Kreps & Scheinkman with product differentiation: an expository note

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April 2000; revised December 2001

Abstract

Kreps and Scheinkman's (1983) celebrated result is that in a two-stage model of a market with homogeneous products in which firms noncooperatively pick capacities in the first stage and set prices in the second stage, the equilibrium outcome is that of a one-shot Cournot game. This note derives capacity best response functions for the first stage and extends the Kreps and Scheinkman result to the case of differentiated products.

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Contents

3

1. Introduction

Kreps and Scheinkman (1983) develop an extension of the Cournot and Bertrand duopoly models in which (1983, p. 327)

> *Capacities* are set in the first stage by the two producers. Demand is then determined by Bertrand-like price competition, and production takes place at zero cost, subject to capacity constraints generated by the first-stage decisions.

Equilibrium in this two-stage game has firms selecting capacities in the first stage that are just sufficient to produce the Cournot equilibrium outputs, and producing those outputs in the second period.

The Kreps and Scheinkman model is frequently cited as providing a justification for use of Cournot model, which is characterized as being differentially unsatisfactory compared with the Bertrand model. For example (Maggi, 1996, p. 240):[1]

> The Cournot model of quantity competition has been the subject of considerable criticism in the theory of industrial organization. If taken literally, the Cournot model assumes that firms dump their production on the market and that an auctioneer determines the price that clears the market. In most industries there is nothing that resembles an auctioneer, and firms use prices as a strategic variable.

Such criticisms may be put forward in part to motivate interest in analyses of the Kreps and Scheinkman type. This does not make them compelling. All models are abstractions from reality. The mechanism by which price is determined is also unspecified in the standard model of a perfectly competitive market, a model that is not set aside on that account.

Nor are such criticisms needed to justify interest in the Kreps and Scheinkman model: it is a seminal example of analysis of rivalry in an imperfectly competitive market in which outcomes depend critically on the sequence in which decisions may be taken, and the way in which earlier decisions condition the payoffs associated with later decisions.

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[1]Friedman (1982, p. 505) compares the Cournot model, the Bertrand model, and a model of price-setting oligopoly with product differentiation, and concludes that the latter is to be preferred on the grounds that it relies on more satisfactory assumptions without being computationally more complex.

Kreps and Scheinkman assume that the product is homogeneous. One implication of this assumption is that they must include in the model an assumed rationing rule that determines the quantity demanded of a higher-price firm if the capacity of a lower-price firm does not allow it to supply the entire quantity demanded at the lower price. Their results depend on the particular form of rationing rule that is used, as they suggest (Kreps and Scheinkman, 1983, p. 328) and as Davidson and Deneckere (1986) show formally.

If one extends the Kreps and Scheinkman model to differentiated products, the quantity demanded of each firm in the second stage of the game is well defined for all price pairs. This avoids the need to have a rationing rule as part of the model.

In these lecture notes, I show that the Kreps and Scheinkman result holds when the model is extended to the case of differentiated products.

There is a sense in which this result is intuitive. One would not expect firms to hold excess capacity in equilibrium. If there is no excess capacity in the second stage, then when firms maximize profit in the first stage, they are maximizing a payoff function that has then same demand and cost structures as in the corresponding one-shot Cournot game, the difference being that in the first stage of the two-stage game, firms select capacities rather than outputs. This leads to the result that with product differentiation, firms' capacity best response functions in the neighborhood of equilibrium are functionally identical to the Cournot quantity best response functions. It is thus to be expected that the equilibrium of the two-stage game should reproduce the Cournot outcome.

This result is obtained by Yin and Ng (1997), who do not present the capacity best response functions.

The analysis reported here is tedious. It has in common with many spatial models of imperfectly competitive markets that a complete treatment requires working through many cases which never occur in equilibrium, and which one knows from the beginning, or at least strongly suspects, will not occur in equilibrium. It turns out nonetheless to be necessary to work these cases out in order to demonstrate that they do not occur in equilibrium, and to verify that the most plausible suspect is in fact an equilibrium.

Section 2 presents the model of demand for differentiated products, due to Bowley (1924), that is used in these notes. Section 3 gives the cost function.

For comparison and background, Sections 4.1 and 4.2 give abbreviated treatments of the Bertrand and Cournot duopoly models with product differentiation.

As is standard in two-stage models, the analysis begins in the second stage and works backward. Section 5 introduces the possible segments of a firm's second-stage price best response function, and Section 6 works out the three possible shapes of the price best response functions.

Section 7 states the results, in the form of four Lemmas and a Theorem. Lemma 1 gives the relationship between the capacity level chosen in the first stage and the shape of the firm's price best response function in the second period. Lemma 2 gives the relationship between the capacity levels chosen in the first period and the nature of the price best response functions in the neighborhood of equilibrium. Lemma 3, the proof of which is entirely mechanical, gives equilibrium prices, quantities, and payoffs for the alternative second-stage equilibria. Lemma 4 gives the properties of the (first-stage) capacity best response functions. The Theorem is that the result of the Kreps and Scheinkman model holds when products are differentiated.

2. Demand

For the Bowley (1924) linear product differentiation model the representative consumer utility function is

$$U(q_1, q_2) = a(q_1 + q_2) - \frac{1}{2}b(q_1^2 + 2\theta q_1 q_2 + q_2^2) + m, \tag{2.1}$$

where $0 \leq \theta \leq 1$.

A Lagrangian function to describe the constrained optimization problem is

$$\mathcal{L}_1 =$$

$$U(q_1, q_2) = a(q_1 + q_2) - \frac{1}{2}b(q_1^2 + 2\theta q_1 q_2 + q_2^2) + m \tag{2.2}$$

$$+\lambda_1 \left(Y - m - p_1 q_1 - p_2 q_2\right),$$

where Y is income, m all other goods, and $p_m = 1$ the price of all other goods.

The Kuhn-Tucker conditions are

$$a - b(q_1 + \theta q_2) - \lambda p_1 \leq 0 \qquad q_1\left[a - b(q_1 + \theta q_2) - \lambda p_1\right] = 0 \qquad q_1 \geq 0 \tag{2.3}$$

$$a - b(\theta q_1 + q_2) - \lambda p_2 \leq 0 \qquad q_2\left[a - b(\theta q_1 + q_2) - \lambda p_2\right] = 0 \qquad q_2 \geq 0 \tag{2.4}$$

$$1 - \lambda_1 \leq 0 \qquad m(1 - \lambda_1) = 0 \qquad m \geq 0 \tag{2.5}$$

$$Y - m - p_1 q_1 - p_2 q_2 \geq 0 \qquad \lambda_1 (Y - m - p_1 q_1 - p_2 q_2) = 0 \qquad \lambda_1 \geq 0. \qquad (2.6)$$

Assume Y is sufficiently large so that $m > 0$. Then (2.5) implies that

$$\lambda_1 = 1 \qquad (2.7)$$

and (2.6) implies

$$m = Y - (p_1 q_1 + p_2 q_2). \qquad (2.8)$$

Substitute (2.7) in (2.3) and (2.4) to obtain

$$a - b(q_1 + \theta q_2) - p_1 \leq 0 \qquad q_1 \left[a - b(q_1 + \theta q_2) - p_1 \right] = 0 \qquad q_1 \geq 0 \qquad (2.9)$$

$$a - b(\theta q_1 + q_2) - p_2 \leq 0 \qquad q_2 \left[a - b(\theta q_1 + q_2) - p_2 \right] = 0 \qquad q_2 \geq 0. \qquad (2.10)$$

Case 1: $q_1 > 0$, $q_2 > 0$.
Then (2.9) and (2.10) imply that the inverse demand curves are

$$p_1 = a - b(q_1 + \theta q_2) \qquad (2.11)$$

$$p_2 = a - b(\theta q_1 + q_2). \qquad (2.12)$$

The equations of the inverse demand curves can be inverted to obtain the equations of the demand curves when consumption of both varieties is positive:,

$$q_1 = \frac{a - p_1 - \theta(a - p_2)}{b(1 - \theta^2)} \qquad (2.13)$$

$$q_2 = \frac{a - p_2 - \theta(a - p_1)}{b(1 - \theta^2)} \qquad (2.14)$$

Case 2: $q_1 > 0$, $q_2 = 0$.
Then (2.9) implies that the inverse demand curve for variety 1 is

$$p_1 = a - b q_1, \qquad (2.15)$$

with corresponding demand curve

$$q_1 = \frac{a - p_1}{b}. \qquad (2.16)$$

(2.10) implies

$$p_2 \geq a - b\theta q_1. \qquad (2.17)$$

7

Case 3: $q_1 = 0$, $q_2 > 0$.
(2.10) implies

$$p_2 = a - b\theta q_2. \tag{2.18}$$

with corresponding demand curve

$$q_2 = \frac{a - p_2}{b}. \tag{2.19}$$

(2.9) implies that

$$p_1 \geq a - b\theta q_2. \tag{2.20}$$

3. Cost

Let

k_i = firm i's capacity

ρ = long-run cost per unit of capacity

The cost of capacity is fixed; once the firm gets to the second period, its cost function is

$$C(q_i; k_i) = cq_i + \rho k_i, \; q_i \leq k_i. \tag{3.1}$$

The units in which capacity is measured are normalized so that one unit of capacity allows production of one unit of output.

