without loss of generality, assume that $r_{01} \leq r_{02}$. As already noted, the optimal static route visits R1 first. Assume this is the default route. Hence, the route that
visits R2 first is the “change” route. For these two alternative routes, Table 1 gives the values that are key in determining \( C^* \).

**Table 1: Comparison of key values for the two routes when N=2.**

<table>
<thead>
<tr>
<th></th>
<th>Default</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B_1 )</td>
<td>( r_{01} )</td>
<td>( r_{02} + r_{12} )</td>
</tr>
<tr>
<td>( B_2 )</td>
<td>( r_{01} + r_{12} )</td>
<td>( r_{02} )</td>
</tr>
<tr>
<td>( m_1 )</td>
<td>( m )</td>
<td>( m - r_{02} - r_{12} + r_{01} )</td>
</tr>
<tr>
<td>( m_2 )</td>
<td>( m )</td>
<td>( m - r_{02} + r_{01} + r_{12} )</td>
</tr>
<tr>
<td>( \mu^C_1 )</td>
<td>( \mu(2m + 2r_{01} + r_{12}) )</td>
<td>( \mu(2m + 2r_{01} + r_{12}) )</td>
</tr>
<tr>
<td>( \sigma^C_1 )</td>
<td>( \sqrt{2m + 2r_{01} + r_{12} + 2\sqrt{m(m + r_{12})}} )</td>
<td>( \sqrt{2m + 2r_{01} + r_{12} + 2\sqrt{(m + r_{01} - r_{02})(m + r_{01} - r_{02} + r_{12})}} )</td>
</tr>
</tbody>
</table>

And in (12), \( H^0 + H^F = hm\mu(m - 1 + 2r_{01} + r_{12}) \) is the same for either route.

Finally, note that for \( r_{01} < r_{02} \), \( Z^* \), is smaller for the change route. This is because the change route postpones the allocation of inventory between the two retailers for \( (r_{02} - r_{01}) \) additional time periods, thereby providing more risk pooling (represented by the lower value of \( \sigma^C_1 \)). This fact alone suggests that a fixed routing policy that alternated the default and change routes might have lower total expected cost/period than the optimal static route. However, this is not necessarily the case. Under an alternating route policy, the expected costs in each cycle that uses the default route will be greater than the \( Z^* \) given above, since in these cycles the change route (not the default route) determines the end periods of the allocation cycles.

7. The Analytical Model

The formulas (A6) and (A7) in Appendix A give the expected cost/cycle for any fixed routing policy; i.e., for any choice of \( B_i \)'s and \( m_i \)'s. These are estimates, of course, because
of the assumptions (Section 2) used in their development. We examine the sensitivity of the estimates to these assumptions in the next section.

Furthermore, given any state-dependent routing decision-rule, these formulas can be used to compute the expected cost/cycle of that routing policy, provided that the steady-state distribution of system status can be determined. For the case of \( N = 2 \) retailers, it can be shown that, given optimal replenishment and allocation, the distribution of system status (i.e., the joint distribution of \((x_1, x_2)\), the inventory levels at the two retailers at the time of routing, used by the decision-rule to choose routes using the change-revert heuristic) follows a joint normal distribution with known parameters. Given this distribution of system status at the time of the routing decision, and, hence, the probabilities of each alternative route, we computed the expected cost/period of the change-revert routing decision-rule (by numerical integration using the Maple Mathematical software package). For any given set of parameters, the difference in the analytical model’s corresponding estimates of the cost/cycle of the optimal-static routing policy and its estimates of the change-revert routing policy provides the analytical model’s estimate of savings. The quality of these estimates is also examined below.

8. Simulation Tests: \( N = 2 \) Retailers

We conducted simulations for two reasons: First, to assess the ability of the analytical model (Section 7) to estimate the cost/period of static and change-revert routing policies, and, in particular, to estimate the savings associated with a change-revert routing decision. Second, to assess the value of the change-revert heuristic under a variety of parameterizations. This assessment was conducted using \( N = 2 \) and \( N = 6 \) retailers with normally-distributed and negative-binomially distributed demand. Our observations follow a description of the simulation model.

The Simulation Model

The simulation was written in VBA (Visual Basic for Application) and run on a Pentium(R) 4, 2.40GHz computer. Each replenishment-allocation cycle was simulated as described in Section 2. In each replenishment-allocation cycle, the system replenishment quantity is the optimal base-stock (A5) less the current system net inventory. If a negative system replenishment is prescribed, then a replenishment of zero is made. Retailer allocations
are prescribed using (A4). If a negative allocation is prescribed, then an allocation of zero is made. Backorders are assessed a backorder-penalty cost regardless of the period in which they occur. For any given parameterization, the same retailer-demand realizations occur regardless of the routing decision-rule being simulated. In simulation tests involving the normal distribution, negative demand realizations are permitted.

