Two-interval Inventory-allocation Policies in a One-warehouse $N$-identical-retailer Distribution System

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For a system of $N$-identical-retailers we construct a model for determining warehouse inventory-allocation policies which minimize system lost sales per retailer between system replenishments. An allocation policy is specified by: (a) the number of withdrawals from warehouse stock; (b) the intervals between successive withdrawals; (c) the quantity of stock to be withdrawn from the warehouse in each interval; and (d) the division of withdrawn stock among the retailers. We show that in the case of two withdrawals, available stock in each interval should be divided to "balance" retailer inventories.

We also develop an infinite-retailer model and use it to determine two-interval allocation heuristics for $N$-retailer systems. Simulation tests suggest that the infinite-retailer heuristic policies are near-optimal for as few as two retailers. Simulation tests indicate that the risk-pooling benefits of allocation policies with two well-chosen intervals are comparable to those of base-stock policies with four equal intervals.

(Multi-echelon Inventory; Distribution; Inventory Allocation)

1. Introduction
This paper examines inventory-allocation policies for a one-warehouse, $N$-identical-retailer distribution system facing stochastic demand for a single product. Specifically, we consider warehouse-to-retailer allocations between successive warehouse replenishments, given a fixed warehouse replenishment schedule (i.e., replenishment cycle). An allocation policy is characterized by a set of four decisions: (a) the number of withdrawals from warehouse stock, where each withdrawal is an opportunity to allocate the withdrawn stock to any (or all) of the $N$ retailers; (b) the times between successive withdrawals, which divide the replenishment cycle into intervals; (c) for each withdrawal, the quantity of stock to be withdrawn from the warehouse; and (d) for each withdrawal, the division of withdrawn stock among the retailers. Our analysis assumes that (a), (b) and (c) are determined when the warehouse is replenished, while (d) is a function of retailer inventories at the time of withdrawal.\footnote{Note: The policy for the division of withdrawn stock is also planned in advance, e.g., Max-Min, Largest Orders First, etc. It is the actual quantity to be received by each retailer which is determined at the time of withdrawal.} We seek an allocation policy which minimizes expected lost sales per retailer (denoted "lost sales / retailer"), subject to a constraint on the total system stock.

Previous work on inventory-allocation policies has either assumed a single withdrawal of all warehouse stock immediately after system replenishment (i.e., a "ship-all" policy), or assumed equal-interval withdrawals. We examine two-interval policies (i.e., policies in which two withdrawals are made from warehouse stock) and focus on the withdrawal quantities and times, and the division of withdrawn stock.

The contribution of this paper is its analysis of optimal two-interval policies for an infinite-retailer system. This
analysis yields two heuristic policies for systems with a finite number of retailers: the infinite-retailer heuristic and the 50/25 heuristic. Computer simulation is used to compare the expected lost sales/retailer of the heuristics to the lost sales/retailer of the "best" two-interval policies and four-equal-interval base-stock policies. Our results suggest that the risk-pooling benefits of allocation policies with two well-chosen (unequal) intervals are comparable to the benefits of the four-equal-interval base-stock-oriented policies.

2. Literature Review

Distribution systems pool risk by stocking retailers through warehouses. Schwarz (1989) identifies two ways that warehouses can be used to pool risk. First, rather than allocating stock directly to individual retailers, the outside-supplier ships unallocated stock to the warehouse, where it is subsequently allocated to the retailers, thereby pooling risk over the outside-supplier leadtime. This type of risk-pooling does not require the warehouse to hold inventory. Second, warehouse inventory can be used between system replenishments to "rebalance" retailer inventories which have become "unbalanced" due to variations in individual retailer demands. This between-replenishment risk-pooling is examined here.

A number of other studies have examined between-replenishment risk-pooling. Schwarz, et al. (1984) and Badinelli and Schwarz (1988) use an approximate model (Deuermeyer and Schwarz 1981) of a one-warehouse N-identical-retailer system to determine where system inventory should be held. The warehouse and retailers follow (Q, R) inventory-replenishment policies and retailer orders on the warehouse are filled on a first-come first-served basis. Schwarz et al. examine the fill-rate maximizing positioning of a fixed quantity of safety stock between the warehouse and retailers. The best policies (found through search) involve very small warehouse on-hand inventories, suggesting that the benefit from between-replenishment risk-pooling is not significant. Badinelli and Schwarz obtain similar results when minimizing backorders subject to a constraint on average system-wide inventory.

Other studies have observed significant between-replenishment risk-pooling benefits. Jönsson and Silver (1987a) consider a periodic (equal-interval) model with no warehouse in which a complete redistribution of retailer stocks occurs in the last period before system replenishment. In the redistribution, every retailer is brought to the same normalized inventory. Computational tests show that this redistribution can substantially reduce the inventory investment required to provide a given service-level compared to a system with no redistribution. Although a warehouse is not used to pool risk, Jönsson and Silver's results indicate that a policy which rebalances inventories late in the replenishment cycle can provide substantial risk-pooling. Jönsson and Silver (1987b) observed similar effects when replacing the redistribution with a second warehouse-retailer shipment in the last period before the system replenishment.

Jackson (1988) examines a periodic model employing equal-interval base-stock-oriented allocation policies to supply N identical retailers. At the beginning of each period, the warehouse stock withdrawn is the quantity required to bring each retailer's inventory position up to a stationary base-stock. If, at the beginning of some period, called the "runout" period, warehouse stock is insufficient to bring all retailers to the base-stock level, then all remaining stock is withdrawn and divided so as to maximize the minimum retailer inventory. (We call this a balancing division.) By comparing the simulated backorder performance of well-chosen four-equal-interval base-stock policies to the performance of "ship-all" policies (in which the warehouse withdraws and divides all system stock at the beginning of the replenishment cycle), Jackson shows that system service-levels can be substantially improved by allowing the warehouse to allocate stock between system replenishments.

Our interpretation of why the Schwarz studies observe small risk-pooling benefits while Jönsson and Silver and Jackson report significant benefits is that the fixed retailer order-quantities and the first-come first-served allocation rule in the Deuermeyer and Schwarz model do not allow the warehouse to balance retailer inventory levels when system stocks are low. To illustrate, suppose that the on-hand inventories of several retailers are small at some time when warehouse on-hand stocks are low. Then the retailer placing the next order might be allocated most or all of the remaining warehouse stock. On the other hand, the balancing-
type redistribution in Jönsson and Silver and the balanced division of remaining warehouse stocks in Jackson's runout period bring retailer inventories to comparable levels late in the replenishment cycle. Indeed, we believe that the risk-pooling benefits observed by Jönsson and Silver and Jackson are mostly due to this single rebalancing late in the replenishment cycle. Hence our interest in two-interval allocation policies.

Of course, the benefits of between-replenishment risk-pooling must ultimately be assessed against the costs which must be incurred to obtain them. Although we do not explicitly consider these costs, they include: (a) the increased cost of a warehouse that holds inventory, compared to a “ship-all” warehouse that is simply a repackaging and transfer point; (b) diseconomies in packaging, shipping, transportation and receiving that would be associated with two or more warehouse-stock withdrawals versus the single withdrawal of a “ship-all” policy, and (c) the increased information costs associated with determining retailer inventory levels at the beginning of each interval.