4. Benchmark cases

4.1. Bertrand duopoly

If firms compete in prices, marginal cost is x, and the quantity demanded of both firms is positive, firm 1's profit function is

$$\pi_1 = (p_1 - x)\frac{(1 - \theta)(a - x) - (p_1 - x) + \theta(p_2 - x)}{b(1 - \theta^2)} \tag{4.1}$$

The first-order condition is

$$2(p_1 - x) - \theta(p_2 - x) = (1 - \theta)(a - x) \tag{4.2}$$

and symmetric Bertrand equilibrium prices are

$$p_B(x) = x + \frac{1 - \theta}{2 - \theta}(a - x) \tag{4.3}$$

8

Substitute in the equation of the demand function to obtain Bertrand equilibrium quantities demanded:

$$q_B(x) = \frac{1}{(1+\theta)(2-\theta)}\frac{a-x}{b}. \tag{4.4}$$

The Bertrand equilibrium payoff with marginal cost x is

$$\pi_B(x) = \frac{[p_B(x)-x]^2}{b(1-\theta^2)} = \frac{1}{1+\theta}\frac{1-\theta}{(2-\theta)^2}\frac{(a-x)^2}{b} \tag{4.5}$$

4.2. Cournot duopoly

If firms compete in quantities and marginal cost is x, firm 1's profit function is

$$\pi_1 = [a - x - b(q_1 + \theta q_2)]q_1 \tag{4.6}$$

The first-order condition is

$$2q_1 + \theta q_2 = \frac{a-x}{b}, \tag{4.7}$$

leading to symmetric equilibrium outputs

$$q_C(x) = \frac{1}{2+\theta}\frac{a-x}{b}. \tag{4.8}$$

The corresponding Cournot equilibrium prices are

$$p_C(x) = x + \frac{a-x}{2+\theta} \tag{4.9}$$

Cournot equilibrium profit per firm is

$$\pi_C(x) = \frac{1}{(2+\theta)^2}\frac{(a-x)^2}{b}. \tag{4.10}$$

5. Segments of the price best response function

Firms first choose capacities, then set prices, then produce the quantities demanded at those prices.

Here we consider the nature of firm 1's price best response function, taking capacity as given. There are at most four segments of the price best response

9

function; the actual number of segments may be two, three, or four, depending on the capacity level chosen in the first stage and on the parameters of the model.

Once k_1 has been chosen, firm 1's profit function for

$$q_1 = \frac{(1-\theta)a - p_1 + \theta p_2}{b(1-\theta^2)} = \frac{(1-\theta)a - (p_1 - c) + \theta(p_2 - c)}{b(1-\theta^2)} \leq k_1 \qquad (5.1)$$

is

$$\pi_1 = (p_1 - c)\frac{(1-\theta)a - p_1 + \theta p_2}{b(1-\theta^2)} - \rho k_1. \qquad (5.2)$$

Call the first-order condition to maximize (5.2) when the capacity constraint (5.1) is not binding *branch one* of firm 1's price best response function; this is the Bertrand best response function with marginal cost equal to c, with equation

$$2(p_1 - c) - \theta(p_2 - c) = (1-\theta)(a - c). \qquad (5.3)$$

If the capacity constraint is binding, firm 1's output equals capacity; the equation of the binding capacity constraint may be variously written

$$q_1 = \frac{(1-\theta)a - p_1 + \theta p_2}{b(1-\theta^2)} = k_1, \qquad (5.4)$$

or

$$p_2 = \frac{p_1 - (1-\theta)a + b(1-\theta^2)k_1}{\theta} \qquad (5.5)$$

or

$$p_1 = \theta p_2 + (1-\theta)a - b(1-\theta^2)k_1. \qquad (5.6)$$

Call this *branch two* of firm 1's price best response function

On branch two of its best response function, firm 1's profit function is

$$\pi_1 = (p_1 - c - \rho)k_1. \qquad (5.7)$$

If p_2 rises sufficiently, the quantity demanded of firm 2 goes to zero. At that point, the quantity demanded of firm 1 is less than $\min(k_1, q_{m(c)})$, where $q_{m(c)}$ is the output of a single-variety monopolist with marginal cost c per unit. The nonnegativity constraint $q_2 \geq 0$ becomes binding, and it is the $q_2 = 0$ equation,

$$-\theta p_1 + p_2 = (1-\theta)a \qquad (5.8)$$

10

(from (2.14)) that is the equation of firm 1's price best response function. Call this segment of the price best response function *branch four* of firm 1's best response function.

In the *KSPD* model, there is no *branch three* of the price best response function. Maggi (1996) develops an extension of the Kreps and Scheinkman model in which firms may expand capacity after demand is realized, at a differentially higher unit cost, if it is profitable to do so. In the Maggi model, branch three is the segment of a firm's price best response function when it maximizes profit, expanding capacity beyond the level chosen in stage one. Branch three does not appear in the *KSPD* model.

Finally, if p_2 rises sufficiently, the quantity demanded of firm 1 equals $\min(k_1, q_{m(c)})$, at which point firm 1's price best response function becomes vertical: firm 2's price is so high that firm 1 can sell as close to monopoly output as its capacity level permits, without creating a positive demand for variety 2. Call this segment of the price best response function *branch five* of firm 1's best response function.

6. Price best response functions

Four configurations are possible for firm 1's price best response function, depending on its capacity level and on the parameters of the model.

(b2,b5): for very low capacity levels (as specified in Lemma 1), firm 1 is capacity constrained until p_2 reaches such a high level that the quantity demanded of firm 2 is zero even when firm 1 sells all it can produce, given its capacity; see Figure 6.1.[2]

(b1,b2,b5):For larger capacity levels, but not exceeding a limit specified in Lemma 1, firm 1's price best response function begins with the unconstrained, branch one segment, then moves on to branch two and branch five. See Figure 6.2.

(b1,b2,b4,b5): For still higher values of k_1, but not exceed a level specified in Lemma 1, firm 1's price best response function has all four segments. See Figure 6.3.

(b1,b4,b5): For very high capacity levels, firm one is not capacity constrained, and sets price along its branch one, until p_2 rises so high that firm 1 is able to set the unconstrained monopoly price and sell the unconstrained monopoly output. See Figure 6.4. Firm 1's price best response function is of form $(b1, b4, b5)$.