The first 5,000 cycles of the simulation were used to allow the system to attain a steady state. Then, every next 10,000 cycles were used to compute a single observation of the simulated results. At least 10 observations were collected to compute the average results. Cost savings are reported with respect to manageable cost; i.e., the average dollar saving as a percentage of the cost/period of the default (i.e., optimal static-route) policy minus its unmanageable pipeline inventory-holding cost.

**Simulation Results**

The first set of assessments were done for an \( N = 2 \) retailer business scenario with normally-distributed retailer demand. The base-case used the following parameters: \( r_{01} = 1 \), \( r_{02} = 2 \), and \( r_{12} = r_{21} = 3 \). Each replenishment cycle contains \( m = 8 \) time periods. Holding cost, \( h \), is $1/unit-period; the backorder cost, \( p \), is $160/unit-period, which provides a composite newsvendor target fractile of \( (p - h(m - 1))/(p + h) = 95.03\% \). Period demand for each retailer is normally distributed with \( \mu = 100 \) units/period, \( \sigma = 120 \) units/period, so that the coefficient of variation in the last period of each static allocation cycle is 0.42.

**Assessing the Analytical Model**

First, as expected, the analytical model under-estimates cost/period observed in the simulation, since its assumptions ignore constraints (e.g., no negative allocations) that can increase cost in reality, and, hence, in the simulation. Nonetheless, its estimates are close to those observed in the simulation. For example, in the base-case, its estimates of cost/period for either the optimal-static or change-revert routing policy is within 5% of the simulated mean. Second, and for the same reason as above, the change-revert decision-rule, under-estimates the savings from changing routes, but these estimates are generally “good”. For example, in the base case, its estimate of cost savings is 3.54%, while the simulation prescribes an average cost savings of 5.66%. Third, the change-revert decision-rule provides “very good” estimates of the “change frequency”; i.e., the percentage of replenishment cycles in which the change route is
chosen. For example, in the base case, the analytical model estimates an 18.86% change frequency and the simulation reported a change frequency of 18.67%. The second and third observations support the assertion that the model’s estimates are managerially meaningful; e.g., that “at least (this) saving will be provided with approximately (this) change frequency”.

As a direct test of the sensitivity of the analytical model’s three major assumptions, the simulation also kept track of the percentage of replenishment cycles: (1) in which a negative allocation was prescribed; (2) in which a negative system replenishment was prescribed; and (3) in which backorders occurred before the last period of any retailer’s allocation cycle. Negative allocations were prescribed in between 1.81% and 1.89%; negative replenishments were prescribed in between 0.049% and 0.055%; and backorders occurred before the end of any retailer’s allocation cycle in between 0.00% and 4.75% of the parameterizations. Hence, based on the simulations we conducted, we assert that the three major assumptions made in the analytical model are innocuous.

**Assessing the Value of the Change-Revert Heuristic**

First, and most important, the expected savings provided by the change-revert heuristic w.r.t. the optimal static route, measured as a percentage of manageable cost, are, statistically significant and managerially meaningful. Savings are 5.66±.41% in the base case. Further, these savings are provided with route changes in only 18.7% of the replenishment cycles. Virtually all of these savings are from reduced backorders. For example, in the base case, 93.1% of the savings provided by change-revert over the optimal static route were due to reduced backorders. This supports the assertion that change-revert manages (virtually) the same system inventory better.

Table 2 reports the 95% confidence intervals for the savings in manageable cost and the percentage of the time the change route was chosen (under the 2 columns labeled as “Original Decision Rule”). Results for the base-case parameterization are in bold. In order to assess sensitivity, the same is reported for alternative $r_{01}$, $r_{02}$, and $r_{12}$ values. Starting with the base case, but increasing $r_{02}$ from 2 to 3 periods reduces both the savings (to 2.34±0.37%) and change-route frequency (to 11.2%). This is to be expected, since increasing $r_2$ increases the “cost” of a change; i.e., increases the variance of both leadtime and allocation-cycle length. Similarly, decreasing $r_{02}$ from 2 to 1 period increases savings (to 8.74±0.48) and, change-route
frequency (to 18.8%). Note, further, that holding $r_{01}$ and $r_{02}$ fixed, but increasing $r_{12}$, first increases and then decreases the estimated cost savings. For example, using the base case as the starting point, decreasing $r_{12}$ decreases estimated savings to $5.29 \pm 0.30\%$ (for $r_{12} = 2$) and to $1.17 \pm 0.21\%$ (for $r_{12} = 1$). However, increasing $r_{12}$ to 4 and then 5 periods also causes a reduction in savings (to $4.61 \pm 0.49\%$ and $3.24 \pm 0.49\%$, respectively). Change-route frequency decreases monotonically as $r_{12}$ increases. Our interpretation of this follows: As $r_{12}$ changes, we observe the combined impact from two effects. For small values of $r_{12}$, there may be little or no reduction in the delivery leadtime to Retailer 2 by using the change route, so the potential to reduce shortages at Retailer 2 is small. For example, in the base case, but with $r_{12} = 1$, the delivery leadtime to Retailer 2 is 2 for either route. Hence, Retailer 2 shortages are never reduced by the change route. On the other hand, as $r_{12}$ increases, the difference in the default route’s delivery leadtimes between the two retailers increases, so the difference in the allocation quantities increases. As the difference in the allocation quantities increases, the frequency of route-change decreases, along with the average savings.
Examining the Frequency of Route Changes