Erkip (1984) develops a two-equal-interval N-retailer model where, at the beginning of the replenishment cycle, the warehouse receives $S_0$ from the outside supplier and must divide that quantity into $\alpha \cdot S_0$, to be divided among the retailers immediately, and $(1 - \alpha) \cdot S_0$, to be divided among the retailers at some period $\tau$ prior to the last period in the replenishment cycle. Erkip provides a dynamic-programming formulation for finding the optimal $\alpha$ and $\tau$, noting that its state space will be extremely large. For the special case of identical retailers, and under the allocation assumption, Erkip proposes, but does not test, an approximate solution which, for fixed $\tau$, finds the $\alpha$ which minimizes the expected costs for a corresponding single retailer.

Jackson and Muckstadt (1989) examine two-interval allocation policies for an N-identical-retailer system with fixed intervals and backordering of excess demand. They prove that, given balancing divisions of withdrawn stock and independent, identically-distributed retailer demands, the expected second-interval balanced inventory level (see below) decreases with increases in the first-interval withdrawal quantity. Further, this expected second-interval balanced inventory converges as $N$ approaches infinity. For the N-identical-retailer model with correlated demands, they use this second property to develop an approximate cost function and a procedure for determining near-optimal quantities of stock to “reserve” at the warehouse for division in the second interval. They reported that preliminary computational experience indicated that the algorithms developed for infinite retailers yield allocation-policy parameters that are within a few units of the optimal values for the case of two retailers. They also propose an approximate cost function and solution procedure when the distributions of retailer demands are independent, but not necessarily identical.

The concept of “inventory” balance was first introduced by Clark and Scarf (1960) (and later refined by Federgruen and Zipkin 1984) to describe when all retailer inventory positions are at the same fractile of demand, i.e., at the same normalized level. (The normalized inventory for a retailer in a given interval is obtained by dividing the difference between its inventory position and mean interval demand by the standard deviation of interval demand.) In the identical-retailer case, perfect “balance” means that each retailer’s inventory position is the same. Many studies have assumed or examined balancing-type warehouse stock divisions, e.g., the “allocation assumption” in Eppen and Schrage (1981) and Federgruen and Zipkin (1984) and the “Run-Out Allocation Rule” in Jackson. To date, only two studies have developed formal measures of inventory balance or proved the optimality of balanced inventories for more than a single interval. Zipkin (1984) proves that for an m-period N-retailer model, the allocation policy which maximizes a proposed measure of balance in every period minimizes an approximation of a dynamic program describing the multiperiod newsboy cost function. McGavin et al. (1992) show that for N-Identical retailers, balancing allocations minimize expected lost sales and backorders over M independent intervals. They also demonstrate that balancing divisions are not necessarily optimal for nonidentical retailers.

This paper is organized as follows: In §3 we describe the N-retailer model and prove that withdrawn warehouse stock in each interval should be divided to “balance” retailer inventories (i.e., to maximize the minimum retailer inventory). The infinite-retailer model and properties of lost-sales minimizing infinite-retailer allocation policies are presented in §4. Section 5 presents numerical results. Simulation tests in §6 demonstrate
that the infinite-retailer model provides "good" allocation policies for N-retailer systems. In §7 we compare results obtained using our two-interval infinite-retailer and 50/25 heuristic policies to those that would be obtained using Jackson's four-equal-interval base-stock policies. This comparison indicates that well-chosen two-interval allocation policies have risk-pooling effects comparable to four-equal-interval base-stock policies. Conclusions and directions for future research are in §8.

3. Two-interval Allocation Policies for N Retailers

The distribution system functions as follows: At the beginning of each replenishment cycle (e.g., each month), warehouse stock is replenished by a shipment of goods from an outside supplier. For simplicity's sake, we assume that all stock in the system comes from the supplier at this time, that is, neither the warehouse nor any retailer has stock when the shipment arrives. Consequently, the first withdrawal of warehouse stock occurs immediately after the system replenishment. This zero-starting-inventory restriction can be relaxed as described below.

After the first withdrawal and division, stock is shipped to each retailer. We assume instantaneous delivery. During the first interval (i.e., the time between the first and second withdrawals), each retailer satisfies random demand from its stock on hand and first-interval lost sales/retailer are counted. At the beginning of the second interval, the warehouse reviews retailer inventory levels and divides the remaining stock among them. Each retailer then experiences random demand, second-interval lost sales are counted, and total lost sales/retailer are determined. Although this system is similar to a periodic-review system, it differs in that the intervals are not necessarily equal. Hence, we use the term "interval" rather than "period" to describe the times between warehouse stock withdrawals.

The demand process at each retailer is assumed to be identical to and independent of the demand process at every other retailer. For each retailer in each interval, demand is generated by a known stochastic process with stationary and independent increments. Well-known demand processes of this type include the Poisson and gamma processes and the Brownian-motion process with drift.

Our objective is to determine the two-interval allocation policy which minimizes expected lost sales per retailer (labelled "lost sales/retailer") between system replenishments for a given quantity of stock/retailer available at the beginning of the replenishment cycle. Inventory holding costs are ignored. We choose to examine the case where unsatisfied demand is lost because of the ambiguity of determining penalty costs for backorders which occur in possibly unequal time intervals. However, Theorems 1 through 4 (below) apply to the backorders case as well.

Although our analysis is limited to a single system-replenishment cycle, the policies we examine can be applied repeatedly for each cycle in a multiple-cycle scenario. In such a scenario, these policies would be called "myopic allocation" policies (see Federgruen and Zipkin 1984), since they would allocate inventory to minimize expected lost sales only in the cycle in which they are allocated. Federgruen and Zipkin found myopic-allocation policies to be near-optimal with respect to minimizing average inventory-associated cost/cycle in multiple-cycle situations with low-variance retailer demand.

In Theorem 1 we prove that, given any withdrawal quantities and interval lengths, the lost-sales-minimizing division balances retailer inventories (i.e., maximizes the minimum inventory level) in both intervals. In a balancing division, the resulting inventory level of all retailers who receive stock in the division is called the balanced inventory level. To illustrate a balancing division in the second withdrawal, consider ordering the

\[ \text{Figure 1 Ogive of Retailer Inventory Levels} \]

\[ \text{Diagram showing retail inventories} \]

\[ \text{Legend: Retailers 1 & 2, Inventory Level X, Retailer 9, Retailer 10} \]
interval-1 post-demand retailer on-hand inventories from smallest to largest as in Figure 1. Figure 1 is the ogive, or empirical distribution function, of retailer inventories. The second-interval balancing division can be represented by rotating Figure 1 about its diagonal axis and "pouring" the second-interval withdrawal quantity into the "well" created by the (new) vertical axis and the retailer inventories. The "height" which the poured stock attains on the (new) vertical axis is the balanced inventory level in interval 2.

**Theorem 1.** Given any withdrawal quantities and interval lengths, the division scheme which minimizes expected lost sales/retailer during the replenishment cycle divides available stock in each interval in such a way as to maximize the minimum retailer inventory.

