The Game Theoretic Approach to Modeling Strategic Behavior: Answers to Problems

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Abstract

GTAans.tex
http://www.fee.uva.nl/fo/sm/aie2/GTAans.pdf

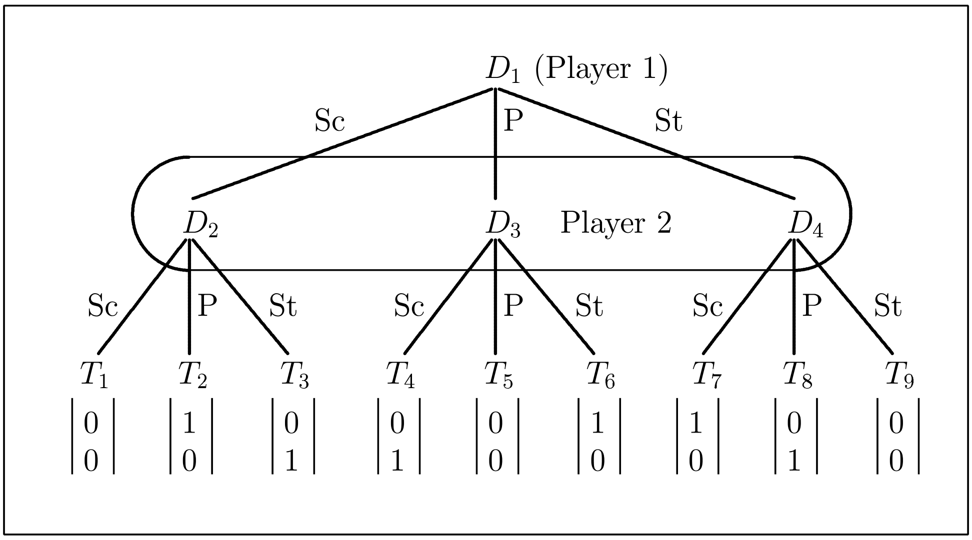

Figure 1: Game tree for one round of scissors-paper-stone

A.1 These are the rules of scissors-paper-stone:

- two players simultaneously make shares with their hands.
- the shapes are
  - scissors: two fingers spread apart;
  - paper: palm facing down;
  - stone: a fist.
- If both players make the same sign, no one wins. Otherwise, the winner is determined according to the rules
  - scissors cuts paper;
  - paper covers stone;
  - stone blunts scissors.

The winner of a round receives €1.

(1) Suppose two players play a single round of scissors-paper-stone. Draw the game tree for the extensive form and the payoff matrix for the strategic form.

		Player 2		
		Sc	P	St
Player 1	Sc	$0, 0$	$1, 0$	$0, 1$
	P	$0, 1$	$0, 0$	$1, 0$
	St	$1, 0$	$0, 1$	$0, 0$

Figure 2: Payoff matrix, one round of scissors-paper-stone. Rows show strategies of player 1, columns show strategies of player 2. The first element in each entry shows the payoff of player 1, the second element in each entry shows the payoff of player 2.

(2) Suppose the players play a supergame that consists of an infinite number of rounds of scissors-paper-stone. Is there an equilibrium strategy for the supergame?

There is no noncooperative equilibrium in pure strategies. The mixed-strategy equilibrium has each player making each move with equal probability:

$$\Pr(Scissors) = \Pr(Paper) = \Pr(Stone) = \frac{1}{3}.$$

Expected payoffs are then zero. See Von Neumann and Morgenstern (1944, p. 94, p. 111).

		Player 2	
		Cooperate	Defect
Player 1	Cooperate	R, R	S, T
	Defect	T, S	P, P

Figure 3: Payoff matrix for the Prisoners' Dilemma game with general cash payoffs.

A2 Consider a Prisoners' Dilemma game with general cash payoffs, as in Figure 3.

Discuss the conditions $T > R > P > S$ and $R > (T + S)/2$. See Axelrod (1984, pp. 9–10).

$T > R$ means that the one-period payoff from defecting exceeds the one-period payoff from cooperating. $R > P$ means that the one-period payoff from cooperating exceeds the one-period payoff if both defect. $P > S$ means that the payoff from being a sucker — cooperating while the other player defects — is even less than the payoff if both players defect.

$R > \frac{1}{2}(T + S)$ means that the payoff from joint cooperation exceeds the average payoff from alternating roles as defector and sucker. If this condition is not met, alternating strategies (C, D) and (D, C) will dominate (C, C).

A.3 Figure 4 (Kreps and Wilson, 1982a, Figure 8) illustrates a game in extensive form.