Table 2: Sensitivity of the Simulated Results to the LeadTimes

| $r_{01}$ | $r_{02}$ | $r_{12}$ | Original Decision Rule | | | Threshold Decision Rule | | |
|---|---|---|---|---|---|---|---|
| | | | Savings (%) | Frequency of Change (%) | Savings (%) | Frequency of Change (%) |
| 1 | 1 | 7.92±0.35 | 37.47 | 6.97±0.26 | 7.22 |
| 2 | 9.39±0.46 | 27.51 | 8.59±0.37 | 6.99 |
| 3 | 8.74±0.48 | 18.81 | 7.95±0.48 | 5.81 |
| 4 | 6.75±0.48 | 12.05 | 6.38±0.45 | 4.32 |
| 5 | 4.95±0.48 | 7.24 | 4.59±0.37 | 2.84 |
| 1 | 2 | 5.29±0.30 | 31.77 | 4.11±0.23 | 4.75 |
| 3 | **5.66±0.41** | **18.67** | **5.12±0.32** | **4.22** |
| 4 | 4.61±0.49 | 10.19 | 4.31±0.30 | 2.94 |
| 5 | 3.24±0.49 | 5.20 | 3.23±0.42 | 1.82 |
| 2 | 2 | 0.55±0.19 | 17.21 | 0.00 | 0.00 |
| 3 | 2.34±0.37 | 11.21 | 1.75±0.14 | 1.86 |
| 4 | 2.39±0.42 | 5.97 | 2.37±0.31 | 1.63 |
| 5 | 2.05±0.37 | 2.94 | 1.84±0.35 | 1.03 |
| 1 | 3 | 7.00±0.37 | 36.94 | 5.89±0.26 | 6.73 |
| 2 | 8.64±0.48 | 25.86 | 7.67±0.39 | 6.57 |
| 3 | 7.39±0.46 | 16.90 | 6.67±0.34 | 5.04 |
| 4 | 5.28±0.48 | 10.17 | 4.91±0.37 | 3.49 |
| 5 | 3.76±0.46 | 5.71 | 3.50±0.45 | 2.14 |
| 2 | 3 | 4.18±0.38 | 26.57 | 3.12±0.19 | 3.51 |
| 3 | 4.39±0.34 | 15.19 | 3.85±0.28 | 3.31 |
| 4 | 3.38±0.45 | 7.86 | 3.27±0.37 | 2.18 |
| 5 | 2.50±0.46 | 3.74 | 2.21±0.38 | 1.24 |
| 3 | 3 | 7.08±0.49 | 23.99 | 6.16±0.41 | 5.55 |
| 3 | 5.86±0.49 | 14.48 | 5.02±0.45 | 4.03 |
| 4 | 3.82±0.48 | 8.01 | 3.44±0.36 | 2.53 |
| 5 | 2.55±0.39 | 4.08 | 2.28±0.49 | 1.36 |

Examining the Frequency of Route Changes

Note (Table 2: Col. 5), that although the change frequency reported is less than 20% in 18 (66.7%) of the 27 parameterizations, it is over 30% in four (15%) of the parameterizations. We attribute this to the fact that the decision-rule prescribes the change route whenever the estimated expected cost of the change route is smaller than that the default route, regardless of the
magnitude of the savings. Table 2 (Col. 7) also provides the corresponding simulation results using a “threshold” decision-rule under which a change is prescribed only if the estimated savings in that particular cycle is greater than or equal to 10% of the estimated costs under the default route. Note, for example, that in the base case, this “threshold” decision rule provides most of the cost savings (5.12±0.32% versus 5.66±0.41%) as the original rule, but reduces the frequency of change (from 18.7 to 4.2%). Similar results apply to other parameterizations.

Table 3 reports the sensitivity of the simulated results to the standard deviation of retailer demand/period. Results for the base case (with $\sigma = 120$ units/period), already reported, are in bold. As expected, decreasing $\sigma$ reduces the estimated savings and change frequency (to zero for $\sigma = 20$) while increasing $\sigma$ increases the savings and the change frequency. To examine the sensitivity of the simulated results to the overall level of system inventory, we also ran simulations for composite retailer fill-rates of 80.5% and 99% and compare them with the 95.03% fill-rate for the base-case. Table 4 reports the results. As expected, systems with higher target fill-rates provide less opportunity for cost savings to change-revert, regardless of the decision rule. For the unmodified decision-rule, more and more frequent change-route decisions are required to provide lower levels of savings. Again, the threshold decision-rule provides most of the savings; and change frequency decreases as target fill-rate increases.

In order to verify that the effectiveness of the change-revert heuristic is not an artifact of the assumption that customer demand was normally distributed, we performed simulations for the base-case travel parameters using negative-binomial customer demand/period with a mean of 100 and three coefficients of variation: 0.2 (low), 0.6 (medium), and 1 (high).