**Proof.** See Appendix.

Henceforth, all stock divisions are considered to be balancing divisions.

Further examination of allocation policies for N-retailer systems is difficult, since one requires ogives of realized retailer demand in order to assess the expected lost sales/retailer resulting from any policy with more than a single interval. In particular, in order to determine expected second-interval lost sales, one requires the second-interval post-division ogive of retailer on-hand inventory levels. We denote this ogive as \( O^2 \).

Unfortunately, given stochastic demand, \( O^2 \) is not known in advance. In principle, one could estimate \( O^2 \) from the expected values of retailer-demand order statistics (see Jönsson and Silver 1987b), but this approach is complex and its estimates are imprecise. Instead, we chose an approximation we call the "infinite-retailer" model.

To understand the applicability of the infinite-retailer model, consider any unknown probability distribution function \( F(\cdot) \). Given \( N \)-independent realizations of \( F(\cdot) \), one can use the ogive, denoted \( O_N(\cdot) \), to estimate \( F(\cdot) \). From elementary statistics we know that \( \lim_{N \rightarrow \infty} O_N(\cdot) = F(\cdot) \). By reversing this logic one can use a known \( F(\cdot) \) to perfectly estimate \( O_N(\cdot) \). Of course, \( F(\cdot) \) is only an approximation to the ogive of interval demands for a system with \( N \) retailers. On the other hand, for large \( N \), it is reasonable to assume that the approximation is a good one. We indirectly measure its goodness below by examining the goodness of the infinite-retailer model's prescribed allocation policies for a finite number of retailers.

**4. Infinite-retailer Model**

The infinite-retailer model is identical to the \( N \)-retailer model in every respect, except for the number of retailers. In addition, Theorem 1 continues to be valid for any given number of withdrawals, interval lengths, and withdrawal quantities. (See McGavin 1990.) However, in order to retain the nature of the allocation problem, it is necessary to define total system stock at the beginning of the order cycle (as well as all other stocking quantities, lost sales and demand) in units per retailer (denoted "units/retailer"). The system stock/retailer is assumed to remain constant as \( N \), the number of retailers, increases.

The following notation will be used:

**Indices.** \( t \) indexes time intervals, \( t = 1, 2 \).

**Parameters.** \( I \) = replenishment-cycle length; i.e., the time between system replenishments; \( A \) = total stock per retailer in the system at the beginning of the replenishment cycle; \( \mu \) = mean demand rate/time at each retailer.

**Allocation Policy Variables.** \( I_1, I_2 \) = the lengths of the two intervals, where \( I_1 + I_2 = I \); \( A_1, A_2 \) = the quantities of stock (on a per-retailer basis) to be withdrawn from the warehouse and divided among the retailers at the beginning of each of the two intervals, where \( A_1 + A_2 = A \).

**Other Notation.** \( U \) = random variable denoting demand at a retailer; \( F_1(u) \), \( F_2(u) \) = the distribution functions of demand at each retailer during the intervals following the first and second withdrawals (identical and independent for all retailers), respectively. \( f_i(u) \) is the p.d.f. associated with \( F_i(u) \). We assume that \( F_i(u) = 0 \) for \( u < 0 \). Note that, since we are considering an infinite number of retailers, \( F_i(u) \) is also the realized ogive of retailer demand during interval \( t \), as explained above; \( D_i(y) \) = the inverse of \( F_i(u) \). At the end of each interval, we define each retailer's end-of-interval net inventory to be the difference between its beginning-of-interval post-division on-hand inventory and its demand during the interval.

\( X \) = random variable denoting net inventory at a retailer.
\( \Phi_1(x), \Phi_2(x) \) = the distribution functions of retailer net inventories after demand has occurred, \( \phi_1(x) \) and \( \phi_2(x) \) are the associated p.d.f.'s. A portion of the domain of \( \Phi_i \) will be negative if lost sales occurred in \( t \). (Note that \( \Phi_1(x) \) is that fraction of the total retailers which have net inventories of \( x \) or less at the end of interval 1.)

\( \Gamma_2(x) \) = the function representing the distribution of retailer on-hand inventories at the beginning of interval 2 before the division of \( A_2 \). \( \gamma_2(x) \) is the associated p.d.f. The domain of \( \Gamma_2 \) is \([0, A_1] \).

\( \Pi_2(x) \) = the function representing the distribution of retailer on-hand inventories at the beginning of interval 2 after the division of \( A_2 \), but before demand. \( \pi_2(x) \) is the associated p.d.f. Since the domain of \( \Pi_2 \) is \([0, \infty) \), \( \pi_2(x) = 0 \) for \( x \leq 0 \). Note that the corresponding distribution function for interval 1 is given by

\[
\Pi_1(x) = \begin{cases} 
1 & \text{for } x \geq A_1, \\
0 & \text{for } x < A_1.
\end{cases}
\]

\( B \) = the balanced inventory level after the division of withdrawn warehouse stock in interval 2.

Note: The p.d.f.'s we use to describe the probability distributions of retailer inventory and demand are not necessarily continuous and may contain atoms, or masses, representing probabilities for discrete values in the domain. For example, \( \gamma_2(x) \) and \( \pi_2(y) \) include atoms at \( x = 0 \) and \( y = B \), each equal in "weight" to the proportion of retailers which had shortages in the first interval and the proportion of retailers which are brought up to \( B \) by the division of \( A_2 \), respectively. It is assumed that the integrations necessary to obtain the respective distribution functions (e.g., \( \Gamma_2 \) and \( \Pi_2 \)) also include these atoms. We present a more precise description of the distribution functions in the Appendix.

Using the notation above, the warehouse receives \( A \) units/retailer from the supplier at the beginning of the replenishment cycle. Each retailer receives \( A_1 \) in the first division. During the first interval, each retailer experiences random demand drawn from \( F_1(u) \). The resulting retailer net inventories are ordered to obtain \( \Phi_1(x) \). Realized lost sales/retailer in the first interval are then given by \( \int_{-\infty}^{0} \Phi_1(x) \, dx \). \( \gamma_2(x) \), is obtained by truncating the domain of \( \phi_1(x) \) below 0 and assigning the probability associated with the truncated domain to \( x = 0 \).

Then \( A_2 \) is withdrawn and divided, yielding \( \Pi_2(x) \) with a balanced inventory level \( B \). During interval 2, each retailer experiences random demand drawn from \( F_2(u) \), giving the post-demand distribution of retailer net inventories \( \Phi_2(x) \). Second-interval lost sales/retailer are given by \( \int_{-\infty}^{0} \Phi_2(x) \, dx \).

The advantage of the infinite-retailer model over the the \( N \)-retailer model can now be illustrated. For finite \( N \), the shape of \( \Phi_i(x) \) is a function of \( N \) random demands. In the footnoted example below, \(^2\) there is a 50% probability that \( \Phi_i(x) \) takes on form (1) and a 25% probability of either (2) or (3). At best, one could derive the expected values of the order statistics of retailer inventories which can then be used to estimate \( \Phi_i(x) \) and

\(^2\) For example, suppose that in a two-retailer system,

\[
\Phi_i(x) = \begin{cases} 
0 & \text{for } x < 1, \\
1 & \text{for } x \geq 1.
\end{cases}
\]

\[
f_i(u) = \begin{cases} 
0.5 & \text{for } u = 0, \\
0.5 & \text{for } u = 1, \\
0 & \text{for } u \neq 0, 1.
\end{cases}
\]

(See figure below.)