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[2]Unless otherwise noted, figures are drawn for $a = 12$, $b = c = \rho = 1$, $\theta = 1/2$.
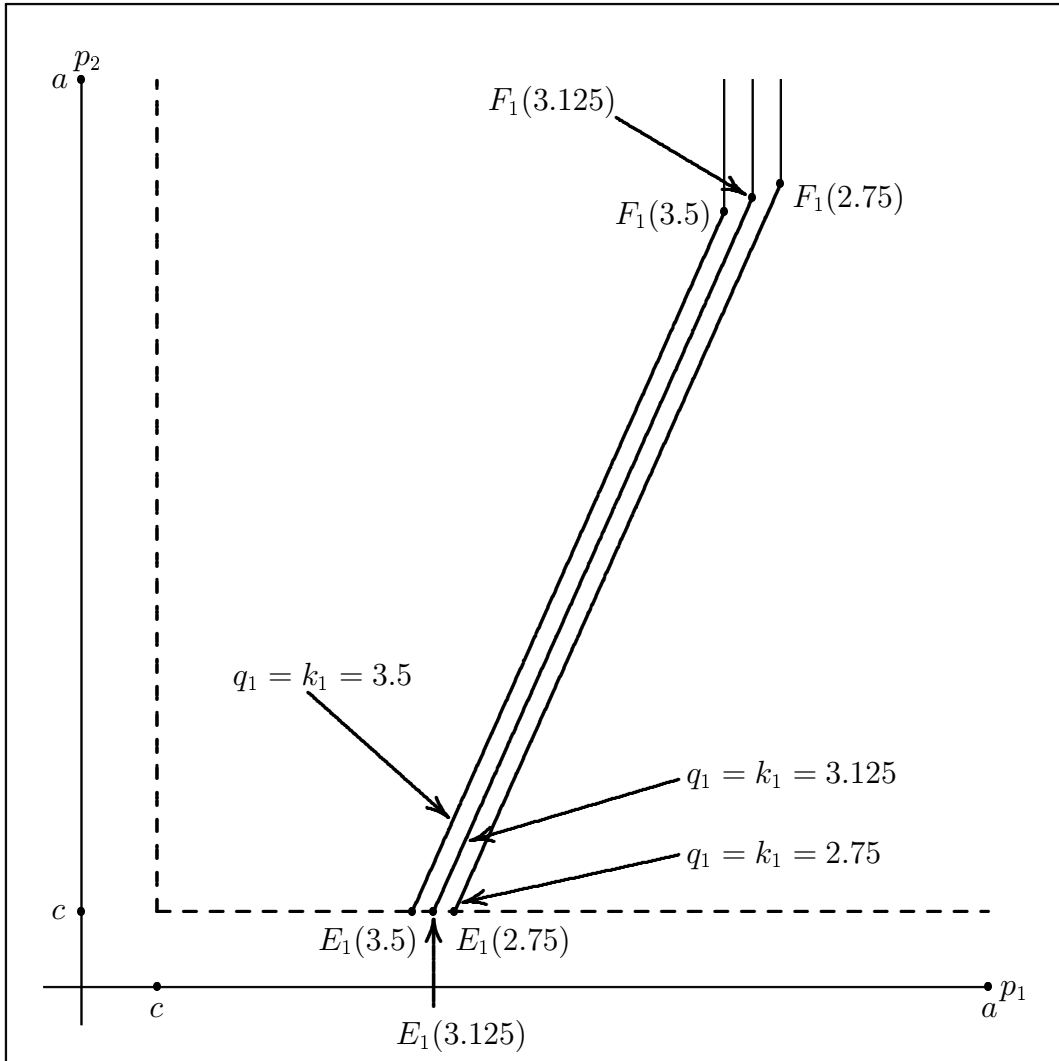
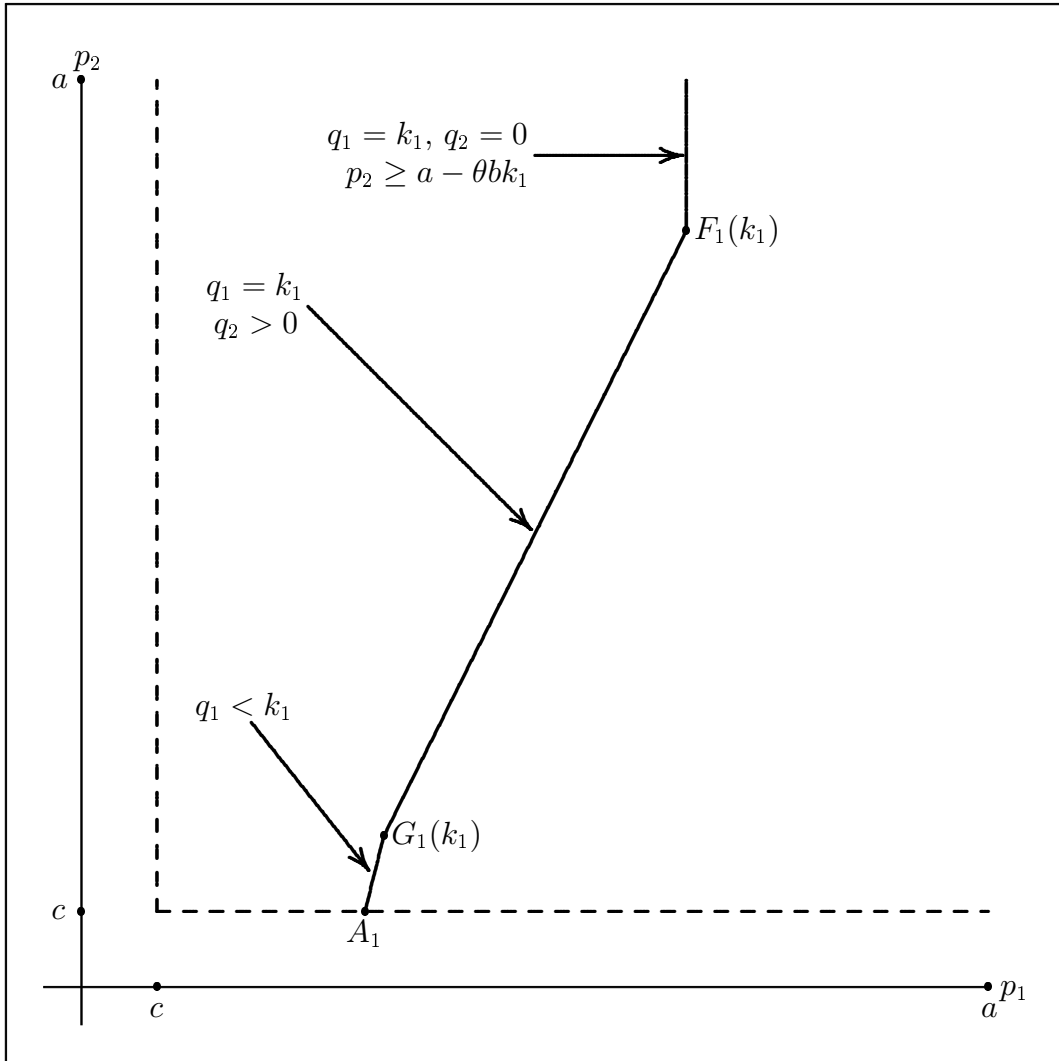Figure 6.1: Firm 1's $(b2, b5)$ price best response function, $k_1 \leq k_A$.

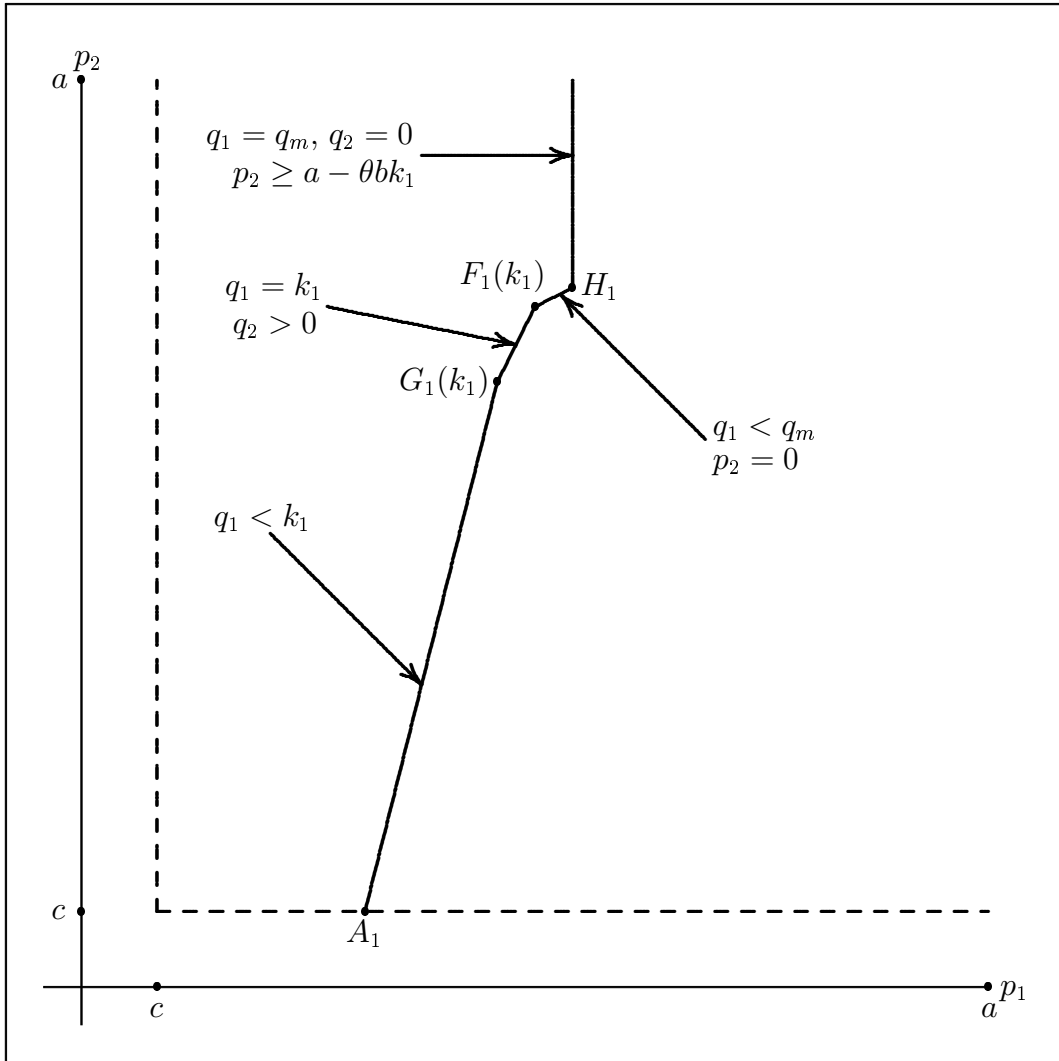Figure 6.2: Firm 1's $(b1, b2, b5)$ price best response function, $k_A \leq k_1 \leq q_m$.