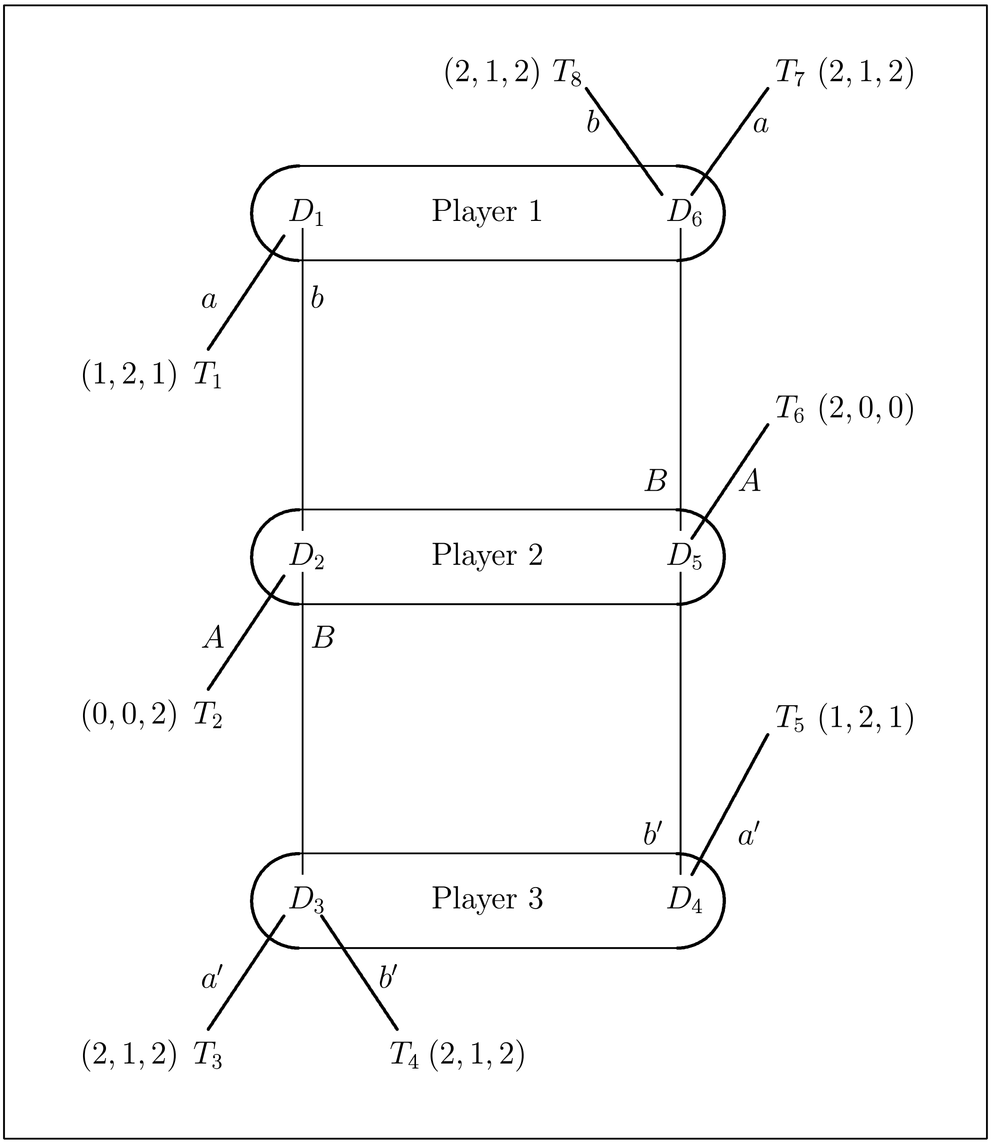

Figure 4: Game tree for problem A3.

(a) Show that the following strategies are Nash equilibria and subgame perfect Nash equilibria:

(1) $\pi^1(a) = 1$, $\pi^2(A) = 1$, $\pi^3(a') = 1$

(2) $\pi^1(b) = 1$, $\pi^2(B) = 1$, $\pi^3(b') = 1$

(Hint: First calculate the realization probabilities – the probabilities of the different terminal outcomes. Then calculate the expected payoffs of the three players. Then show, for (1) and (2) above, that if any two players make the indicated choices, the remaining player maximizes his expected payoff by making the indicated choice.)

(b) Show that $\pi^1(b) = 1$, $\pi^2(B) = 1$, $\pi^3(b') = 1$ is a trembling-hand perfect equilibrium and a sequential equilibrium.

(Hint: for the latter, use the realization probabilities of the terminal nodes from the answer to (a) to compute the probabilities of the decision nodes according to Bayes' rule. Then compute the expected payoffs of the three players and show that each player maximizes his expected payoff by making the indicated choice.)

Remarks: The game may begin at Decision node D_1 or at decision node D_4. The initial assessment of the probability that the game begins at D_1 is $\rho_1 = 1/2$. The initial assessment of the probability that the game begins at D_4 is $\rho_4 = 1 - \rho_1 = 1/2$. Row vectors give payoffs.

(a) Show that the following strategies are Nash equilibria and subgame perfect Nash equilibria:

(1) $\pi^1(a) = 1$, $\pi^2(A) = 1$, $\pi^3(a') = 1$

(2) $\pi^1(b) = 1$, $\pi^2(B) = 1$, $\pi^3(b') = 1$

The realization probabilities of the terminal nodes that can be reached from D_1 are

$$\Pr(T_1) = \rho_1\pi_1(a),$$

the probability that the game begins in player 1's information set times the probability according to strategy π that player 1 chooses a;

$$\Pr(T_2) = \rho_1\pi_1(b)\pi_2(A);$$

$$\Pr(T_3) = \rho_1\pi_1(b)\pi_2(B)\pi_3(a');$$

$$\Pr(T_4) = \rho_1\pi_1(b)\pi_2(B)\pi_3(b').$$

The realization probabilities of the terminal nodes that can be reached from D_2 are

$$\Pr(T_5) = \rho_2\pi_3(a');$$

$$\Pr(T_6) = \rho_2\pi_3(b')\pi_2(A);$$

$$\Pr(T_7) = \rho_2\pi_3(b')\pi_2(B)\pi_1(a);$$

$$\Pr(T_8) = \rho_2 \pi_3(b')\pi_2(B)\pi_1(b).$$

Calculate the expected payoffs of the three players.

Expected payoff of player 1:

If the game starts at D_1, then player 1 receives 1 if he plays a, 0 if he plays b and player 2 plays A, and 2 if he plays b and player 2 plays B (no matter what the play of player 3).

If the game starts at D_4, then player 1 receives 1 if player 3 plays a', 0 if player 3 plays b' and player 2 plays A, and 2 if player 3 plays b' and player 2 plays B (no matter what the play of player 1). Player 1's expected payoff is therefore

$$H_1 = \rho_1[1\pi_1(a) + 0\pi_1(b)\pi_2(A) + 2\pi_1(b)\pi_2(B)]$$

$$= \rho_1[\pi_1(a) + 2\pi_1(b)\pi_2(B)] + \pi_2[1 + \pi_3(b')]$$

If the game starts at D_1 and player 1 plays a, then player 2's payoff is 2. If the game starts at D_1, player 1 plays b, and player 2 plays A, then player 2 receives 0. If the game starts at D_1, player 1 plays b, and player 2 plays B, then player 2 receives 1 (no matter what the choice of player 3).

If the game starts at D_4, then player 2 receives 2 if player 3 plays a', 0 if player 3 plays b' and player 2 plays A, and 1 if player 3 plays b' and player 2 plays B (no matter what the choice of player 1). Player 2's expected payoff is therefore

$$H_2 = \rho_1[2\pi_1(a) + 0\pi_1(b)\pi_2(A) + 1\pi_1(b)\pi_2(B)]$$

$$+ \rho_2[2\pi_3(a') + 0\pi_3(b')\pi_2(A) + 1\pi_3(b')\pi_2(B)]$$

$$= \rho_1[2\pi_1(a) + \pi_1(b)\pi_2(B)] + \ \rho_2[2\pi_3(a') + \pi_3(b')\pi_2(B)]$$

If the game starts at D_1 and player 1 plays a, then player 3 receives 1. If the game starts at D_1 and player 1 plays b, then player 3 receives 2, no matter what the choices of players 2 or 3.