As expected, the analytical model (using the normal approximation to the negative-binomial) underestimates the cost/period observed in the simulation, but, except in the high-variance case, its estimate was within 5% of the simulated mean. And, as in the scenarios with normally-distributed demand, the major assumptions of the analytical model were seldom violated.

Table 3: Sensitivity of the Simulated Results to the Standard Deviation of Retailer Demand/Period
Original Decision Rule | Threshold Decision Rule
\[ \sigma \]
\begin{tabular}{|c|c|c|c|c|}
\hline
 & Savings (%) & Frequency of Change (%) & Savings (%) & Frequency of Change (%) \\
\hline
20 & 0.00 & 0.00 & 0.00 & 0.00 \\
60 & 1.53±0.37 & 2.56 & 1.35±0.30 & 0.82 \\
120 & 5.66±0.41 & 18.67 & 5.12±0.32 & 4.22 \\
180 & 8.43±0.47 & 30.79 & 7.68±0.33 & 6.60 \\
240 & 10.44±0.47 & 38.53 & 9.34±0.37 & 8.49 \\
\hline
\end{tabular}

Table 4  Sensitivity of the Simulated Results to the Composite Retailer Fill-Rates

<table>
<thead>
<tr>
<th>Composite Retailer Fill-Rate</th>
<th>Original Decision Rule</th>
<th>Threshold Decision Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Savings (%)</td>
<td>Frequency of Change (%)</td>
</tr>
<tr>
<td>80.5%</td>
<td>6.83±0.35</td>
<td>15.04</td>
</tr>
<tr>
<td>95.03%</td>
<td>5.66±0.41</td>
<td>18.67</td>
</tr>
<tr>
<td>99%</td>
<td>4.64±0.48</td>
<td>23.61</td>
</tr>
</tbody>
</table>

Finally, and most significantly, the change-revert heuristic generated observed savings/period of 2.01 (low) and 7.29% of manageable cost with change frequencies of 2.0 and 12.6%. As expected, the threshold heuristic provided most of these savings, 2.04 and 6.92%, respectively, with significantly reduced change frequencies of 0.9 and 4.2%.

9. The Change-Revert Heuristic for \( N > 2 \) Retailers

In order to assess the value of the change-revert heuristic for systems with more than two retailers, we conducted simulations for two configurations of a 6-retailer system. The simulations were conducted as described above. The change-revert heuristic was applied as follows: As in the case of \( N = 2 \) retailers, the heuristic’s decision-rule is applied before the vehicle leaves the warehouse, and yields one of two decisions for the current cycle: (1) use the (optimal static) default route; or (2) choose another of the \((N!-1 =)\) 719 possible routes. In either case, the decision-rule assumes that the route in the next replenishment cycle will be the default route. The chosen route is then implemented, without change, during the current cycle. The
“original” decision rule chooses the route to minimize $C^*$, in (13); in the “threshold” version, the estimated savings from using the non-default route must be greater than or equal to 10% of $C^*$ for the default route.

In both examples, retailer demand each period is independently, identically, normally distributed with $\mu = 100$ and $\sigma = 120$ units, $h = $1/unit-period. The number of periods in the replenishment cycle, $m$, was set equal to one period larger than the longest possible delivery route in that configuration (so that one vehicle could make all the deliveries); and $p$ was chosen so that the critical ratio for the corresponding composite retailer, (3), which depends on $m$, was 0.95.

Configuration 1 was a star, with each retailer 2 time periods (travel time) from the warehouse (i.e., $r_{i0} = 2$), and either 2, 3, or 4 periods of travel time from the others. For this example, we used $m = 21$ periods and $p = $420/unit-period. The results are summarized in column 2 of Table 5. Note that the change-revert heuristic saved 18.9% of the manageable cost compared with the default route, and had a change frequency of 83.9%. The threshold policy provided more than half of these savings (11.8% vs. 18.9%), but dramatically reduced the change frequency to 10.5%. We also kept track of which non-default routes were chosen. For the original decision rule, of the 719 non-default routes, 561 were chosen at least once (78%). Of these 561 non-default routes, 21 (3%) were chosen in at least 1% of the cycles. Finally, 70 routes accounted for 80% of the non-default choices made. The corresponding number of routes for the threshold decision-rule were 307, 1, and 54, respectively.

Configurations 2 was generated randomly, with travel times integer and uniformly distributed between 1 and 5 periods. The results are summarized in columns 3 of Table 5.

We believe that these example results support the following assertions, all of them consistent with the results for $N = 2$ retailers: First, that dynamic routing provides statistically significant and managerially meaningful savings for systems with more than 2 retailers. Second, that although the “original” routing decision rule might prescribe very frequent route changes; the threshold policy provides more than half of the manageable-cost savings of the original, but with dramatically fewer route changes. Third, the statistics collected on the choices of non-default routes suggest that change-revert can still be effective even if the number of “candidate” non-default routes is relatively small. For example, note that in the random configuration 2 (in Table 5), only 20 of the 719 possible non-default routes accounted for 80% of...
the routes chosen by the threshold heuristic. As expected, these particular non-default routes were among those with the smallest traveling-salesman (total) travel times (e.g., for Configuration 2, in which the minimum (maximum) TSP time was 13 (26) periods, these 10 routes had total travel times of 13-17 periods).