![Graph](image)

Then, depending on realized demand, \( \Phi_i(x) \) takes one of the following three forms:

\(^1\) \( \Phi_i(x) = \begin{cases} 
0 & \text{for } x < 0, \\
0.5 & \text{for } 0 \leq x < 1, \\
1 & \text{for } x \geq 1.
\end{cases} \)

if one retailer experiences demand of 0 and the other demand of 1;

\(^2\) \( \Phi_i(x) = \begin{cases} 
0 & \text{for } x < 0, \\
1 & \text{for } x \geq 0.
\end{cases} \)

if each retailer has demand of 1; and

\(^3\) \( \Phi_i(x) = \begin{cases} 
0 & \text{for } x < 1, \\
1 & \text{for } x \geq 1,
\end{cases} \)

if each retailer has demand of 0.
its lost sales. With an infinite number of retailers, on the other hand, \( F_t(u) \) is the distribution of realized demand in \( t \). Thus, if every retailer receives \( A_1 \) at the beginning of the first interval, then \( \Phi_t(x) \) is the realized distribution of end-of-interval net inventory. Further, given any predemand distribution of on-hand inventory \( \Pi_t(x) \), and any demand distribution \( F_t(u) \), one can derive the exact distribution of end-of-interval \( (t) \) retailer net inventories \( \Phi_t(x) \), through a convolution of \( \pi_t \) and the negative of \( f_t \). Specifically, for any interval \( t \), \( \phi_t(x) \) is given by

\[
\phi_t(x) = \int_{u=-\infty}^{x} \pi_t(x + u)f_t(u)du \tag{1}
\]

and the distribution function is given by

\[
\Phi_t(x) = \int_{v=-\infty}^{x} \phi_t(v)dv. \tag{2}
\]

The infinite-retailer model also provides a means for separating: (1) the lost sales resulting when realized average demand/retailer exceeds the average stock/retailer, and from (2) the lost sales resulting when realized individual retailer demands exceed individual retailer stocks. In particular, the quality of any allocation policy can only be accurately measured when the effect of (1) can be accounted for. In the infinite-retailer case, the average realized demand/retailer equals \( \mu \), and hence, can be compared to the given average stock/retailer \( A \), to precisely determine (1). On the other hand, in the finite-retailer case, since the average realized demand/retailer is stochastic, (1) is unknown.

Properties of Optimal Two-interval Allocation Policies for the Infinite-retailer Model

**Theorem 2.** Given a replenishment cycle divided into fixed intervals \( l_1 \) and \( l_2 \), there exists an optimal allocation policy with the property that \( B \leq A_1^* \).

**Proof.** The intuition behind the proof is that if every retailer has inventory of at least \( B \) after the division of \( A_2 \), then every retailer may as well have been stocked up to at least \( B \) after the division of \( A_1 \), thereby avoiding potential shortages in the first interval. See Appendix for details.

**Corollary to Theorem 2.** Given a replenishment cycle divided into fixed intervals \( l_1 \) and \( l_2 \); there exists an optimal allocation policy with the property that \( A_1^* \geq A_2^* \).

**Proof.** See Appendix.

Finally, given the initial distribution of retailer inventories in interval 1,

\[
\Pi_1(x) = \begin{cases} 1 & \text{for } x \geq A_1, \\ 0 & \text{for } x < A_1, \end{cases}
\]

for any fixed \( I_1 \) and \( I_2 \), we provide the expression for the lost sales/retailer over the replenishment cycle and prove that the determination of \( A_1^* \), the optimal value of \( A_1 \), is a convex optimization problem.

Define \( LS(A_1) \) to be the lost sales/retailer given withdrawal quantities \( (A_1, A_2 = A - A_1) \) for any fixed intervals \( (l_1, l_2) \). It can be shown that

\[
LS(A_1) = \int_{A_1}^{\infty} [1 - F_1(u)]du + (1 - Q) \cdot \int_{\Pi}^{\infty} [1 - F_2(u)]du + \int_{0}^{\Pi} \int_{A_1}^{\infty} [1 - F_2(u)]dudy \tag{3}
\]

where \( F_1(A_1) = \) fraction of the retailers with leftovers at the end of interval 1. \( Q = \) fraction of the retailers which receive no stock in the division of \( A_2 = A - A_1 \) in interval 2. \( Q \) uniquely solves

\[
\int_{Q}^{\Pi} [D_1(y) - D_1(Q)]dy + (A_1 - D_1(Q)) \cdot (1 - F_1(A_1)) = A - A_1,
\]

\[
B = \frac{(A - A_1) + \int_{Q}^{\Pi} [A_1 - D_1(y)]dy}{(1 - Q)}. \tag{4}
\]

(Note that \( B \) is also given by \( B = A_2 - D_1(Q) \).)

Expressions (3) and (4) may be interpreted as follows, using Figure 2: Given that each retailer starts interval 1 with \( A_1 \), the area under \( F_1(u) \) for \( 0 \leq u \leq A_1 \) is the interval-1 leftovers/retailer, and the fraction of retailers with leftovers is \( F_1(A_1) \). Correspondingly, the area above \( F_1(u) \) for \( A_1 \leq u \leq \infty \) is the interval-1 lost sales/retailer. This is the first term in expression (3). The fraction of retailers with lost sales is \( 1 - F_1(A_1) \). These retailers have no inventory at the end of interval 1. Also
note that those retailers with the smallest end-of-interval-1 on-hand inventories experienced the largest interval-1 demands. Consequently, end-of-interval-1 retailer inventories should be read from right to left using the axis on the top of Figure 2, with 0 corresponding to $A_1$ on the lower axis.

The division of $A_2$ is used to raise the minimum retailer inventory to $B$. Consequently, $A_2$ is the area above $F_1(u)$ between 0 and $B$ on the upper axis. The fraction of retailers receiving stock is $(1 - Q)$. Note that not all of these $(1 - Q)$ of the retailers have stocked out. In fact, every fractile of retailers $y$, where $Q < y < F_1(A_1)$, has leftovers at the end of interval 1 equal to $A_1 - D_1(y)$, where $D_1(\cdot)$ is the inverse of the demand distribution $F_1(\cdot)$. Consequently, the expression for $B$ can be obtained by dividing $A_2 = (A - A_1)$ plus the total leftovers at the $F_1(A_1) - Q$ retailers by $(1 - Q)$, the total number of retailers brought up to $B$. This yields expression (4).

The second term of expression (3) represents the second-interval lost sales for these $(1 - Q)$ retailers.

The remaining $Q$ retailers receive no stock in the division of $A_2$ since their interval-1 leftovers exceed $B$. For each fractile of these retailers $y$, its leftovers at the end of interval 1 is given by $A_1 - D_1(y)$. The third term of expression (3) represents the second-interval lost sales for these $Q$ retailers.