If the game starts at D_4, player 3 receives 1 if player 3 plays a'. If the game starts at D_4, and player 3 plays b', then player 3 receives 0 if player 2 plays A and 2 if player 2 plays B (no matter what the play of player 1).

Player 3's expected payoff is therefore

$$H_3 = \rho_1[1\pi_1(a) + 2\pi_1(b)] + \rho_2[1\pi_3(a') + 0\pi_3(b')\pi_2(A) + 2\pi_3(b')\pi_2(B)]$$

$$= \rho_1[1 + \pi_1(b)] + \rho_2[\pi_3(a') + 2\pi_3(b')\pi_2(B)]$$

Thus we have

$$H_1 = \rho_1[\pi_1(a) + 2\pi_1(b)\pi_2(B)] + \pi_2[1 + \pi_3(b')]$$

$$H_2 = \rho_1[2\pi_1(a) + \pi_1(b)\pi_2(B)] + \rho_2[2\pi_3(a') + \pi_3(b')\pi_2(B)]$$

$$H_3 = \rho_1[1 + \pi_1(b)] + \rho_2[\pi_3(a') + 2\pi_3(b')\pi_2(B)]$$

Now show that $\pi_1(a) = \pi_2(A) = \pi_3(a') = 1$ is a Nash equilibrium. We do this by showing that if any two of the players select the specified strategy, the remaining player maximizes his expected payoff by choosing the specified strategy.

Let $\pi_2(A) = \pi_3(a') = 1$. Then player 1's expected payoff is

$$H_1 = \rho_1\pi_1(a) + \rho_2.$$

Player 1 maximizes his expected payoff by making $\pi_1(a)$ as large as possible, i.e., by setting $\pi_1(a) = 1$.

Now let $\pi_1(a) = \pi_3(a') = 1$. Player 2's payoff is then

$$H_2 = 2.$$

Any strategy $0 < \pi_2(A) < 1$ maximizes player 2's expected payoff. In particular, $\pi_2(A) = 1$ maximizes player 2's expected payoff.

Finally, let $\pi_1(a) = \pi_2(A) = 1$. Player 3's expected payoff becomes

$$H_3 = \rho_1 + \rho_2\pi_3(a').$$

Player 3 therefore maximizes his expected payoff by making $\pi_3(a')$ as large as possible, which is to say $\pi_3(a') = 1$. It follows that $\pi_1(a) = \pi_2(A) = \pi_3(a') = 1$ is a Nash equilibrium. If any two of the players make the indicated choice, the remaining player maximizes his expected payoff by selecting the indicated strategy. It is also, trivially, a subgame perfect equilibrium: there is no subgame of the original game except the original game itself.

Using the same methodology, show that $\pi_1(a) = \pi_2(A) = \pi_3(a') = 0$ is a Nash equilibrium.

Let $\pi_2(A) = \pi_3(a') = 0$. Then player 1's expected payoff is

$$H_1 = \rho_1[\pi_1(a) + 2\pi_1(b)] + 2\rho_2 = 2 - \rho_1\pi_1(a).$$

Player 1 therefore maximizes his expected payoff by making $\pi_1(a)$ as small as possible, which is to say by setting $\pi_1(a) = 0$. Now let $\pi_1(a) = \pi_3(a') = 0$. Player 2's payoff is then

$$H_2 = \pi_2(B) = 1 - \pi_2(A).$$

Player 2 therefore maximizes his expected profit by making $\pi_2(A)$ as small as possible. If $\pi_1(a) = \pi_3(a') = 0$, player 2's expected payoff-maximizing strategy is $\pi_2(A) = 0$.

Let $\pi_1(a) = \pi_2(A) = 0$. Player 3's expected payoff becomes

$$H_3 = 2\rho_1 + \rho_2[\pi_3(a') + 2\pi_3(b')] = 2 - \rho_2\pi_3(a').$$

Player 3 therefore maximizes his expected payoff by making $\pi_3(a')$ as small as possible, which is to say $\pi_3(a') = 0$.

It follows that $\pi_1(a) = \pi_2(A) = \pi_3(a') = 0$ is a Nash equilibrium. If any two of the players make the indicated choice, the remaining player maximizes his expected payoff by selecting the indicated strategy. Since the only subgame of the original game is the original game itself, this is also a subgame perfect equilibrium.

(b) Show that $\pi_1(b) = 1, \pi_2(B) = 1, \pi_3(b') = 1$ is a trembling hand perfect equilibrium. Strategies in the perturbed game differ from strategies in the underlying game by terms that reflect the possibility that players will make a mistake. Thus

$$\widehat{\pi}_1(a) = (1 - \varepsilon)\pi_1(a) + \varepsilon q_1(a)$$

$$\widehat{\pi}_1(b) = (1 - \varepsilon)\pi_1(b) + \varepsilon q_1(b)$$

where $q_1(a) + q_1(b) = 1$;

$$\widehat{\pi}_2(A) = (1 - \varepsilon)\pi_2(A) + \varepsilon q_2(A)$$

$$\widehat{\pi}_2(B) = (1 - \varepsilon)\pi_2(B) + \varepsilon q_2(B)$$

where $q_2(A) + q_2(B) = 1$; and

$$\widehat{\pi}_3(a') = (1 - \varepsilon)\pi_3(a') + \varepsilon q_3(a')$$

$$\widehat{\pi}_3(b') = (1 - \varepsilon)\pi_3(b') + \varepsilon q_3(b')$$

where $q_3(a') + q_3(b') = 1$.