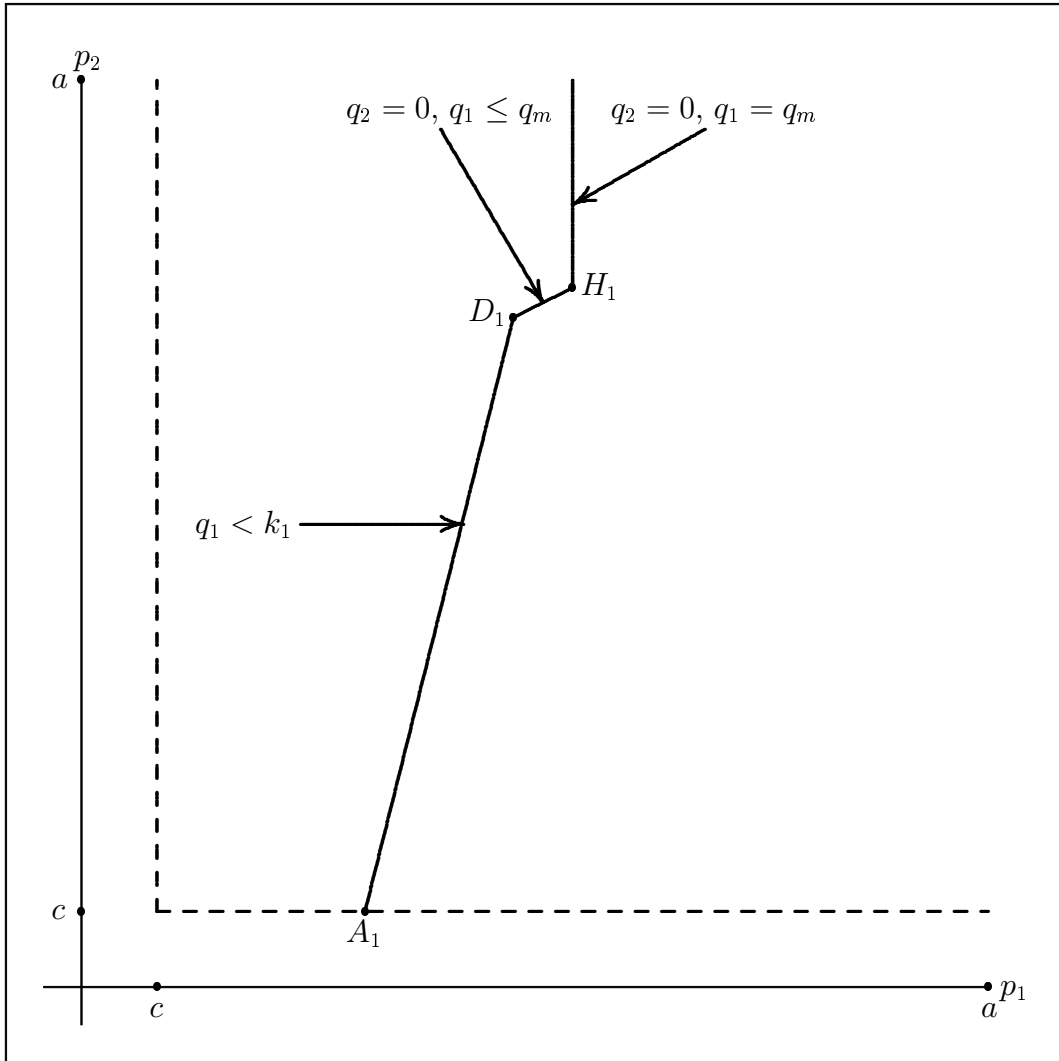Figure 6.3: Firm 1's $(b1, b2, b4, b5)$ price best response function, $q_m \leq k_1 \leq k_D$.

Figure 6.4: Firm 1's $(b1, b4, b5)$ price best response function, $k_1 \geq k_D$

15

7. Results

7.1. Lemma 1

Lemma 1: Let

$$k_A \equiv \frac{1}{1+\theta}\frac{a-c}{2b}, \tag{7.1}$$

$$k_D \equiv \frac{1}{2-\theta^2}\frac{a-c}{b}. \tag{7.2}$$

Then

(a) the relation between first-stage capacity k_i and the configuration of the second-stage price best response function is

$$
\begin{array}{lll}
k_i \leq k_A & \Rightarrow & \text{firm } i\text{'s best response function is of the form (b2,b5)} \\
k_A \leq k_i \leq q_{m(c)} & \Rightarrow & \text{firm } i\text{'s best response function is of the form (b1,b2,b5)} \\
q_{m(c)} \leq k_i \leq k_D & \Rightarrow & \text{firm } i\text{'s best response function is of the form (b1,b2,b4,b5)} \\
k_D \leq k_i \leq \frac{a-c-\rho}{b} & \Rightarrow & \text{firm } i\text{'s best response function is of the form (b1,b4,b5)}
\end{array}.
$$

(b) in the second stage, there are 16 possible combinations of price best response functions of the two firms, one combination for each of the 16 regions shown Figure 7.1.
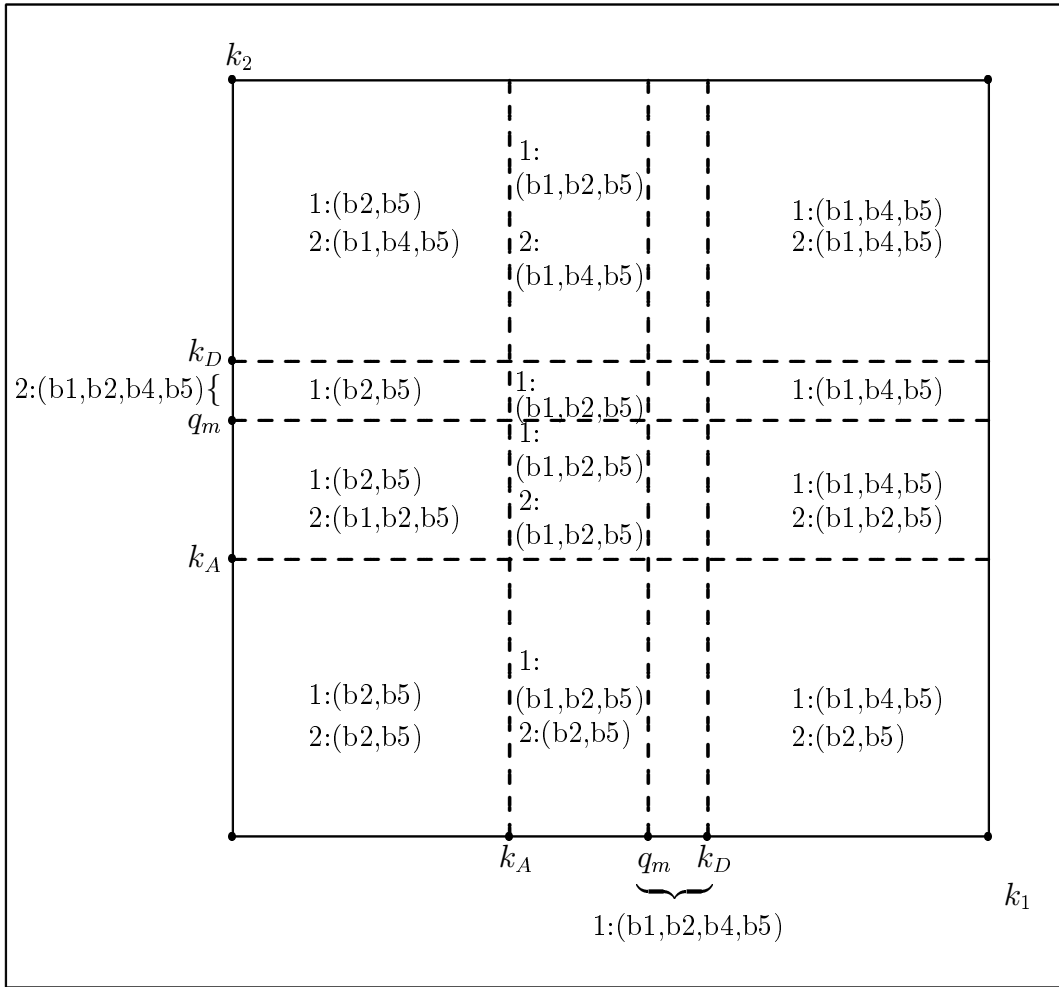
Figure 7.1: Price best response function configurations, capacity space