**Theorem 3.** $LS(A_1)$ is convex in $A_1$ for fixed $I_1$ and $I_2$.

**Proof.** See Appendix.

**Using Theorems 1–3 to Determine Optimal Infinite-Retailer Policies**

The properties derived in Theorems 1–3 can be used to simplify the search for optimal two-interval infinite-retailer allocation policies. In particular, given Theorem 1, only balancing divisions need be considered, and from Theorem 2, only policies where $A_1 > A_2$ need to be examined. Theorem 3 simplifies the search for $A_1^*$ given $I_1$ and $I_2$. Details of our numerical search are given in §5.

**The 50/25 Heuristic Allocation Policy**

The 50/25 policy is a heuristic which has performed well in our computational tests. The heuristic has a first interval equal to 50% of the replenishment cycle and a second-interval withdrawal quantity equal to 25% of mean replenishment-cycle demand. It is based on the following rationale: (1) Set $I_1 = I_2 = 50% \cdot I$. Such a choice of $I_1$ and $I_2$ is particularly appropriate in multi-item situations where there are economic incentives for joint allocation. (2) Every retailer should begin the second interval with an inventory of at least $\mu \cdot I_2$, the mean second-interval demand/retailer. Theorem 4 and its corollary below show how, for symmetric demand, the value of $A_2$ necessary to achieve this desirable second-interval inventory level is satisfied by the heuristic.

**Theorem 4.** For fixed $I_1$ and $I_2$, $A \geq \mu$, and symmetric $f_1(u)$, $A_2 = \mu \cdot I_2$ is sufficient to guarantee $B \geq \mu \cdot I_2$, with equality holding when $A = \mu$.

**Proof.** See Appendix.

**Corollary to Theorem 4.** For $I_1 = I_2 = 50% \cdot I$, $A = \mu$, and symmetric $f_1(u)$, $A_2 = 25% \cdot \mu$ is the interval-2 withdrawal quantity which yields $B$, the minimum interval-2 retailer inventory, equal to $\mu \cdot I_2$, the mean second-interval demand/retailer.

**Proof.** The corollary follows directly from Theorem 4 when $I_1 = I_2$. Q.E.D.

In §6, we evaluate the 50/25 policy as a heuristic for an $N$-retailer system.

5. **Numerical Study of the Infinite-retailer Model**

In this section we describe the calculation of optimal infinite-retailer policies and the characteristics we observed for a sample of scenarios.
The Scenarios

A, the initial system stock used in each scenario, was chosen to be the stock/retailer required for a “ship-all” policy (i.e., $A_1 = A$, $A_2 = 0$) to yield lost sales/retailer equal to a specified percentage $P$, of mean replenishment-cycle demand/retailer. For example, $P = 10\%$ means that the initial system stock would yield expected lost sales/retailer equal to $10\%$ of the mean replenishment-cycle demand/retailer ($\mu$) under a ship-all policy. $P$ values of $10\%$ (low service), $5\%$ (medium service), and $1\%$ (high service) were examined. $A$ is expressed as a percentage of mean demand/retailer ($\mu$). Similarly, withdrawal quantities/retailer ($A_1$ and $A_2$) and interval lengths ($l_1$ and $l_2$) are stated as percentages of initial stock/retailer ($A$), and of length of the replenishment cycle ($l$, respectively.

Retailer demand was generated by a gamma process. The gamma process is a more practical model for high-variability demand than either the Poisson or Brownian-motion processes. The Poisson is an impractical choice because as mean replenishment-cycle demand decreases, its coefficient of variation increases. This, combined with the fact that Poisson demand is integer, often forces the optimal first withdrawal to be $100\%$ (i.e., $A_1 = A$) for almost any choice of $l_1$ or $A$. On the other hand, a high-variability Brownian-motion process (i.e., normally-distributed demand) often yields negative demand. Since our results are all scaled to $\mu$, the mean value of the gamma process is arbitrary. The variance of the gamma process was set to yield coefficients of variation, $K = 0.1$ (low), 0.33 (medium), or 1 (high).

The Infinite-retailer Heuristic Allocation Policy

In the simulations described below, we test allocation policies computed from the infinite-retailer heuristic model as heuristic allocation policies for finite-retailer systems. For each finite-retailer scenario examined, the infinite-retailer heuristic policy is the $(A^*_1, I^*_1)$ obtained by optimizing the corresponding infinite-retailer model. In order to determine the optimal two-interval infinite-retailer allocation policy for each scenario $(K, P, A)$ we searched a numerical representation of the infinite-retailer model (see Appendix) using a two-stage process: (a) For a given $I_1$, we determined the lost-sales minimizing $A_1^*$, denoted $A^*_1$ and the corresponding $LS(A^*_1 | I_1)$ by an enumerative search of $LS(A | I_1)$ on $A_1$ in the range $[0\%, 100\%]$ of $A$. (b) We determined the optimal $(A^*_1, I^*_1)$ pair by an enumerative search of $LS(A^*_1 | I_1)$ values for $I_1$, over the range $[0\%, 100\%]$ using a step-size of 3.125%.

Columns 1–3 of Table 1 describe the seven scenarios examined. Columns 4 and 5 report the optimal infinite-retailer policy, $(A^*_1, I^*_1)$, for each scenario. Observe that $A^*_1$ ranges from approximately 87 to 98% of $A$ and $I^*_1$ ranges from approximately 43 to 78% of $l$. Column 6 reports the lost sales/retailer associated with the optimal allocation policy $(A^*_1, I^*_1)$.

Measuring the Risk-pooling Benefits of the Optimal Two-interval Allocation Policy

The risk-pooling benefits of the optimal allocation policy can be measured in several ways. One measure is the decrease in lost sales/retailer provided by the policy over the ship-all policy. Column 7 reports this difference as a percentage of $P$. This measure of risk-pooling ranges from 8% (scenario 1: low service, low $K$) to 88% of $P$ (scenario 5: high service, low $K$).

Columns 8 and 9 report an alternative, inventory-oriented measure of the risk-pooling. Column 8 reports the initial system stock which would be required for a ship-all policy to match the improvement in service provided by the optimal allocation policy. In scenario 6, for example, initial system stock would have to increase from 160.73 to 187.47% of $\mu$ in order for a ship-all policy to provide the optimal allocation policy’s lost sales/retailer of 0.25%. Column 9 reports the relative increase. For example, in scenario 6, the increase in $A$ from 160.73 to 187.47% of $\mu$ is a 16.63% increase in initial system stock.