Strategies in the underlying game, being probabilities, must lie between zero and 1. These bounds on the strategies in the underlying game imply the following bounds on the strategies in the perturbed game:

$$1 - \varepsilon q_1(b) \geq \widehat{\pi}_1(a) \geq \varepsilon q_1(a)$$

$$1 - \varepsilon q_1(a) \geq \widehat{\pi}_1(b) \geq \varepsilon q_1(b)$$

$$1 - \varepsilon q_2(B) \geq \widehat{\pi}_2(A) \geq \varepsilon q_2(A)$$

$$1 - \varepsilon q_2(A) \geq \widehat{\pi}_2(B) \geq \varepsilon q_2(B)$$

$$1 - \varepsilon q_3(b') \geq \widehat{\pi}_3(a') \geq \varepsilon q_3(a')$$

$$1 - \varepsilon q_3(a') \geq \widehat{\pi}_3(b') \geq \varepsilon q_3(b').$$

Observe that all strategies in the perturbed game have strictly positive probability.

The realization probabilities and expected payoffs in the perturbed game have the same form as those in the underlying game, substituting the strategies in the perturbed game for those in the underlying game. (In writing player 1's expected payoff, substitute $\widehat{\pi}_1(a)$ for $\pi_1(a)$, and so on.) The expected payoffs of the players in the perturbed game are therefore

$$\widehat{H}_1 = \rho_1[\widehat{\pi}_1(a) + 2\widehat{\pi}_1(b)] + \rho_2[1 + \widehat{\pi}_3(b')$$

$$\widehat{H}_2 = \rho_1[2\widehat{\pi}_1(a) + \widehat{\pi}_1(b)\widehat{\pi}_2(B)] + \ \rho_2[2\widehat{\pi}_3(a') + \widehat{\pi}_3(b')\widehat{\pi}_2(B)]$$

$$\widehat{H}_3 = \rho_1[1 + \widehat{\pi}_1(b)] + \rho_2[\widehat{\pi}_3(a') + 2\widehat{\pi}_3(b')\widehat{\pi}_2(B)].$$

Take the derivative of each player's expected payoff with respect to his own strategy:

$$\frac{\partial \widehat{H}_1}{\partial \widehat{\pi}_1(b)} = \rho_1[\widehat{\pi}_2(B) - \widehat{\pi}_2(A)]$$

$$\frac{\partial \widehat{H}_2}{\partial \widehat{\pi}_2(B)} = \rho_1\widehat{\pi}_1(b) + \rho_2\widehat{\pi}_3(b')$$

$$\frac{\partial \widehat{H}_3}{\partial \widehat{\pi}_3(b')} = \rho_2[\widehat{\pi}_2(B) - \widehat{\pi}_2(A)].$$

Because strategies in the perturbed game are bounded away from zero, $\partial \widehat{H}_2/\partial \widehat{\pi}_2(B)$ is strictly positive. It follows that to maximize his expected payoff, player 2 should make $\widehat{\pi}_2(B)$ as large as possible. Thus player 2's optimal strategies in the perturbed game are

$$\widehat{\pi}_2(A) = \varepsilon q_2(A)$$

$$\widehat{\pi}_2(B) = 1 - \varepsilon q_2(A).$$

From these values, we obtain

$$\widehat{\pi}_2(B) - \widehat{\pi}_2(A) = 1 - 2\varepsilon q_2(A).$$

which will be positive for ε sufficiently small. This in turn implies that $\partial \widehat{H}_1/\partial \widehat{\pi}_1(b)$ and $\partial \widehat{H}_3/\partial \widehat{\pi}_3(b')$ are positive. To maximize their expected payoffs, players 1 and 3 should make $\widehat{\pi}_1(b)$ and $\widehat{\pi}_3(b')$, respectively, as large as possible. Their optimal strategies are

$$\widehat{\pi}_1(a) = \varepsilon q_1(a) \quad \widehat{\pi}_1(b) = 1 - \varepsilon q_1(a)$$

$$\widehat{\pi}_3(a') = \varepsilon q_1(a') \quad \widehat{\pi}_1(b') = 1 - \varepsilon q_3(a')$$

These are the Nash equilibrium strategies and subgame perfect Nash equilibrium strategies of the perturbed game. As ε goes to zero, the probability of mistaken behavior goes to zero. The limit of the Nash equilibrium strategies in the perturbed game is $\pi_1(b) = \pi_2(B) = \pi_3(b') = 1$, which is therefore a trembling hand perfect equilibrium

(c) Show that $\pi_1(b) = \pi_2(B) = \pi_3(b') = 1$ is a sequential equilibrium.

The realization probabilities of the terminal nodes are given in the answer to (a). Using these realization probabilities and the initial assessments (concerning the starting point of the game), compute the probabilities of the decision nodes, according to Bayes' rule.

Decision nodes in player 1's information set:

$$\mu(D_1) = \frac{\textit{Probability of terminal nodes that follow } D_1}{\textit{Probability of terminal nodes that follow Player 1's information set}}$$

$$= \frac{\pi(T_1) + \pi(T_2) + \pi(T_3) + \pi(T_4)}{\pi(T_1) + \pi(T_2) + \pi(T_3) + \pi(T_4) + \pi(T_7) + \pi(T_8)}$$

$$= \frac{\pi(T_1) + \pi(T_2) + \pi(T_3) + \pi(T_4)}{1 - \pi(T_5) - \pi(T_6)}$$

$$= \frac{\rho_1}{1 - \rho_2 \left[\pi_3(a') + \pi_3(b')\pi_2(A)\right]}.$$

$$\mu(D_6) = \frac{\textit{Probability of terminal nodes that follow } D_6}{\textit{Probability of terminal nodes that follow Player 1's information set}}$$

$$= \frac{\pi(T_7) + \pi(T_8)}{\pi(T_1) + \pi(T_2) + \pi(T_3) + \pi(T_4) + \pi(T_7) + \pi(T_8)}$$

$$= \frac{\pi(T_7) + \pi(T_8)}{1 - \pi(T_5) - \pi(T_6)}$$

$$= \frac{\rho_2 \pi_3(b')\pi_2(B)}{1 - \rho_2 \left[\pi_3(a') + \pi_3(b')\pi_2(A)\right]}$$

It must be the case that $\mu(D_1)$ and $\mu(D_6)$ add up to 1, since they are the subjective beliefs assigned to the two nodes in player 1's information set if player 1's information set is reached. Using this fact, or by direct computation, one can show that

$$1 - \rho_2 \left[\pi_3(a') + \pi_3(b')\pi_2(A)\right] = \rho_1 + \rho_2 \pi_3(b')\pi_2(B).$$

Substituting this expression for the denominator in the expressions for $\mu(D_1)$ and $\mu(D_6)$, one obtains

$$\mu(D_1) = \frac{\rho_1}{\rho_1 + \rho_2 \pi_3(b')\pi_2(B)}$$

$$\mu(D_6) = \frac{\rho_2 \pi_3(b')\pi_2(B)}{\rho_1 + \rho_2 \pi_3(b')\pi_2(B)}.$$

Written in this way, it is evident that $\mu(D_1)$ and $\mu(D_6)$ do in fact add up to 1.