17

7.2. Lemma 2

Lemma 2: Let

$$k^*_{B(c)} = \frac{1}{(1+\theta)(2-\theta)}\frac{a-c}{b} \tag{7.3}$$

denote the capacity that is just sufficient to allow a firm to produce the Bertrand equilibrium output with marginal cost c and let

$$k^{br}_{B(c)}(k_j) = \frac{1}{2-\theta^2}\left(\frac{a-c}{b} - \theta k_j\right) \tag{7.4}$$

be the capacity level that just allows firm i to produce its Bertrand best-response output when both firms have marginal cost c and firm j produces output level k_j.

There are four second-stage equilibrium types:

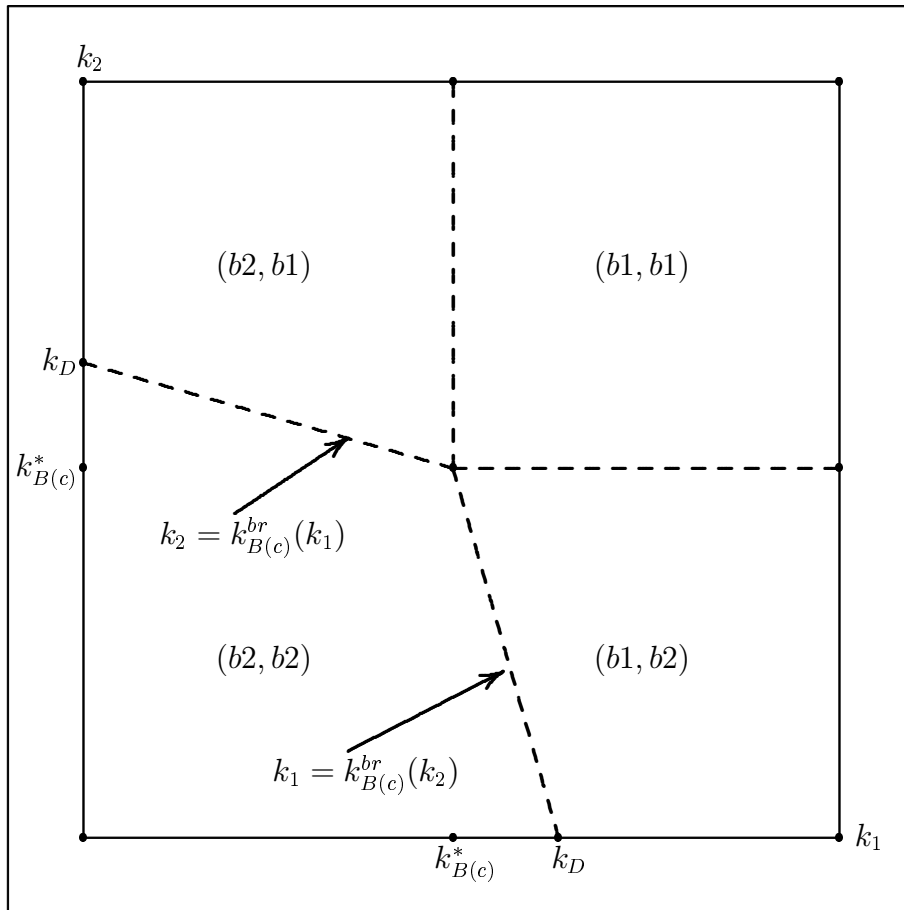|	Region of capacity space	Firm 1	Firm 2
$(b1, b1)$	$k_1 \geq k^*_{B(c)}, \ k_{2)} \geq k^*_{B(c)}$	branch one	branch one
$(b1, b2)$	$k_1 \geq k^{br}_{B(c)}(k_2), \ k_2 \leq k^*_{B(c)}$	branch one	branch two
$(b2, b1)$	$k_1 \leq k^*_{B(c)}, \ k_2 \geq k^{br}_{B(c)}(k_1)$	branch two	branch one
$(b2, b2)$	$k_1 \leq k^{br}_{B(c)}(k_2), \ k_2 \leq k^{br}_{B(c)}(k_1)$	branch two	branch two

These regions are shown in Figure 7.2.

Figure 7.2: Segments of price best response functions that intersect in equilibrium, capacity space; (bi, bj) indicates that in second-stage equilibrium, firm 1's branch i intersects with firm 2's branch j.

7.3. Lemma 3

Lemma 3: Second-stage equilibrium prices, outputs, and payoffs for the four equilibrium types described in *Lemma 2* are

(a) $(b1, b1)$

$$
\begin{array}{cc}
\textit{Firm 1} & \textit{Firm 2} \\
p_{1b1b1} = c + \frac{1-\theta}{2-\theta}(a - c) & p_{2b1b1} = c + \frac{1-\theta}{2-\theta}(a - c) \\
q_{1b1b1} = \frac{1}{(1+\theta)(2-\theta)}\frac{a-c}{b} & q_{2b1b1} = \frac{1}{(1+\theta)(2-\theta)}\frac{a-c}{b} \\
\pi_{1b1b1} = \frac{1-\theta}{(1+\theta)(2-\theta)^2}\frac{(a-c)^2}{b} - \rho k_1 & \pi_{2b1b1} = \frac{1-\theta}{(1+\theta)(2-\theta)^2}\frac{(a-c)^2}{b} - \rho k_2
\end{array}.
$$

(b) $(b1, b2)$

$$
\begin{array}{cc}
\textit{Firm 1} & \textit{Firm 2} \\
p_{1b1b2} = c + \frac{1-\theta^2}{2-\theta^2}\left(a - c - \theta b k_2\right) & p_{2b1b2} = c + \frac{(1-\theta)(2+\theta)(a-c)-2(1-\theta^2)bk_1}{2-\theta^2} \\
q_{1b1b2} = \frac{1}{2-\theta^2}\frac{a-c-\theta b k_2}{b} & q_{2b1b2} = k_2 \\
\pi_{1b1b2} = \frac{1-\theta^2}{(2-\theta^2)^2}\frac{(a-c-\theta b k_2)^2}{b} - \rho k_1 & \pi_{2b1b2} = \left(\frac{(1-\theta)(2+\theta)(a-c)-2(1-\theta^2)bk_2}{2-\theta^2} - \rho\right) k_2
\end{array}.
$$

(c) $(b2, b1)$

$$
\begin{array}{cc}
\textit{Firm 1} & \textit{Firm 2} \\
p_{1b2b1} = c + \frac{(1-\theta)(2+\theta)(a-c)-2(1-\theta^2)bk_1}{2-\theta^2} & p_{2b2b1} = c + \frac{1-\theta^2}{2-\theta^2}\left(a - c - \theta b k_1\right) \\
q_{1b2b1} = k_1 & q_{2b2b1} = \frac{1}{2-\theta^2}\frac{a-c-\theta b k_1}{b} \\
\pi_{1b2b1} = \left(\frac{(1-\theta)(2+\theta)(a-c)-2(1-\theta^2)bk_1}{2-\theta^2} - \rho\right) k_1 & \pi_{2b2b1} = \frac{1-\theta^2}{(2-\theta^2)^2}\frac{(a-c-\theta b k_1)^2}{b} - \rho k_2
\end{array}.
$$