These sample results display two general characteristics that we observed in a larger numerical study of the infinite-retailer model. First, the benefits of the optimal two-interval allocation policy over a ship-all policy are scenario specific. That is, scenarios with very high demand uncertainty ($K$) and high service levels (i.e., very small $P$ values) usually display significant risk-
Table 1  Results of Optimizing the Infinite-retailer Model

| Scenario | Coeff of Variation (K) | Shop-All Initial System | Optimal Optimal Lost Sales of Optimal Decrease in Lost Sales Initial System Stock (A) Req Relative Excess of Col 8 Over Col 9 |
|----------|------------------------|--------------------------|----------------|---------------------|------------------------|------------------------|
| 1        | 0.100                  | 10.00%                   | 90.91%        | 87.40%              | 71.88%                  | 9.20%                  | 8.03%                  | 91.90%               | 1.10%                |
| 2        | 0.333                  | 10.00%                   | 107.72%       | 87.00%              | 56.25%                  | 7.12%                  | 28.76%                 | 116.64%              | 8.28%                |
| 3        | 1.000                  | 10.00%                   | 230.26%       | 93.50%              | 43.75%                  | 8.01%                  | 19.91%                 | 252.46%              | 9.64%                |
| 4        | 0.333                  | 5.00%                    | 125.40%       | 89.00%              | 59.38%                  | 2.73%                  | 45.40%                 | 139.41%              | 11.17%               |
| 5        | 0.100                  | 1.00%                    | 109.46%       | 93.00%              | 78.13%                  | 0.12%                  | 88.32%                 | 120.24%              | 9.84%                |
| 6        | 0.333                  | 1.00%                    | 160.73%       | 95.00%              | 65.63%                  | 0.25%                  | 74.80%                 | 187.47%              | 16.63%               |
| 7        | 1.000                  | 1.00%                    | 460.52%       | 98.00%              | 43.75%                  | 0.27%                  | 72.57%                 | 589.87%              | 26.09%               |

pooling benefits. This observation is consistent with Jackson’s (1988) simulation studies involving finite retailers. Second, in describing the optimal allocation policy ($A_1^*$, $I_1^*$) itself, we generally observed that increasing demand uncertainty (increasing K) increases $A_1^*$ and decreases $I_1^*$. However, increasing service (decreasing P) generally increases both $A_1^*$ and $I_1^*$.

Finally, it is interesting to note that, even though gamma demand is not symmetric and $A < \mu$ for Scenario 1, we always observed $A_1^* \leq \mu \cdot (I_1^*/2)$, as suggested by Theorem 4.


Our simulation study was broken into four steps. First we searched a computer simulation of each scenario ($N, K, P, A$) to find the allocation policy $(A_1, I_1)$ which gave the lowest average lost sales / retailer. The grid of the search included all values of $A_1$ between 0% and 100% using a step size of 1% and all values of $I_1$ between 0% and 100% using a step size of 3.125%. Then, a second simulation of this “best” allocation policy was run to obtain an unbiased estimate of its expected lost sales / retailer. A second simulation of the “best” policy was necessary since the first simulation search yields downwardly-biased estimates of the lost sales / retailer for the chosen policy. Third, we simulated the performance of the corresponding infinite-retailer heuristic policy to estimate its expected lost sales / retailer. Finally, we simulated the performance of the 50 / 25 heuristic; that is, $I_1 = 50\% \cdot I$ and $A_2 = 25\% \cdot \mu$. Retailer demand was generated using a gamma process. More details of the simulation are provided in the appendix.

Columns 1–4 of Table 2 describe the 12 sample scenarios. These scenarios use the $K, P$ and $A$ values already described. Hence, the infinite-retailer ($A_1^*, I_1^*$) values are the same as those reported in Table 1. (The corresponding Table 1 scenarios are listed in the column adjacent to the scenario numbers in Table 2.)

The “Best” Simulation Policy

Columns 5 and 6 of Table 2 describe the “best” allocation policy from the simulation search. Column 7 provides the simulation-based point estimates of the lost sales / retailer incurred by the “best” allocation policy and their 95% confidence intervals.

Test of the Infinite-retailer Heuristic

For each scenario, the corresponding optimal infinite-retailer ($A_1^*, I_1^*$) values are reported in Columns 8 and

4 Note that the “best policy” is chosen by finding the minimum estimated (simulated) lost sales / retailer from a set of candidate policies.

Thus, if the simulation happens to yield an estimate of some policy’s expected lost sales / retailer which is below its true expected lost sales / retailer, then that policy is more likely to be chosen as the “best” than another policy whose simulated lost sales / retailer accurately estimates (or overestimates) its true expected lost sales / retailer. Consequently, the simulated lost sales / retailer which are reported for the “best policy” are a downwardly-biased estimated of that policy’s true expected lost sales / retailer.
9. Column 10 provides the simulation-based estimates of the lost sales/retailer incurred by the infinite-retailer heuristic, and their 95% confidence limits. Observe that the infinite-retailer $A_1^*$ values are higher than the "best" $A_1$ values in 11 of 12 scenarios, and range from 3.62% of $\mu$ lower (Scenario 1) to 6.00% of $\mu$ higher (Scenarios 5 and 11). The infinite-retailer $I_1^*$ values are higher than the "best" $I_1$ values in 6 of the 12 scenarios, equal in 4 scenarios, and lower than the "best" $I_1$ in 2 scenarios; and range from 9.38% of $I$ lower (Scenario 12) to 25.00% higher (Scenario 5) than the "best" $I_1$.

More important, a comparison of Column 7 with Column 10 shows that the expected lost sales/retailer resulting from using the infinite-retailer heuristic are within the 95% confidence limits on the lost sales/retailer of the "best" policy in 11 of 12 scenarios. In other words, the difference in lost sales between the "best" policy and infinite-retailer heuristic policy is insignificant in all but one case; and the "best" policy, on average, had 26.88% fewer lost sales than the ship-all policy. In fact, in several cases, the estimated lost sales/retailer under a heuristic policy was actually lower than the estimated lost sales/retailer under the "best" policy found by simulation search. Evidently, the heuristics perform very well compared to the computationally-intensive search.

On the other hand, the infinite-retailer model is not a good predictor of the lost sales/retailer incurred in a finite-retailer scenario: it consistently underpredicts the lost sales/retailer, as is evident from comparing Column 6 of Table 1 with Column 10 of Table 2.

Test of the 50/25 Heuristic

Column 12 of Table 2 reports the simulation-based estimates of the lost sales/retailer incurred by the 50/25 heuristic and their 95% confidence limits. The 50/25 heuristic’s expected lost sales/retailer are higher than those incurred by the infinite-retailer heuristic. More importantly, the 50/25 heuristic’s estimated lost sales/retailer are outside the 95% confidence limits for the lost sales/retailer of (i.e., significantly worse than) the "best" policy in 5 of the 12 scenarios.

On the other hand, the 50/25 heuristic achieves an average of 88.66% of the reduction in lost sales/retailer of the "best" allocation policy over the ship-all policy (excluding Scenario 11, where the "best" policy showed no improvement over ship-all). This indicates that the 50/25 heuristic may be a cost-effective allocation policy in situations where the cost or complexity of either a simulation search or the calculation of the infinite-retailer policy is prohibitive. In multi-item situations, where there are economic incentives for the joint allocation of all items, the 50/25 heuristic may be near-optimal.

Overall, the results of the simulation study suggest that the infinite-retailer approximation generally provides excellent allocation policies for systems with $N$ retailers, except in scenarios with a small number of retailers, very high service levels, and high-variance demand. Finally, the 50/25 heuristic, while not as effective as the infinite-retailer heuristic, may warrant consideration, particularly in multi-item, joint-allocation situations.