Decision nodes in player 2's information set:

$$\mu(D_2) = \frac{\textit{Probability of terminal nodes that follow } D_2}{\textit{Probability of terminal nodes that follow Player 2's information set}}$$

$$= \frac{\pi(T_2) + \pi(T_3) + \pi(T_4)}{\pi(T_2) + \pi(T_3) + \pi(T_4) + \pi(T_6) + \pi(T_7) + \pi(T_8)}$$

$$= \frac{\pi(T_2) + \pi(T_3) + \pi(T_4)}{1 - \pi(T_1) - \pi(T_5)}$$

$$= \frac{\rho_1 \pi_1(b)}{1 - [\rho_1 \pi_1(a) + \rho_2 \pi_3(a')]}.$$

$$\mu(D_5) = \frac{\textit{Probability of terminal nodes that follow } D_5}{\textit{Probability of terminal nodes that follow Player 2's information set}}$$

$$= \frac{\pi(T_6) + \pi(T_7) + \pi(T_8)}{\pi(T_2) + \pi(T_3) + \pi(T_4) + \pi(T_6) + \pi(T_7) + \pi(T_8)}$$

$$= \frac{\pi(T_6) + \pi(T_7) + \pi(T_8)}{1 - \pi(T_1) - \pi(T_5)}$$

$$= \frac{\rho_2 \pi_3(b')}{1 - [\rho_1 \pi_1(a) + \rho_2 \pi_3(a')]}.$$

Either directly or using the fact that $\mu(D_2) + \mu(D_5) = 1$, one obtains

$$1 - [\rho_1 \pi_1(a) + \rho_2 \pi_3(a')] = \rho_1 \pi_1(b) + \rho_2 \pi_3(b').$$

Substituting this expression for the denominators, one obtains

$$\mu(D_2) = \frac{\rho_1 \pi_1(b)}{\rho_1 \pi_1(b) + \rho_2 \pi_3(b')}$$

$$\mu(D_5) = \frac{\rho_2 \pi_3(b')}{\rho_1 \pi_1(b) + \rho_2 \pi_3(b')}.$$

Decision nodes in player 3's information set:

$$\mu(D_3) = \frac{\textit{Probability of terminal nodes that follow } D_3}{\textit{Probability of terminal nodes that follow Player 3's information set}}$$

$$= \frac{\pi(T_3) + \pi(T_4)}{\pi(T_3) + \pi(T_4) + \pi(T_5) + \pi(T_6) + \pi(T_7) + \pi(T_8)}$$

$$= \frac{\pi(T_3) + \pi(T_4)}{1 - \pi(T_1) - \pi(T_2)}$$

$$= \frac{\rho_1 \pi_1(b)\pi_2(B)}{1 - \rho_1[\pi_1(a) + \pi_1(b)\pi_2(A)]}.$$

$$\mu(D_4) = \frac{\textit{Probability of terminal nodes that follow } D_4}{\textit{Probability of terminal nodes that follow Player 3's information set}}$$

$$= \frac{\pi(T_5) + \pi(T_6) + \pi(T_7) + \pi(T_8}{\pi(T_3) + \pi(T_4) + \pi(T_5) + \pi(T_6) + \pi(T_7) + \pi(T_8)}$$

$$= \frac{\pi(T_5) + \pi(T_6) + \pi(T_7) + \pi(T_8}{1 - \pi(T_1) - \pi(T_2)}$$

$$= \frac{\rho_2}{1 - \rho_1[\pi_1(a) + \pi_1(b)\pi_2(A)]}.$$

Either directly or using the fact that $\mu(D_3) + \mu(D_4) = 1$, one obtains

$$1 - \rho_1[\pi_1(a) + \pi_1(b)\pi_2(A)] = \rho_2 + \rho_1\pi_1(b)\pi_2(B).$$

Substituting for the denominators in the above expressions for $\mu(D_3)$ and $\mu(D_4)$, one obtains

$$\mu(D_3) = \frac{\rho_1\pi_1(b)\pi_2(B)}{\rho_2 + \rho_1\pi_1(b)\pi_2(B)}$$

$$\mu(D_4) = \frac{\rho_2}{\rho_2 + \rho_1\pi_1(b)\pi_2(B)}$$

Now compute the expected payoffs of the three players.

Player 1's payoff from strategy a is 1 if player 1 is at D_1 and 2 if player 1 is at D_6. Player 1's expected payoff from strategy a is therefore

$$1\mu(D_1) + 2\mu(D_6) = 2 - \mu(D_1).$$

If player 1 plays b from D_1, he receives 0 if player 2 subsequently plays A and he receives 2 if player 2 subsequently plays B. (Observe that if from D_1 player 1 plays b and player 2 plays B, then player 1 receives 2 no matter what player 3 does.) If player 1 plays b from D_6, player 1 receives 2. Player 1's expected payoff from strategy b is therefore

$$[0\pi_2(A) + 2\pi_2(B)]\mu(D_1) + 2\mu(D_6) = 2[1 - \mu(D_1)\pi_2(A)].$$

Player 1's expected payoff from the strategy $[\pi_1(a), \pi_1(b)]$ is therefore

$$[2 - \mu(D_1)]\,\pi_1(a) + \{2\,[1 - \mu(D_1)\pi_2(A)]\}\,\pi_1(b) = 1 - \mu(D_1)\,\{1 - \pi_1(b)\,[1 - 2\pi_2(A)]\}\,.$$

Player 2's payoff from strategy A is zero, whether player 2 is at D_2 or at D_6.