Finally, these example results indicate that the potential for dynamic routing to provide savings increases with the number of retailers. This is to be expected, since the opportunity for inventory imbalances increases with the number of retailers.

Table 5: Simulation Results for $N = 6$ Retailers

<table>
<thead>
<tr>
<th></th>
<th>Configuration 1: Star</th>
<th>Configuration 2: Random</th>
</tr>
</thead>
<tbody>
<tr>
<td>Savings (%): Original/Threshold</td>
<td>18.9/11.8</td>
<td>14.3/8.0</td>
</tr>
<tr>
<td>Frequency of Change (%): Original/Threshold</td>
<td>83.9/10.5</td>
<td>78.7/6.1</td>
</tr>
<tr>
<td>No. of Routes Used: Original/Threshold</td>
<td>561/307</td>
<td>413/160</td>
</tr>
<tr>
<td>No. of Routes used with Freq. $\geq 1%$: Original/Threshold</td>
<td>21/1</td>
<td>18/0</td>
</tr>
<tr>
<td>No. of Routes used for 80% of the Time: Original/Threshold</td>
<td>70/54</td>
<td>35/20</td>
</tr>
</tbody>
</table>

10. Conclusion and Suggestions for Further Work

In our research we have developed an analytical model to determine a combined system inventory-replenishment, routing, and inventory-allocation policy that minimizes the total expected cost/period of one-warehouse $N$-retailer distribution system over an infinite time horizon. We have employed this model: (1) to determine the optimal static route; (2) to examine the interactions between routing and inventory-management policies, particularly the interaction between dynamic routing and inventory allocation; and (3) to develop a heuristic routing policy: the change-revert heuristic.

We used simulation to examine the impact of the major assumptions used to develop the analytical model. The results indicate, first, that these assumptions are seldom violated; and, second, even when they are, the insights provided by the analytical model — in particular, the model’s estimates of cost/period and the desirability of changing routes — appear to be
insensitive to them. In particular, the simulation results provide evidence that the change-revert heuristic provides statistically significant and managerially meaningful savings for multi-retailer systems.

Of course, in order to focus on the potential of dynamic routing to reduce costs, our analysis deliberately excluded explicit consideration of the cost of alternative routes (e.g., mileage-, or driver-time related costs). In Section 4 we explained how such costs are easily accommodated in choosing the optimal static route. In a similar fashion, transportation-related cost can be added to the decision rule(s) used in the change-revert heuristic.

Our analysis also excluded consideration of other costs associated with dynamic routing. However, in some systems this involves no more than adding a feature to a system already used to monitor system status and communicate with its operatives (e.g., vehicle drivers). Our model also ignores other costs that might be associated with dynamic routing. The quality-management literature, for example, suggests that variety itself can increase cost and reduce quality. Hence, it is possible that, by increasing route variety, dynamic routing may increase system costs. However, it is also possible, as indicated by our sample results for 6 retailers, that considerably fewer than $N!$ routes will provide most of the savings of $N!$ variety.

Our analysis only considered a single product. However, we believe the extension to multiple products is straightforward, provided that our assumption of a single (uncapacitated) vehicle remains. If multiple vehicles, each with possibly a different route, are used, an additional element is added to the problem: the assignment of products to vehicles.

Finally, we deliberately leave two questions open for future research: First, is an examination of what might be called on-route dynamic routing. This form of dynamic routing would decide which retailer to visit next as part of the allocation decision made at every retailer (except the last) in each replenishment cycle. (The policy we examine makes these $N-1$ decisions once and for all at the time the vehicle first leaves the warehouse.) Second, is a closer examination of change-revert routing policies. Perhaps the most interesting question here is: “What is the best default route?” We arbitrarily chose the optimal static route to be the default route, but, given a change-revert policy, it is certainly possible that a different default route would provide even better results.
Appendix A: Optimal replenishment and dynamic allocation with fixed routes

We consider the joint replenishment-allocation-routing problem described in Section 2 in the case when the routes are fixed (i.e., chosen in advance) for all cycles. We show that under the assumptions of Section 2, the infinite horizon problem decomposes into independent single-cycle problems. For the single-cycle problem we use the composite retailer defined in Appendix B to give the cost minimizing replenishment–and-allocation policy. These optimal policies are the basis for the heuristic approach to the problem that includes route selection given in Section 6.