(d) $(b2, b2)$

$$
\begin{array}{cc}
\textit{Firm 1} & \textit{Firm 2} \\
p_{1b2b2} = a - b(k_1 + \theta k_2) & p_{2b2b2} = a - b(\theta k_1 + k_2) \\
q_{1b2b2} = k_1 & q_{2b2b2} = k_2 \\
\pi_{1b2b2} = [a - c - \rho - b(k_1 + \theta k_2)] k_1 & \pi_{2b2b2} = [(a - c - \rho - b(\theta k_1 + k_2)] k_2
\end{array}.
$$

7.4. Lemma 4

Lemma 4: Let

$$k_S = \frac{1}{\theta} \left\{ \frac{a-c-\rho}{b} - 2\sqrt{2\frac{1-\theta^2}{2-\theta^2}\left[\frac{(1-\theta)(2+\theta)(a-c) - (2-\theta^2)\rho}{4(1-\theta^2)b}\right]} \right\}. \quad (7.5)$$

Equilibrium capacity best response functions are

$$
\begin{array}{ll}
\textit{Firm } j\text{'s capacity} & \textit{Firm } i\text{'s best response capacity} \\
0 \le k_j \le k_S & k_i(k_j) = k_{C(c+\rho)}^{br}(k_2) = \frac{1}{2}\left(\frac{a-c-\rho}{b} - \theta k_j\right) \\
k_S \le k_j \le \frac{a-c-\rho}{b} & k_i(k_j) = \frac{(1-\theta)(2+\theta)(a-c)-(2-\theta^2)\rho}{4(1-\theta^2)b}
\end{array} ,
$$

where $k_{C(c+\rho)}^{br}(k_2)$ is the capacity level that just allows firm i to produce its Cournot best-response output if both firms have unit cost $c + \rho$ and firm j is producing output k_j.
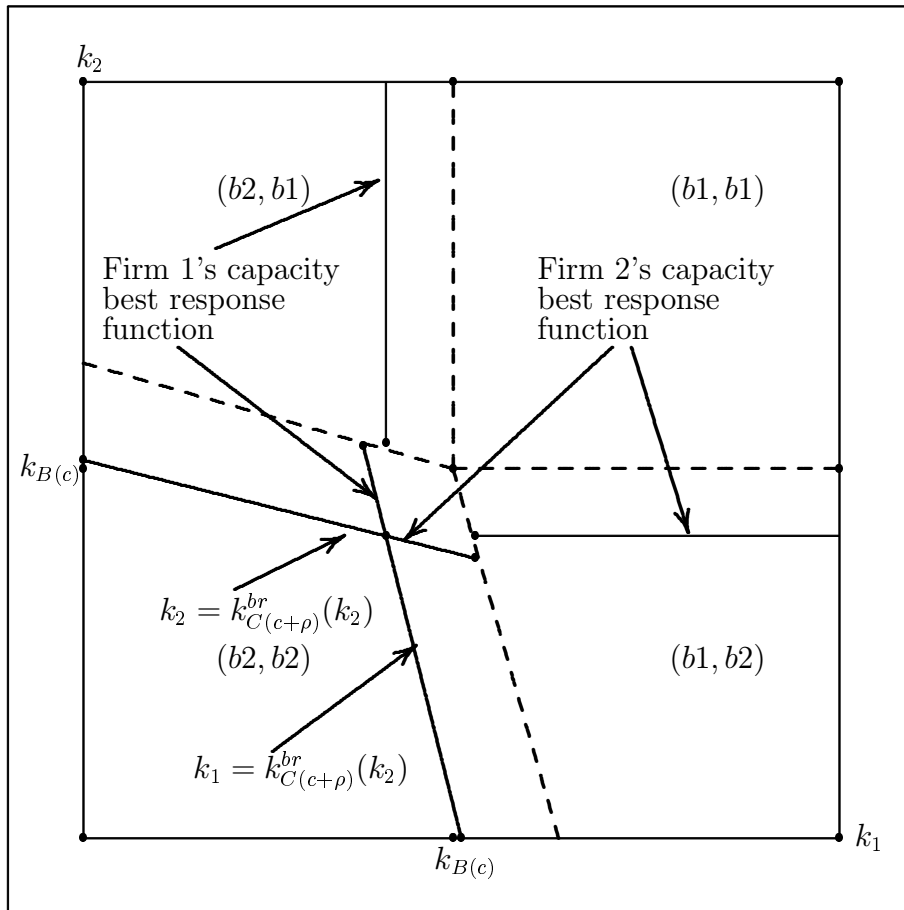
The best response functions are shown in Figure 7.3.

21

Figure 7.3: Capacity best response functions, Kreps & Scheinkman model with product differentiation, $a = 12$, $b = c = \rho = 1$, $\theta = 1/2$.

7.5. Theorem

Let

$$k_{C(c+\rho)} = \frac{1}{2+\theta} \frac{a-c-\rho}{b} \tag{7.6}$$

denote the minimum capacity that permits a firm to produce the Cournot equilibrium output of the one-shot game when both firms have marginal cost $c + \rho$.

Theorem: In the unique noncooperative equilibrium of the Kreps and Scheinkman model with product differentiation, firms select capacities $k_i = k_{C(c+\rho)}$ in the first stage and set the Cournot equilibrium prices in the second stage.

Proof: this follows from the facts that the segment of the capacity best response function that is functionally identical to the Cournot quantity best response function rises above $k_{B(c)}^*$, the capacity level that permits a firm to produce the Bertrand equilibrium output when marginal cost is c,

$$k_S > k_{B(c)}^*$$

and that Bertrand equilibrium output with marginal cost c is greater than Cournot equilibrium output with marginal cost $c + \rho$:

$$k_{B(c)}^* > k_{C(c+\rho)}.$$

The best response functions are shown in Figure 7.3. Figure 7.4 shows the price best response functions for the continuation game when the noncooperative equilibrium capacity levels are chosen in the first stage.
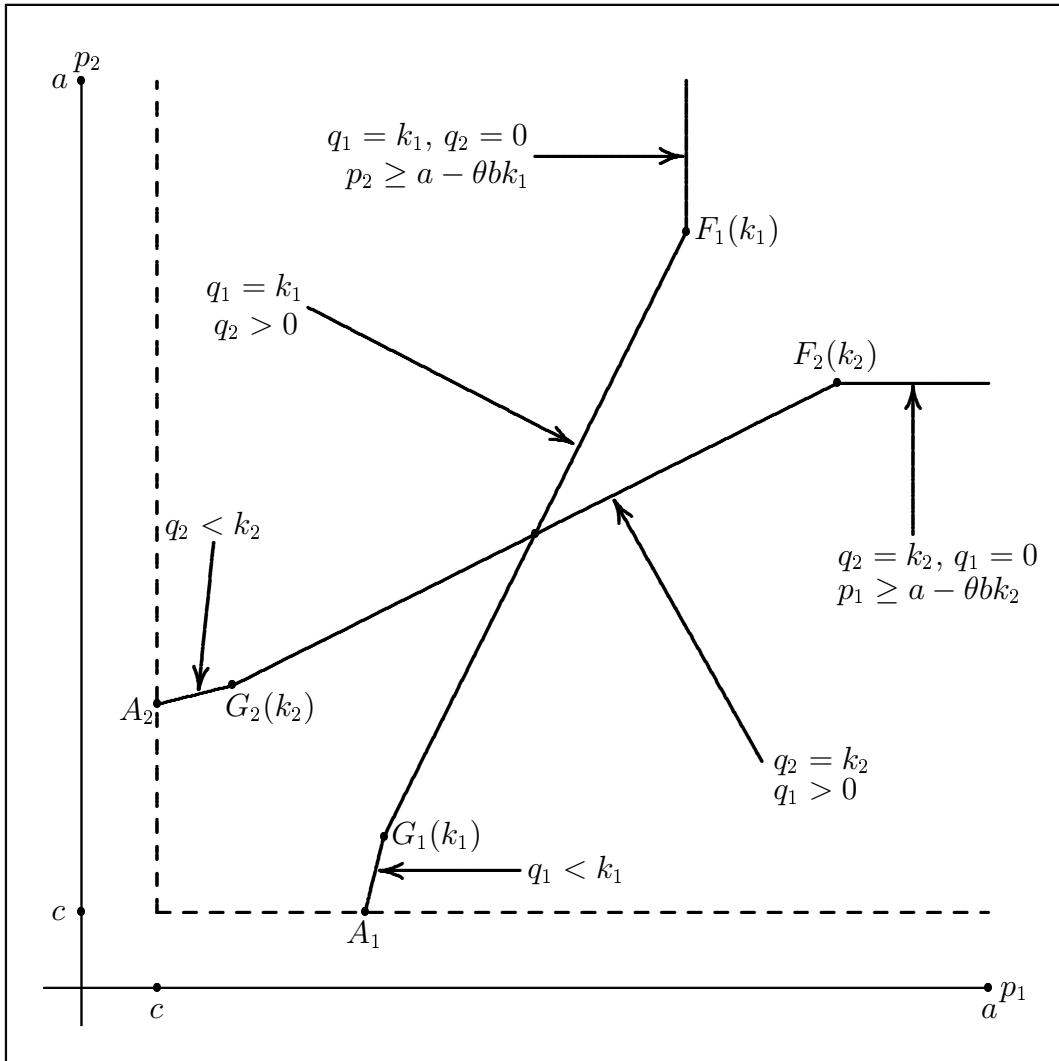
Figure 7.4: Second-stage $(b2, b2)$ equilibrium for equilibrium capacities; both firms $(b1, b2, b5)$ price best response functions.