If player 2 is at D_2 and plays B, then the game moves to D_3 and player 2 receives 1 no matter what play is chosen by player 3. If player 2 is at D_5 and plays B, then the game moves to D_6 and player 2 receives 1 no matter what play is chosen by player 1. Thus player 2's expected payoff from B is 1.

Formally, one can proceed to write player 2's expected payoff as

$$0\pi_2(A) + 1\pi_2(B) = \pi_2(B)$$

and argue that 2 should therefore make $\pi_2(B)$ as large as possible, which is 1. Less formally, it is evident that if player 2 gets 0 from playing A and 1 from playing B, then player 2 should always play B.

Player 3's payoff from strategy a' is 2 if the game is at D_3 and 1 If the game is at D_4. Player 3's expected payoff from a' is therefore

$$2\mu(D_3) + 1\mu(D_4) = 1 + \mu(D_3).$$

Player 3's payoff from strategy b' is 2 if the game is at D_3. If the game is at D_4 and player 3 chooses b', the game moves to D_5. If player 2 chooses A, player 3 receives 0. If player 2 chooses B, the game moves to D_6. From D_6, player 3 receives 2 no matter what player 1 does. Hence player 3's expected payoff from b' is

$$2\mu(D_3) + [0\pi_2(A) + 2\pi_2(B)]\mu(D_4) = 2[\mu(D_3) + \mu(D_4)\pi_2(B)].$$

Player 3's expected payoff is therefore

$$\pi_3(a')[1 + \mu(D_3)] + 2\pi_3(b')[\mu(D_3) + \mu(D_4)\pi_2(B)].$$

Now examine the sequentially optimal strategies of the three players. From above, we know that player 2's optimal strategy is $\pi_2(A) = 0$, $\pi_2(B) = 1$.

From this, player 1's expected payoff is

$$1 - \mu(D_1)[1 - \pi_1(b)] = 1 - \mu(D_1)\pi_1(a).$$

It follows that to maximize his expected payoff, player 1 should make $\pi_1(a)$ as small as possible, i.e., should set $\pi_1(a) = 0$.

If $\pi_2(B) = 1$, player 3's expected payoff is

$$\pi_3(a')[1 + \mu(D_3)] + 2\pi_3(b')[\mu(D_3) + \mu(D_4)] = 2 - \mu(D_4)\pi_3(a').$$

It follows that to maximize his expected payoff, player 3 should make $\pi_3(a')$ as small as possible, i.e., should set $\pi_3(a') = 0$.

The optimal strategies are therefore

$$\pi_1(a) = \pi_2(A) = \pi_3(a') = 0$$

$$\pi_1(b) = \pi_2(B) = \pi_3(b') = 1.$$

Substituting these values into the optimal sequential beliefs, one obtains the equilibrium values

$$\mu(D_1) = \mu(D_2) = \mu(D_3) = \rho_1$$

$$\mu(D_4) = \mu(D_5) = \mu(D_6) = \rho_2.$$

A.4 Figure 5 (Kreps and Wilson, 1982a, Figure 14) illustrates a game in extensive form.

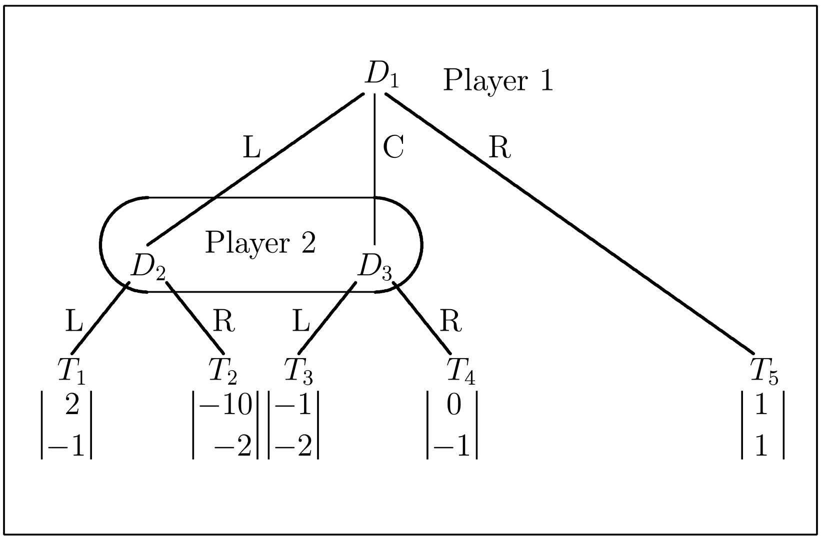

Figure 5: Game tree for problem A4.

(a) Show that $\pi_1(R)$, $\pi_2(r) = 1$ is a sequential equilibrium.
(b) Show that $\pi_1(L)$, $\pi_2(l) = 1$ is a sequential equilibrium.
Discuss.
Payoffs are

$$H_1 = 2\pi_1^L\pi_2^L - 10\pi_1^L\pi_2^R - \pi_1^C\pi_2^L + \pi_1^R$$

$$H_2 = \mu(D_2)(-\pi_2^L - 2\pi_2^R) + \mu(D_3)(-2\pi_2^L - \pi_2^R)$$

$$= -[2 - \mu(D_2)] + \pi_2^R[1 - 2\mu(D_2)].$$

If beliefs concerning decision nodes in player 2's information set can be defined by Bayes' rule, they are

$$\mu(D_2) = \frac{\pi_1^L}{\pi_1^L + \pi_1^C}$$

$$\mu(D_3) = \frac{\pi_1^C}{\pi_1^L + \pi_1^C}$$

This will only be possible if $\pi_1^R < 1$.

Proceeding formally, player 1 must maximize his payoff subject to the constraint that $\pi_1^R + \pi_1^L + \pi_1^C = 1$. The Lagrangian is

$$L = 2\pi_1^L\pi_2^L - 10\pi_1^L\pi_2^R - \pi_1^C\pi_2^L + \lambda\left(1 - \pi_1^L - \pi_1^C - \pi_1^R\right)$$

The Kuhn-Tucker conditions for player 1's optimization problem are

$$\pi_2^L - 10\pi_2^R - \lambda \leq 0 \quad \pi_1^L\left(\pi_2^L - 10\pi_2^R - \lambda\right) = 0 \quad \pi_1^L \geq 0;$$

$$-\pi_2^L - \lambda \leq 0 \quad -\pi_1^C\left(\pi_2^L + \lambda\right) = 0 \quad \pi_1^C \geq 0;$$

$$1 - \lambda \leq 0 \quad \pi_1^R(1 - \lambda) = 0 \quad \pi_1^R \geq 0;$$

$$1 - \pi_1^L - \pi_1^C - \pi_1^R = 0.$$

First show $\pi_1^R = \pi_2^R = 1$ is a sequential equilibrium.