Relative to some particular replenishment-allocation cycle, denoted the “current” cycle, let $t = 0$ denote the period when the replenishment occurs, and we use the following:

- $B_i$ is the delivery leadtime (start of the allocation cycle) for retailer $i$ in the cycle.
- Let $B_i^c$ be the delivery leadtime for retailer $i$ in the next allocation cycle. Then $(m + B_i^c - 1)$ is the last period in retailer $i$’s current allocation cycle, and $m_i = m + B_i^c - B_i$ is the number of periods in retailer $i$’s current allocation cycle.
- Let $[j]$ denote the index of the $j$th retailer on the current route, and $h_{[j]} = B_{[j]} - B_{[j-1]}$ is the incremental leadtime to retailer $[j]$. $B_{[0]} = B_0 = 0$.
- Let $d_{i,t}$ denote the demand at retailer $i$ in period $t$. It is drawn from a normal distribution with mean and standard deviation of $\mu_i$ and $\sigma_i$.

The Single-Cycle Problem

Consider the operation of our system over an infinite horizon. We wish to determine the replenishment-and-allocation policies that will minimize the average cost/period. To this end we divide the problem into a series of single-cycle problems. As described in Section 2, all costs over the infinite horizon are uniquely assigned to a cycle. The current cycle is assigned costs
\[ h\sum_{t=0}^{m-1} E_t + (h + p)\sum_{t=0}^{N} \{s_{t,j} : t = B_i + m_i\}, \quad (A1) \]

where the \( s_{t,i} \) terms are the backorders incurred by each retailer at the end of their allocation cycles. The single-cycle problem is to choose the replenishment quantity and the \( N \) (dynamic) allocation quantities, within the current cycle, that minimize the expected value of (A1).

The expectation of the first term in (A1) depends only upon the system inventory after the replenishment quantity has been added. Expected backorders at retailer \( i \) depend only upon the sum of retailer \( i \)'s inventory at \( t = 0 \) and the amount allocated to retailer \( i \) in the current cycle. The only constraint on the replenishment and allocation quantities is that the sum of the allocations equal the replenishment quantity. Therefore, each single-cycle problem is equivalent to the problem in which (1) all initial inventories are zero (2) the replenishment decision is to choose \( Y \), the system inventory at \( t = 0 \), and (3) each allocation decision, \( y_i \) for \( i = 1 \) to \( N \), is an allocation from \( Y \), constrained only by \( \sum y_i = Y \). In particular, the replenishment and allocation decisions in previous cycles have no impact on the optimal solution, or its expected costs, in the current cycle problem. Hereafter, we denote this “equivalent” single-cycle problem as SCP. It follows that an optimal policy for the infinite horizon is to solve each single-cycle problem independently.

Optimal Replenishment and Allocation using the Composite Retailer in SCP

In each cycle, there is a sequence of \( N \) dynamic allocation decisions. Given any \( Y \), optimal dynamic allocations minimizes the sum of the retailers expected shortages, given the information available at the time of each allocation. We apply the composite retailer, defined in Appendix B, to derive expressions for optimal allocations and replenishment in SCP. The composite retailer was first defined in Kumar (1995) for dynamic routing in a single-cycle problem with retailer-only holding costs and static routing. Although we use a slightly different definition of the composite retailer, the allocation quantities are identical with Kumar. However, one important difference is the allocation results given in Kumar only apply to static routes. Our results, for system-based costs, apply to any fixed routes. (We do not believe the composite retailer results are generalizable to fixed routes under retailer-only holding cost.) For \( j = 1 \) to...
N-1, the \( j \)th allocation decision is made in period \( B_{ij1} \), and is viewed as an allocation between two retailers; retailer \( [j] \) and a composite retailer that represents retailers \( [j+1] \) to \([N]\). The \( N \)th allocation decision allocates \( y_{[N]} = Y - \sum_{j=1}^{N-1} y_{[j]} \) to Retailer \([N]\).

Consider the \( N-1 \)th allocation decision. The values of \( y_{[k]} \), \( k=1 \) to \( N-2 \), have already been determined, so there are \( Y - \sum_{k=1}^{N-2} y_{[k]} \) units to divide between retailers \([N-1]\) and \([N]\). At the time of allocation these retailers have experienced \( B_{[N-1]} \) periods of demand, that in effect reduce the amount of units to allocate between the two retailers to

\[
X_{[N-1]} = Y - \sum_{k=1}^{N-1} y_{[k]} - \sum_{r=0}^{B_{[N-1]-1}} d_{r,[N-1]} - \sum_{r=0}^{B_{[N-1]-1}} d_{r,[N]},
\]

where the demand terms are now known realizations. Since the objective of the allocation is to minimize the resulting expected shortages, we use the composite retailer defined in Appendix B. Specifically, let Retailer \([N-1]\) ( \([N]\) ) play the role of Retailer A (B) with \( X_{[N-1]} \) the value of X. This gives:

\[
\mu_A = m_{[N-1]} \mu_{[N-1]} \quad (\sigma_A)^2 = m_{[N-1]}^2 \sigma_{[N-1]}^2
\]

\[
\mu_B = (b_{[N]} + m_{[N]}) \mu_{[N]} \quad (\sigma_B)^2 = (b_{[N]} + m_{[N]})^2 \sigma_{[N]}^2
\]