8. Proof of *Lemma 1*

8.1. (b1,b4,b5)

Rewrite firm 1's branch one profit function, (5.2), in terms of deviations from marginal cost:

$$\pi_1 = (p_1 - c)\frac{(1 - \theta)(a - c) - (p_1 - c) + \theta(p_2 - c)}{b(1 - \theta^2)} - \rho k_1. \tag{8.1}$$

The first-order condition to maximize (8.1) with respect to p_1 is:

$$\frac{\partial \pi_1}{\partial p_1} = \frac{(1 - \theta)(a - c) - 2(p_1 - c) + \theta(p_2 - c)}{b(1 - \theta^2)} = 0. \tag{8.2}$$

Note that (8.2) implies that when the first-order condition holds, firm 1's output is

$$q_1 = \frac{(1 - \theta)(a - c) - (p_1 - c) + \theta(p_2 - c)}{b(1 - \theta^2)} = \frac{p_1 - c}{b(1 - \theta^2)} \tag{8.3}$$

so that along this segment of its best response function, firm 1's payoff is

$$\pi_1 = \frac{(p_1 - c)^2}{b(1 - \theta^2)} - \rho k_1. \tag{8.4}$$

Solving (8.2) for p_1 gives the equation of the branch one ($q_1 < k_1$) segment of firm 1's best response function:

$$p_1 = c + \frac{1}{2}[(1 - \theta)(a - c) + \theta(p_2 - c)]. \tag{8.5}$$

Firm 2 would never charge a price below $p_2 = c$. For $p_2 = c$,

$$p_1^{br}(c) = c + \frac{1}{2}(1 - \theta)(a - c)$$

In the (b1,b4,b5) case, the initial point of firm 1's price best response function is

$$A_1\colon (p_{1A}, p_{2A}) = \left(c + \frac{1}{2}(1 - \theta)(a - c), c\right) \tag{8.6}$$

(Figure 6.4).

Even though firm 1's branch one best response price rises as p_2 rises from $p_2 = 0$, p_1 rises relatively less than p_2, with the result that q_1 rises, and q_2 falls, moving up along firm 1's branch one price best response function. This continues until p_2 reaches such a high level that q_2 falls to zero. As p_2 rises from this point, D_1 in Figure 6.4, firm 1's price rises, and q_1 falls, moving along the $q_2 = 0$ line (firm 1's branch four).

It follows that for firm 1's price best response function to have the (b1,b4,b5) configuration, its capacity k_1 must permit it to produce the quantity demanded of it at point D_1, the intersection of firm 1's branch one and firm 1's branch four.

The system of equations formed by the equation of branch one, (8.5), and the equation of branch four, (5.8), both rewritten in terms of deviations from c, is

$$\begin{pmatrix} 2 & -\theta \\ -\theta & 1 \end{pmatrix} \begin{pmatrix} p_1 - c \\ p_2 - c \end{pmatrix} = (1 - \theta)(a - c) \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \tag{8.7}$$

with solution

$$\begin{pmatrix} p_1 - c \\ p_2 - c \end{pmatrix} = \frac{1 - \theta}{2 - \theta^2} \begin{pmatrix} 1 + \theta \\ 2 + \theta \end{pmatrix} (a - c). \tag{8.8}$$

Firm 1's branch one and branch four intersect at point

$$D_1: (p_{1D}, p_{2D}) = \left(c + \frac{1 - \theta^2}{2 - \theta^2}(a - c), c + \frac{(1 - \theta)(2 + \theta)}{2 - \theta^2}(a - c) \right). \tag{8.9}$$

From (8.3), the quantity demanded of firm 1 at point D_1 is

$$q_{1D} = \frac{p_1 - c}{b(1 - \theta^2)} = \frac{1}{b(1 - \theta^2)} \frac{1 - \theta^2}{2 - \theta^2}(a - c) = \frac{1}{2 - \theta^2} \frac{a - c}{b}. \tag{8.10}$$

Note that $q_{1D} > q_{m(c)}$, the unconstrained monopoly output with marginal cost c per unit

$$\frac{1}{2 - \theta^2} \frac{a - c}{b} - \frac{a - c}{2b} = \frac{\theta^2}{2(2 - \theta^2)} \frac{a - c}{b} \geq 0. \tag{8.11}$$

The condition for firm 1's price best response function to have the (b1,b4,b5) configuration is then

$$k_1 \geq k_D. \tag{8.12}$$

Firm 1's best response price runs along the $q_2 = 0$ line, (5.8), until p_1 reaches the unconstrained monopoly price, which occurs for

$$p_1 = c + \frac{(p_2 - c) - (1 - \theta)(a - c)}{\theta} = c + \frac{1}{2}(a - c)$$

26

$$p_2 = c + \frac{1}{2}(2 - \theta)(a - c) \equiv p_{2H}.$$

The second segment of firm 1's Bertrand best response function is the straight line connecting point D_1: (p_{1D}, p_{2D}) and point

$$H_1: (p_{1H}, p_{2H}) = \left(c + \frac{1}{2}(a - c), c + \frac{1}{2}(2 - \theta)(a - c) \right). \tag{8.13}$$

For higher values of p_2, firm 1 charges the unconstrained monopoly price and sells the unconstrained monopoly quantity.

If (8.12) is met, firm 1's best response function has three segments, branch one from point A_1 to point D_1, branch four from point D_1 to point H_1, and vertical at $p_1 = p_{1H}$ thereafter.

8.2. (b2,b5)

At point A_1, $p_2 = c$; the quantity demanded of firm 1 at point A_1 along firm 1's branch one $(q_1 < k_1)$ is

$$q_{1A} = \frac{p_{1A} - c}{b(1 - \theta^2)} = \frac{1}{1 + \theta} \frac{a - c}{2b}. \tag{8.14}$$

q_{1A} is less than the unconstrained monopoly output of a single variety:

$$\frac{a - c}{2b} - \frac{1}{1 + \theta} \frac{a - c}{2b} = \frac{\theta}{1 + \theta} \frac{a - c}{2b} > 0.$$

Let

$$k_A = \frac{1}{1 + \theta} \frac{a - c}{2b} \tag{8.15}$$

be the capacity that is just sufficient to allow the firm to produce q_{1A}.

If

$$k_1 \leq k_A, \tag{8.16}$$

firm 1 is on the capacity constrained (branch two) segment of its price best response function until p_2 rises so much that q_2 falls to zero.

If $p_2 = c$ on firm 1's branch two, then from (5.6) firm 1's price is

$$p_1 = \theta c + (1 - \theta)a - b(1 - \theta^2)k_1 =$$

$$c + (1 - \theta)(a - c) - b(1 - \theta^2)k_1 \equiv p_{1E}. \tag{8.17}$$

27

If $k_1 \leq k_A$, firm 1's best response function begins at point

$$E_1(k_1) : (p_{1E}, p_{2E}) = \left(c + (1-\theta)a - b(1-\theta^2)k_1, c\right) \qquad (8.18)$$

(Figure 6.1).