If $\pi_2^R = 1$, then the Kuhn-Tucker conditions for player 1's optimization problem become

$$-10 - \lambda \leq 0 \quad \pi_1^L(-10 - \lambda) = 0 \quad \pi_1^L \geq 0;$$

$$-\lambda \leq 0 \quad \pi_1^C + \lambda = 0 \quad \pi_1^C \geq 0;$$

$$1 - \lambda \leq 0 \quad \pi_1^R(1 - \lambda) = 0 \quad \pi_1^R \geq 0;$$

$$1 - \pi_1^L - \pi_1^C - \pi_1^R = 0.$$

Suppose $\pi_1^L > 0$. Then $\lambda = -10 < 0$,which contradicts $-\lambda \leq 0$.Hence $\pi_1^L = 0$.

Suppose $\pi_1^C > 0$. Then $\lambda = 0$, which contradicts $1-\lambda \leq 0$. Hence $\pi_1^C = 0$.

It follows that $\pi_1^R = 1$ and $\lambda = 1$.

From the expression for player 2's payoff, we obtain

$$\frac{\partial H_2}{\partial \pi_2^R} = 1 - 2\mu(D_2).$$

If $\mu(D_2) \leq 1/2$, $\partial H_2/\partial \pi_2^R \geq 0$ and player 2 will maximize his expected payoff by setting $\pi_2^R = 1$.

Now show that $\pi_1^L = \pi_2^L = 1$ is a sequential equilibrium.

In this case, beliefs can be defined by Bayes' rule: $\mu(D_2) = 1$, $\mu(D_3) = 0$.
If $\pi_2^L = 1$, player 1's Kuhn-Tucker conditions become

$$1 - \lambda \leq 0 \quad \pi_1^L(1 - \lambda) = 0 \quad \pi_1^L \geq 0;$$

$$-1 - \lambda \leq 0 \quad \pi_1^C(1 + \lambda) = 0 \quad \pi_1^C \geq 0;$$

$$1 - \lambda \leq 0 \quad \pi_1^R(1 - \lambda) = 0 \quad \pi_1^R \geq 0;$$

$$1 - \pi_1^L - \pi_1^C - \pi_1^R = 0.$$

Suppose $\pi_1^R > 0$. Then $\lambda = 1$, which contradicts $-1 - \lambda \leq 0$. Hence $\pi_1^R = 0$.

Suppose $\pi_1^C > 0$. Then $\lambda = -1$, which contradicts $1 - \lambda \leq 0$. Hence $\pi_1^C = 0$.

It follows that $\pi_1^L = 1, \lambda = 1$.

Given $\mu(D_2) = 1$, player 2's expected payoff is

$$H_2 = \pi_2^L - 2.$$

It follows that

$$\frac{\partial H_2}{\partial \pi_2^L} = 1,$$

so player 2 should make π_2^L as large as possible, and set $\pi_2^L = 1$.

(c) Show that the same two strategy combinations may be trembling-hand equilibria.

Define the following probabilities for the perturbed game:

$$\widehat{\pi}_1^L = (1 - \varepsilon)\pi_1^L + \varepsilon q_1^L$$

$$\widehat{\pi}_1^C = (1 - \varepsilon)\pi_1^C + \varepsilon q_1^C$$

$$\widehat{\pi}_1^R = (1 - \varepsilon)\pi_1^R + \varepsilon q_1^R$$

$$\widehat{\pi}_2^L = (1 - \varepsilon)\pi_2^L + \varepsilon q_2^L$$

$$\widehat{\pi}_2^R = (1-\varepsilon)\pi_2^R + \varepsilon q_2^R$$

where $q_1^L + q_1^C + q_1^R = 1$ and $q_2^L + q_2^R = 1$.

Probabilities in the perturbed game are subject to the following bounds:

$$1-\varepsilon\left(1-q_1^L\right) \geq \widehat{\pi}_1^L \geq \varepsilon q_1^L$$

$$1-\varepsilon\left(1-q_1^C\right) \geq \widehat{\pi}_1^C \geq \varepsilon q_1^C$$

$$1-\varepsilon\left(1-q_1^R\right) \geq \widehat{\pi}_1^R \geq \varepsilon q_1^R$$

$$1-\varepsilon q_2^R \geq \widehat{\pi}_2^L \geq \varepsilon q_2^L$$

$$1-\varepsilon q_2^L \geq \widehat{\pi}_2^R \geq \varepsilon q_2^R$$

Payoffs in the perturbed game are

$$\widehat{H}_1 = 2\widehat{\pi}_1^L\widehat{\pi}_2^L - 10\widehat{\pi}_1^L\widehat{\pi}_2^R - \widehat{\pi}_1^C\widehat{\pi}_2^L + \widehat{\pi}_1^R$$

$$\widehat{H}_2 = -\widehat{\pi}_1^L\widehat{\pi}_2^L - 2\widehat{\pi}_1^L\widehat{\pi}_2^R - 2\widehat{\pi}_1^C\widehat{\pi}_2^L - \widehat{\pi}_1^C\widehat{\pi}_2^R + \widehat{\pi}_1^R$$

First show that $\widehat{\pi}_1^L = 1-\varepsilon\left(1-q_1^L\right)$, $\widehat{\pi}_1^C = \varepsilon q_1^C$ is a noncooperative equilibrium for the perturbed game.