The optimal allocation to retailer \([N-1]\) is given by (B1) and (B2). The sum of the expected shortages from the two retailers is given by (B3), and is equivalent to the shortages of a single “combined” retailer with inventory \( X_{[N-1]} \) facing normal demands characterized by (B4) and (B5). Further, given this optimal allocation in period \( B_{[N-1]} \), the expected shortages of these two retailers as viewed in period \( B_{[N-2]} \) is captured by the composite retailer with mean and variance given by (B6) and (B7) where

\[
\mu_0 = b_{[N-1]} (\mu_{[N-1]} + \mu_{[N]}) \quad (\sigma_0)^2 = b_{[N-1]}^2 (\sigma_{[N-1]}^2 + \sigma_{[N]}^2).
\]

That is, the mean and variance of the \( N-1 \)th composite retailer, as viewed in period \( B_{[N-2]} \), is given by

\[
\mu_{[N-1]}^C = (b_{[N-1]} + m_{[N-1]}) \mu_{[N-1]} + (b_{[N]} + b_{[N]}) \mu_{[N]}
\]
\[(\sigma_{N-1}^C)^2 = b_{[N-1]}(\sigma_{[N-1]}^2 + \sigma_{[N]}^2) + \left(\sqrt{m_{[N-1]}\sigma_{[N-1]}} + \sqrt{h_{[N]} + m_{[N]}\sigma_{[N]}}\right)^2\]

Given that an optimal allocation will be made in period \(B_{[N-1]}\), the \(N-2^{nd}\) allocation is equivalent to a two retailer problem composed of Retailer \([N-2]\) and the \(N-1^{st}\) composite retailer. Given an optimal allocation in period \(B_{[N-2]}\), the \(N-2^{nd}\) composite retailer can be constructed to use in the \(N-3^{rd}\) allocation decision, and so on. This leads to

\[\mu_i^C = \sum_{k=j}^{N} (B_{[k]} - B_{[j-1]} + m_{[k]})\mu_{[k]}\]  \hspace{1cm} (A2)

and \((\sigma_i^C)^2\) defined recursively by

\[(\sigma_N^C)^2 = (b_{[N]} + m_{[N]}\sigma_{[N]})\]

\[(\sigma_j^C)^2 = b_{[j]}\sum_{k=j}^{N} \sigma_{[k]}^2 + \left(\sqrt{m_{[j]}\sigma_{[j]} + \sigma_{[j+1]}^C}\right)^2, \text{ for } i = 1 \text{ to } N-1\]  \hspace{1cm} (A3)

To calculate the optimal dynamic allocation quantities, \(y_{[j]}\), for \(j=1\) to \(N-1\):

1) Let \(X_{[j]} = Y - \sum_{k=1}^{j-1} y_{[k]} - \sum_{p=j}^{N} \sum_{t=0}^{p-1} d_{t[p]}\)

2) Calculate \(z_j^* = (X_{[j]} - m_{[j]}\mu_{[j]} - \mu_{k+1}^C)/(\sqrt{m_{[j]}\sigma_{[j]} + \sigma_{[j+1]}^C})\)

3) \(y_{[j]} = m_{[j]}\mu_{[j]} + \sum_{t=0}^{p-1} d_{t[p]} + z_j^*\sqrt{m_{[j]}\sigma_{[j]}}\)

The first composite retailer represents the entire system of \(N\) retailers with \(\mu_i^C\) equal to the sum of the mean demands from the replenishment decision to the end of each retailer’s allocation cycle. The optimal replenishment decision is equivalent to a single retailer newsvendor problem with demand characterized by \(\mu_i^C\) and \(\sigma_i^C\), an overstocking cost of \(hm\) (the holding cost per unit incurred by leftovers) and an understocking cost of \(p - (h-1)m\) (an additional unit that prevents a backorder saves the \((p+h)\) of the second term in (A1), but incurs a cost of \(hm\) in the first term). Thus the optimal replenishment is given by

\[Y^* = \mu_i^C + K^*\sigma_i^C\]  \hspace{1cm} (A5)

where \(K^*\) is such that \(\Phi(K^*) = (p - h(m-1))/(p + h)\).
The resulting expected backorders in the cycle are \( R(K^\ast)\sigma_1^C \), where \( R \) is the standard normal loss function. Let \( \mu^S = \sum_{j=1}^N \mu_j \). The expected values of the \( E_i \) terms in (A1) are \( Y^\ast(t+1)\mu^S \), for \( t = 0 \) to \( m-1 \). Hence, the optimal expected costs in the cycle is

\[
Z^\ast = h[m(Y^\ast - \mu^S(m+1)/2)] + (p + h)\sigma_1^C R(K^\ast)
\]

\[
Z^\ast = h[m(\mu_1^C + K^\ast \sigma_1^C - \mu^S(m+1)/2)] + (p + h)\sigma_1^C R(K^\ast)
\]