As already noted, we hypothesize that the risk-pooling benefits observed by Jönsson and Silver (1987a,b) and Jackson (1988) are mostly due to the fact that their (different) policies each provide the opportunity for a single rebalancing of retailer stocks late in the replenishment cycle. In order to test this hypothesis we compared the risk-pooling benefits of our heuristic, two-interval allocation policies with the benefits of a four equal-interval base-stock type allocation policy. Specifically, we compared the lost sales/retailer obtained from simulations of: (a) the infinite-retailer heuristic with two unequally spaced withdrawals; and (b) the "best" four equal-interval base-stock type allocation policy, determined by search of the simulation.

The scenarios examined in Columns 1–3 of Table 3 are those of Jackson (1988). However, two differences should be noted. First, in Jackson’s study retailer demands were drawn from a Poisson process in scenarios 1–8, 12 and 13; and from a Brownian-motion process in scenarios 9, 10 and 11. Our demand process was gamma. Second, in Jackson’s study, demand in excess of supply was backordered. In our simulation study, shortages resulted in lost sales. (We omitted Jackson scenarios which had nonzero lead times or nonbalancing divisions of warehouse stock.)
**Table 2** Simulation-based Comparison of “Best” Policies Found Through Search to Infinite-retailer and 50/25 Heuristic Policies

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The lost sales/retailer for underlined scenarios in Columns 10 and 12 fall within the 95% confidence limits of the lost sales/retailer of the “best” policy (Col 7).

**Table 3** Comparison of Well-chosen Two-interval Policies to Four-equal-interval Base-stock Oriented Policies

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The lost sales/retailer for underlined scenarios in Columns 10 and 13 fall within the 95% confidence limits of the lost sales/retailer associated with the “best” policy (Col 6).
The Best Four Equal-interval Base-stock Policy
For each scenario, we determined the “best” base-
stock/retailer (Column 5) by searching a simulation
for the base-stock which achieved the lowest average
lost sales/retailer. A second simulation of this “best”
policy yielded the estimated lost sales/retailer and 95%
confidence limits reported in Column 6. Column 7 pro-
vides the percentage reduction in lost sales achieved by
the optimal base-stock policy over the ship-all policy.
Reductions range from 5% to 84%, with an average of
36%. For comparison purposes, we provide Jackson’s
reported results in Columns 15–17. The results are quite
similar despite the differences noted.

The Infinite-retailer Heuristic
For each scenario, the optimal two-interval infinite-re-
tailer policy, \((A^*_1, I^*_1)\), is given in Columns 8 and 9. Col-
umn 10 reports the simulation-based estimates of the
lost sales/retailer and their 95% confidence limits. Ob-
serves that lost sales/retailer for the infinite-retailer
heuristic are above the 95% confidence interval for the
best four-interval policy (Column 6) in 4 of the 13
scenarios, within the confidence interval in 7 scenarios,
and below it in 2. However, in all scenarios, the infinite-
retailer heuristic yields a large percentage of the risk-
pooling benefit provided by the best base-stock policy.
Column 11 reports the ratio of the reduction in lost
sales/retailer of the two-interval infinite-retailer policy
(from the ship-all policy) to the corresponding reduction
in lost sales/retailer of the best base-stock policy. Per-
centages are between 75% and 123% (100% on aver-
age). This suggests that policies with two “well-chosen”
withdrawals can provide risk-pooling benefits compara-
tible to policies using four equal intervals.

The 50/25 Heuristic
In order to measure the benefit of additional withdraw-
als when equal intervals are desirable, we also simulated
the performance of the 50/25 heuristic using Jackson’s
(modified) scenarios. The withdrawal quantities and
estimated lost sales are reported in Columns 12 and 13,
respectively. Not surprisingly, Column 13 reports that
the simulation-based estimate of the 50/25 heuristic’s
lost sales/retailer are above the 95% confidence interval
of the best four-interval policy in all but one of the 13
scenarios (Scenario 6). However, Column 14 shows that
the 50/25 heuristic achieved between 58% and 91%
(with an average of 70%) of the reduction in lost sales/
retailer (over the ship-all policy) attained by the four-
equal-interval policy. In other words, if one regards the
ship-all policy as a “single withdrawal” policy, then the
addition of one additional withdrawal appears to pro-
vide most of the reduction in lost sales/retailer provided
by three additional withdrawals.

Since the costs to operate a warehouse-retailer system
are likely to be increasing in the number of warehouse
withdrawals, it is reasonable to suppose that, in many
instances, inventory-allocation policies with only a few
“well-chosen” withdrawals would, on an overall-cost
basis, out-perform equal-interval base-stock allocation
policies with more withdrawals, even though these pol-
cies have fewer lost sales/retailer.

8. Conclusions and Directions for
Future Research
In this study we constructed a simple warehouse-retailer
model of a distribution system to examine between-
replenishment risk-pooling. We have demonstrated that
the risk-pooling benefit can be significant when allo-
cation policies are chosen carefully. We have shown
that, when the objective is to minimize lost sales be-
tween system replenishments, optimal divisions are
balancing divisions. We have also indicated that policies
with two well-chosen withdrawals provide service-
levels comparable to four-equal-interval, base-stock-
oriented allocation policies. This study has also de-
monstrated how the infinite-retailer model can be used
to provide “good” heuristic policies.

We believe that the biggest challenge (and benefit)
in extending this research lies in the area of developing
additional analytical relationships for the infinite-retailer
model. In particular, analytical expressions describing
the relationships between the service-level or demand
variance and the optimal withdrawal quantities, with-
drawal times, lost sales or risk-pooling effects are
needed.

Appendix
This appendix contains the proofs of Theorems 1–4, a more precise
definition of the distribution functions introduced in §4, and descrip-
tions of the numerical representation of the infinite-retailer model
discussed in §5 and the simulation used to test the heuristics discussed
in §6 and 7.
Proofs

**Theorem 1.** Given any withdrawal quantities and interval lengths, the division scheme which minimizes expected lost sales/retailer during the replenishment cycle divides available stock in each interval in such a way as to maximize the minimum retailer inventory.

**Proof.** McGavin et al. (1992) show that the lost-sales minimizing division of warehouse stock for N identical retailers over M intervals is a balancing division. The two-interval policy considered here is simply a special case of the M-interval case. Q.E.D.

**Theorem 2.** Given a replenishment cycle divided into fixed intervals $I_1$ and $I_2$, there exists an optimal allocation policy with the property that $B > A_1^*$. The proof is analogous to the proof of Theorem 1. Q.E.D.

**Proof.** Consider an arbitrary allocation policy and the corresponding expected lost sales/retailer. Define $A_i^* = A_i^* = A_i^* - A_i$. Then

$$ A_1^* < A_2^* $$

implies $B > A_1^*$, which violates Theorem 1. Q.E.D.