Suppose $\widehat{\pi}_1^L = 1-\varepsilon\left(1-q_1^L\right)$, $\widehat{\pi}_1^C = \varepsilon q_1^C$. From the expression for player 2's payoff, we obtain the derivative of 2's payoff with respect to $\widehat{\pi}_2^L$:

$$\frac{\partial \widehat{H}_2}{\partial \widehat{\pi}_2^L} = \widehat{\pi}_1^L - \widehat{\pi}_1^C = 1-\varepsilon\left(1-q_1^L+q_1^C\right),$$

which will be positive for ε sufficiently small. Hence player 2 should make $\widehat{\pi}_2^L$ as large as possible, and should set $\widehat{\pi}_2^L = 1-\varepsilon q_1^R$.

If $\widehat{\pi}_1^L = 1-\varepsilon_1^R$ and $\widehat{\pi}_2^R = q_2^R$, player 1's payoff becomes

$$\widehat{H}_1 = 2\left(1-6\varepsilon q_2^R\right)\widehat{\pi}_1^L - \left(1-\varepsilon q_2^R\right)\widehat{\pi}_1^C + \widehat{\pi}_1^R.$$

Ignoring constraints, the derivative of $\widehat{H}_1$ with respect to $\widehat{\pi}_1^L$ is near 2 for ε sufficiently small, the derivative of $\widehat{H}_1$ with respect to $\widehat{\pi}_1^C$ is negative, and the derivative of $\widehat{H}_1$ with respect to $\widehat{\pi}_1^R$ is 1. It follows that player 1 should make $\widehat{\pi}_1^L$ as large as possible.

Hence $\widehat{\pi}_1^L = 1 - \varepsilon\left(1 - q_1^L\right)$, $\widehat{\pi}_2^L = 1 - \varepsilon q_1^R$ is a noncooperative equilibrium for the perturbed game. Letting ε go to zero, $\widehat{\pi}_1^L = 1$, $\widehat{\pi}_2^L = 1$ is a trembling hand equilibrium for the original game.

Now show that $\widehat{\pi}_1^R = 1 - \varepsilon(1 - q_1^R)$, $\widehat{\pi}_2^R = 1 - \varepsilon q_1^L$ is a noncooperative equilibrium for the perturbed game.

If $\widehat{\pi}_1^R$ is as large as possible, $\widehat{\pi}_1^C$ and $\widehat{\pi}_1^L$ are as small as possible: $\widehat{\pi}_1^C = \varepsilon q_1^C$ and $\widehat{\pi}_1^L = \varepsilon q_1^L$. The derivative of $\widehat{H}_2$ with respect to $\widehat{\pi}_2^L$ becomes

$$\frac{\partial \widehat{H}_2}{\partial \widehat{\pi}_2^L} = \varepsilon\left(q_1^L - q_1^C\right),$$

which is of ambiguous sign. If the probability that player 1 makes a mistake to center is greater than the probability that player 1 makes a mistake to the left, given that player 1 makes a mistake at all, then $\partial \widehat{H}_2 / \partial \widehat{\pi}_2^L$ will be negative, and player 2 will maximize his payoff by making $\widehat{\pi}_2^L$ as small as possible. Player 2 would then set $\widehat{\pi}_2^L = \varepsilon q_1^L$, $\widehat{\pi}_2^R = 1 - \varepsilon q_1^L$.

It is instructive to interpret the requirement $q_1^L \leq q_1^C$. If we were to define beliefs in the perturbed game using Bayes' rule, the beliefs assigned to D_2 and D_3 would be, in general,

$$\widehat{\mu}(D_2) = \frac{q_1^L}{q_1^C + q_1^L}$$

$$\widehat{\mu}(D_3) = \frac{q_1^C}{q_1^C + q_1^L}$$

Hence $q_1^L \leq q_1^C$ is equivalent to $\widehat{\mu}(D_3) \geq \widehat{\mu}(D_2)$, which in turn can be written $\widehat{\mu}(D_2) \leq 1/2$. But this is the condition on beliefs which is necessary for $\widehat{\pi}_1^R = \widehat{\pi}_2^R = 1$ to be a sequential equilibrium. Once again, the requirements for sequential and trembling hand equilibrium turn out to be essentially the same.

Now return to the analysis of the second trembling hand equilibrium. If $\widehat{\pi}_2^R = 1 - \varepsilon q_1^L$, $\widehat{\pi}_2^L = \varepsilon q_1^L$, player 1's payoff is

$$\widehat{H}_1 = -8\varepsilon q_2^L \widehat{\pi}_1^L - \varepsilon q_2^L \widehat{\pi}_1^C + \widehat{\pi}_1^R.$$

Thus the derivatives of $\widehat{H}_1$ with respect to $\widehat{\pi}_1^L$ and $\widehat{\pi}_1^C$ are negative, while the derivative of $\widehat{H}_1$ with respect to $\widehat{\pi}_1^R$ is 1. Thus player 1 should make $\widehat{\pi}_1^R$ as large as possible, setting $\widehat{\pi}_1^R = 1 - \varepsilon\left(1 - q_1^L\right)$.

Hence if $q_1^L \leq q_1^C$, $\widehat{\pi}_1^R = 1 - \varepsilon\left(1 - q_1^L\right)$, $\widehat{\pi}_2^R = 1 - \varepsilon q_1^L$ is a noncooperative equilibrium for the perturbed game. Letting ε go to zero, $\widehat{\pi}_1^R = 1$, $\widehat{\pi}_2^R = 1$ is a trembling hand equilibrium for the original game.

(c) Discuss.

Kreps and Wilson (1982a, p. 884) make the following remarks. For $\widehat{\pi}_1^R = 1$, $\widehat{\pi}_2^R = 1$ to be a sequential equilibrium, player 2, finding himself in his information set, must believe that he is more likely to be at D_3 than at D_4. We have seen that the same constraint on beliefs is required if these strategies are to be a trembling hand equilibrium. But if this constraint is met, player 2 must believe that player 1 is more likely to have played C than L. But if player 1 plays C, the best he can hope to do is break even, while if player 1 plays R, he is guaranteed €1. Player 1 may win €2 by playing L, but he cannot do better than €1 by playing C — indeed, he is guaranteed to do worse than €1 by playing C. Given this analysis of player 1's possible payoffs, player 2 ought to expect that if player 1 has made a move leading to player 2's information set, then that move was L, not C. Although $\widehat{\pi}_1^R = 1$, $\widehat{\pi}_2^R = 1$ is a sequential equilibrium, it is a sequential equilibrium only if players hold an unreasonable set of beliefs.