Using \( R(K^\ast) = K^\ast \Phi(K^\ast) - K^\ast + \phi(K^\ast) \) and \( \Phi(K^\ast) = (p - h(m-1))/(p + h) \) we get

\[
(p + h)\sigma_1^C R(K^\ast) = \sigma_1^C (p + h)[K^\ast(p - h(m-1))/(p + h) - K^\ast + \phi(K^\ast)]
\]

\[
= \sigma_1^C[K^\ast(p - h(m-1) - p - h)] + (p + h)\phi(K^\ast))
\]

\[
= \sigma_1^C[-hmK^\ast + (p + h)\phi(K^\ast)]
\]

This leads to

\[
Z^\ast = h[m\mu_1^C - \mu^S(m+1)/2] + (p + h)\phi(K^\ast)\sigma_1^C.
\]

Using \( \mu_1^C = \sum_{j=1}^N (m_j + B_j)\mu_j = \sum_{j=1}^N (m_j + B_j^F)\mu_j = m\mu^S + \sum_{j=1}^N B_j^F \mu_j \),

\[
Z^\ast = H^0 + H^F + (p + h)\phi(K^\ast)\sigma_1^C,
\]

(A6)

where

\[
H^0 = hm(m - 1)\mu^S / 2 \quad \text{and} \quad H^F = hm\sum_{j=1}^N B_j^F \mu_j.
\]

(A7)

Note that \( H^0 \) is the holding cost of the units needed to satisfy average demand over \( m \) periods, and is the same regardless of either the current or next route. \( H^F \) depends only on the route used in the next cycle and is the additional holding cost incurred on on-vehicle inventory.
Appendix B: The Composite Retailer

This appendix defines “composite retailer” and gives its fundamental properties. The composite retailer concept was defined in Kumar (1995). The construction of the composite retailer given here differs slightly from that in Kumar, but conceptually is the same. The composite retailer plays a key role in determining optimal allocations and replenishment under fixed routes given in Appendix A and referenced in Sections 4, 6, and 7.

Consider the following general case involving a system of two retailers. Initially, the system has $U$ units in stock. A system demand, $G_0$, occurs that reduces the stock to $X = U - G_0$. $X$ is then allocated to two retailers, denoted $A$ and $B$, so as to equalize the resulting runout probabilities after each experiences additional demand. Retailer $A$ experiences additional demand of $G_A$, and Retailer $B$ experiences $G_B$. Let $\mu_i$ and $\sigma_i$ be the mean and standard deviations of $G_i$, $i = 0, A, B$. We wish to express the sum of the expected shortages at the two retailers as a function of $U$ and these demand parameters.

Given $X$, let $y_A$ and $y_B$ be the allocations to the two retailer, where $y_A + y_B = X$. The allocation of $X$ that equalizes runout probabilities is given by

$$y_A = \mu_A + z^* \sigma_A \quad \text{and} \quad y_B = \mu_B + z^* \sigma_B$$

where

$$z^* = (X - (\mu_A + \mu_B)) / (\sigma_A + \sigma_B)$$

with expected shortages equal to

$$\sigma_A R(z^*) + \sigma_B R(z^*) = (\sigma_A + \sigma_B) R(z^*)$$

where $R(z)$ is the unit normal loss function. That is, the expected shortages resulting from the allocation of $X$ equal the expected shortages from a single “combined” retailer that has stock $X$ prior to experiencing demand with

$$\text{mean} = (\mu_A + \mu_B)$$

$$\text{standard deviation} = (\sigma_A + \sigma_B)$$.
Now consider the expected shortages, as a function of \( U \), prior to the occurrence of \( G_0 \).

Under the assumption that an allocation as prescribed above will be made after \( G_0 \) occurs, expected shortages are equivalent to the shortages from a single retailer that has inventory of \( U \) and experiences two “sets” of demands, \( G_0 \) and \( G^+ \), where \( G^+ \) has the mean and standard deviations given by (B4) and (B5). Convoluting these two normal variables gives a “total demand” that is normal with

\[
\mu^C = (\mu_0 + \mu_A + \mu_B) \quad \text{(B6)}
\]

\[
(\sigma^C)^2 = \sigma_0^2 + (\sigma_A + \sigma_B)^2 \quad \text{(B7)}
\]

and expected shortages given by

\[
\sigma^C R((U - \mu^C) / \sigma^C).
\]

That is, expected shortages, as a function of \( U \), equal the expected shortages from a hypothetical single retailer that has stock \( U \), and experiences normal demand characterized by \( \mu^C \) and \( \sigma^C \).

We call this single retailer a “composite retailer”.

Property 1: In the definition of a composite retailer, the allocation of \( X \) that equalizes runout probabilities also minimizes the sum of the expected shortages from the two retailers.

Comment: Property 1 is based upon well know results. It is derived from the first-order condition obtained by setting the derivative of expected shortages to zero. In particular, the derivative of \( R(x) \) is \(-\Phi(x)\).
Figure 1: A replenishment-allocation cycle
References


S. Park, L.B. Schwarz, and J. Ward, “Dynamic Routing, Inventory Allocation, and Replenishment in a Distribution System with One Warehouse and N Symmetric Retailers”, Working Paper, Krannert School of Management, Purdue University, West Lafayette, IN, 2002.