**Theorem 3.** LS($A_i$) is convex in $A_i$ for fixed $I_1$ and $I_2$.

**Proof.** Let $R = F_1(1)$. By taking the first and second derivatives of LS and $B$ with respect to $A_i$, and noting that all $dA_i/dQ$ terms cancel, one obtains the following expressions:

$$ \frac{dB}{dA_i} = \frac{-1 + \int_0^1 f_1(y) \cdot (1 - Q)}{(1 - Q)^2} = \frac{(R - Q - 1)}{(1 - Q)^2}, $$

$$ \frac{d(LS(A_i))}{dA_i} = \frac{(R - 1) + (R - Q - 1) \cdot f_2(B - 1)}{(1 - Q)^2} + \int_0^Q f_2[A_1 - D_i(y) - 1] dy, $$

$$ \frac{d^2(LS(A_i))}{dA_i^2} = \frac{dR}{dA_i} \cdot [f_2(B)] + (R - Q - 1)^2, $$

$$ \frac{f_2(B)}{(1 - Q)} + \int_0^Q f_2[A_1 - D_i(y)] dy \geq 0. \text{ Q.E.D.} $$

This theorem can be generalized to show that the division of any fixed quantity of stock between any two consecutive withdrawals for a multi-interval allocation policy is a convex optimization problem. (See McGavin 1990.)

**Theorem 4.** For fixed $I_1$ and $I_2$, $A_i = \mu \cdot I_i$ and symmetric $f_1(u)$, $A_2 = \mu \cdot I_2$ is sufficient to guarantee $B \geq \mu \cdot I_2$ with equality holding when $B = \mu$.

**Proof.** We begin by proving that

$$ \{A_1, A_2\} = \left\{ \mu \cdot \left[ I_1 + I_2 \right], \mu \cdot \left[ I_2 \right] \right\} $$

gives $B = \mu \cdot I_2$ for $B = \mu$. Note that the first-interval lost sales/retailer resulting from $A_1^*$ is given by $f_1^1 - [1 - F_1(u)] du$. The $B$ resulting from dividing $A_2^*$ is given by $z$, where $z$ satisfies

$$ \int_{A_2}^{A_2^*} [1 - F_1(u)] du = A_2^*. $$

Since $f_1(u)$ is symmetric, it can be shown that

$$ \int_{A_2}^{A_2^*} [1 - F_1(u)] du = \mu \cdot \left[ I_2 \right], $$

or $z = \mu \cdot I_2$. Thus, $B = \mu \cdot I_2$. The fact that $A_2 = \mu \cdot I_2$ is sufficient to guarantee $B \geq \mu \cdot I_2$ for $A > \mu$ follows from the fact that if $A_1^*$, the interval-1 withdrawal quantity in the first part of the proof, is increased by $A_1 - B_1$, no retailer will have fewer leftovers before the division of $A_1$ than they had when $A = \mu$. Thus, $A_2 = \mu \cdot [I_2/2]$ will be sufficient to bring $B$ to at least $\mu \cdot I_2$. Q.E.D.

Discussion of Distribution Functions with Probability Mass and Their Convolutions

Assume that $f_1$ and $f_2$ are continuous distributions of demand defined on the domain $[0, \infty)$. From the discussion in §4,
\[
\phi_t(x) = \begin{cases} 
  f_t(A_t - x) & \text{for } x \leq A_t, \\
  0 & \text{for } x > A_t.
\end{cases}
\]

Then the \( \gamma_t \) which we describe in §4 is made up of two parts: a probability mass of weight \( \gamma_{2m} = \Phi_t(0) \) at \( x = 0 \), and the function

\[
\gamma_t(x) = \begin{cases} 
  0 & \text{for } x \leq 0, \\
  \phi_t(x) & \text{for } x > 0.
\end{cases}
\]

Thus,

\[
\Gamma_t(x) = \begin{cases} 
  0 & \text{for } x < 0, \\
  \gamma_{2m} + \int_0^x \gamma_t(v) dv & \text{for } x \geq 0.
\end{cases}
\]

Similarly, the described \( \pi_t \) is made up of a probability mass of weight \( \pi_{2m} = \Gamma_t(B) \) at \( x = B \), and the function

\[
\pi_t(x) = \begin{cases} 
  0 & \text{for } x \leq B, \\
  \gamma_t(x) & \text{for } x > B.
\end{cases}
\]

and

\[
\Pi_t(x) = \begin{cases} 
  0 & \text{for } x < B, \\
  \pi_{2m} + \int_0^x \pi_t(v) dv & \text{for } x \geq B.
\end{cases}
\]

Finally,

\[
\phi_t(x) = \begin{cases} 
  (1-Q) \cdot f_t(B - x) & \text{for } x \leq B, \\
  0 & \text{for } x > B,
\end{cases}
\]

\[+ \int_{-\infty}^x \pi_t(x + u) f_t(u) du.\]

Note that, if any demand distribution \( f_t \) is discrete, then the resulting \( \phi_t \) will consist of probability masses, which may be treated similarly to the \( \gamma_t \) and \( \pi_t \) above.

**Numerical Representation of the Infinite-retailer Model**

The numerical representation of the infinite-retailer model was written in Fortran and run on a VAX 8800 with double-digit precision. All calculations are based on digitized demand and inventory distributions with points in the domain corresponding to every 0.004 (i.e., 1/250th) of the range of the distribution function. Distributions of gamma demand (i.e., \( F_t(u) \) and \( F_t(u) \)) were generated by a modified regular-false search using the IMSL subroutine DGAMDF. For each \( (K, P) \) and \( (A_t, I_t) \) combination, each of the 250 fractiles initially received \( A_t \). This initial inventory "distribution" was convoluted with \( F_t(u) \), the demand distribution for \( I_t \), generating a distribution with \( (250)^3 \) points in the range. This distribution was divided into 250 equal fractiles whose medians were chosen to represent \( \phi_t(x) \). Lost sales in interval 1 were obtained by numerically integrating the determined \( \phi_t(x) \) between \(-\infty \) and 0. \( \Phi_t(x) \) was then truncated at \( x = 0 \) to get \( \Gamma_t(x) \). \( A_t \) was divided among the fractiles of retailers in \( F_t(x) \) with the least stock to get \( \Pi_t(x) \). Lost sales in the second interval were obtained by taking the convolution of \( \Pi_t(x) \) with \( F_t(x) \), choosing the midpoints of the fractiles to get \( \Phi_t(x) \), and integrating \( \Phi_t(x) \) below 0.

**Construction of the Simulation for Testing Heuristics**

The simulation was written in Fortran and run on a VAX 8800 with double-digit precision. For each \( (N, K, P) \) we generated a set of 512,000 (i.e., 32·16,000) i.i.d. pseudorandom gamma demands using the IMSL subroutine DRNGAM. For each \( (A_t, I_t) \) we tested we divided these demands into two sets. Set 1 consisted of 16,000 groups of size \( I_t \), forming 16,000 pseudorandom gamma demands distributed with mean and variance proportional to \( I_t \), and Set 2 consisted of 16,000 groups of size \( I_t \), forming 16,000 pseudorandom gamma demands distributed with mean and variance proportional to \( I_t \). We conducted \([16,000 / N] \) separate trials, where each trial was carried out in the following fashion. Each of the \( N \) retailers initially received \( A_t \). Demand was selected from Set 1 and lost sales were assessed. \( A_t \) was divided among the retailers, interval-2 demands selected from Set 2, and lost sales assessed again. Expected lost sales/retailer were determined by dividing the lost sales/trial by \( N \).

**References**


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