At what point do firm 1's branch two and branch four intersect? Solve the system of equations formed by (5.8) ($q_2 = 0$) and (5.6) ($q_1 = k_1$)

$$\begin{pmatrix} 1 & -\theta \\ -\theta & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = (1-\theta)a \begin{pmatrix} 1 \\ 1 \end{pmatrix} - b(1-\theta^2)k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (8.19)$$

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \end{pmatrix} - bk_1 \begin{pmatrix} 1 \\ \theta \end{pmatrix}. \qquad (8.20)$$

The $q_2 = 0$ line and the $q_1 = k_1$ line intersect at point

$$F_1(k_1) : (p_{1F}, p_{2F}) = (a - bk_1, a - \theta bk_1) \qquad (8.21)$$

(Figure 6.1).

Firm 1's branch two and branch four segments intersect at point $F_1(k_1)$, with coordinates given by (8.21).

If $q_2 = 0$, we obtain the same results if firm 1's constrained optimization problem is formulated with price or with quantity as firm 1's decision variable. We proceed in terms of quantity.

For $p_2 \geq p_{2F}$ and $k_1 \leq k_A$, firm 1 maximizes

$$(a - c - bq_1)q_1$$

subject to the capacity constraints

$$k_1 \geq q_1$$

and subject to the Kuhn-Tucker inequality for $q_2 = 0$ for the representative consumer constrained optimization problem,

$$p_2 \geq a - \theta bq_1.$$

A Lagrangian for firm 1's constrained optimization problem is

$$\mathcal{L}_1 = (a - c - bq_1)q_1 + \lambda_1(k_1 - q_1) + \lambda_2(p_2 - a + \theta bq_1).$$

28

The Kuhn-Tucker conditions are

$$a - c - 2bq_1 - \lambda_1 + \theta b \lambda_2 \leq 0 \quad q_1[a - c - 2bq_1 - \lambda_1 + \theta b \lambda_2] = 0 \quad q_1 \geq 0$$

$$k_1 - q_1 \geq 0 \quad \lambda_1(k_1 - q_1) = 0 \quad \lambda_1 \geq 0$$

$$p_2 - a + \theta b q_1 \geq 0 \quad \lambda_2(p_2 - a + \theta b q_1) = 0 \quad \lambda_2 \geq 0.$$

Suppose $q_1 = k_1 > 0$. Then

$$a - c - 2bq_1 - \lambda_1 + \theta b \lambda_2 = 0.$$

For this to be a solution, we must also have

$$p_2 \geq a - \theta b k_1 = p_{2F},$$

which condition is met. For $p_2 > a - \theta b k_1$, $\lambda_2 = 0$; then

$$\lambda_1 = 2b\left(\frac{a - c}{2b} - k_1\right) > 0.$$

When $k_1 \leq k_A$ and $p_2 \geq p_{2F}$, firm 1's best response is to set price $p_{1F} = a - bk_1$ and sell at capacity.

When capacity satisfies (8.16), the price best response function has a branch two segment connecting point $E_1(k_1)$ to point $F_1(k_1)$ and is vertical thereafter.

Figure 6.1 shows firm 1's $(b2, b5)$ price best response functions for three alternative levels of k_1. As k_1 falls, firm 1's branch two segment shifts right, and the point at which firm 1 shifts from its branch two to its branch four moves up the $q_2 = 0$ line.

8.3. (b1,b2,b5) and (b1,b2,b4,b5)

If $k_A \leq k_1 \leq k_D$, firm 1's best response function begins on branch one, but its capacity is not sufficient to allow it to produce along branch one until it reaches branch four. If

$$k_A \leq k_1 \leq k_D, \tag{8.22}$$

then when p_2 rises sufficiently from $p_2 = c$, firm one moves from branch one to branch two.

What is the point of intersection of branch one and branch two? The equations of branch one and branch two are

$$2(p_1 - c) - \theta(p_2 - c) = (1 - \theta)(a - c)$$

and

$$p_1 - \theta p_2 = (1 - \theta)a - b(1 - \theta^2)k_1$$

respectively.

Rewritten in terms of deviations from c, the system of equations formed by the equations of firm 1's branch one and branch two is

$$\begin{pmatrix} 2 & -\theta \\ 1 & -\theta \end{pmatrix} \begin{pmatrix} p_1 - c \\ p_2 - c \end{pmatrix} = (1 - \theta)(a - c) \begin{pmatrix} 1 \\ 1 \end{pmatrix} - b(1 - \theta^2) \begin{pmatrix} 0 \\ k_1 \end{pmatrix}, \qquad (8.23)$$

with solution

$$\begin{pmatrix} p_1 - c \\ p_2 - c \end{pmatrix} = b(1 - \theta^2)k_1 \begin{pmatrix} 1 \\ 2/\theta \end{pmatrix} - (1 - \theta)(a - c) \begin{pmatrix} 0 \\ 1/\theta \end{pmatrix}. \qquad (8.24)$$

The point of intersection of branch one and branch two is

$$G_1(k_1): \ (p_{1G}, p_{2G}) =$$

$$\left(c + b(1 - \theta^2)k_1, c + \frac{2b(1 - \theta^2)k_1 - (1 - \theta)(a - c)}{\theta} \right) \qquad (8.25)$$

(Figures 6.2 and 6.3).

By (8.11), $k_D > q_{m(c)}$. On the other hand, k_A is less than $q_{m(c)}$:

$$q_{m(c)} - k_A = \frac{a - c}{2b} - \frac{1}{1 + \theta}\frac{a - c}{2b} =$$

$$\left(1 - \frac{1}{1 + \theta} \right)\frac{a - c}{2b} = \frac{\theta}{1 + \theta}\frac{a - c}{2b} > 0.$$

For $k_A \leq k_1 \leq q_{m(c)}$, firm 1's best response function has three segments, branch one from point A_1 to point $G_1(k_1)$, branch two from point $G_1(k_1)$ to point $F_1(k_1)$, and vertical (branch five) thereafter; see Figure 6.2.

For $q_{m(c)} \leq k_1 \leq k_D$, firm 1's best response function has four segments, branch one from point A_1 to point $G_1(k_1)$, branch two from point $G_1(k_1)$ to point $F_1(k_1)$, branch four from point $F_1(k_1)$ to point H_1, and vertical (branch five) thereafter; see Figure 6.3.

8.4. Summary

The results obtained above are summarized in Table 8.1.

$0 \leq k_1 \leq k_A$	(b2,b5)	$E_1(k_1) - F_1(k_1)-$ vertical
$k_A \leq k_1 \leq q_{m(c)}$	(b1,b2,b5)	$A_1 - G_1(k_1) - F_1(k_1)-$ vertical
$q_{m(c)} \leq k_1 \leq k_D$	(b1,b2,b4,b5)	$A_1 - G_1(k_1) - F_1(k_1)_1 - H_1-$ vertical
$k_D \leq k_1$	(b1,b4,b5)	$A_1 - D_1 - H_1-$ vertical

Table 8.1: Capacity and configuration of price best response function

9. Proof of *Lemma 2*

9.1. Cell (4,4)

Cell (4,4) — row 4, column 4 in Figure 7.1 — is defined by the inequalities

$$k_D \leq k_1, \ k_D \leq k_2.$$

In cell (4,4) both firms have best response functions with configuration (b1,b4,b5).

Figure 9.1 shows an equilibrium with both firms on branch one of their price best response functions when both

When both firms have (b1,b4,b5) best response functions, equilibrium occurs at the intersection of the branch one segments if point D_1 lies above firm 2's branch one and point D_2 lies to the right of firm 1's branch one.

From (8.9), the coordinates of point D_1 are

$$(p_{1D}, p_{2D}) = \left(c + \frac{1 - \theta^2}{2 - \theta^2}(a - c), c + \frac{(1 - \theta)(2 + \theta)}{2 - \theta^2}(a - c) \right).$$

The equation of the branch one segment of firm 2's price best response function is

$$-\theta(p_1 - c) + 2(p_2 - c) = (1 - \theta)(a - c). \tag{9.1}$$

The condition for point D_1 to lie above firm 2's branch one is

$$-\theta(p_{1D} - c) + 2(p_{2D} - c) \geq (1 - \theta)(a - c). \tag{9.2}$$

Substituting the coordinates of point D_1, the condition is met if

$$-\theta \left[\frac{1 - \theta^2}{2 - \theta^2}(a - c) \right] + 2 \left[\frac{(1 - \theta)(2 + \theta)}{2 - \theta^2}(a - c) \right] \geq (1 - \theta)(a - c)$$
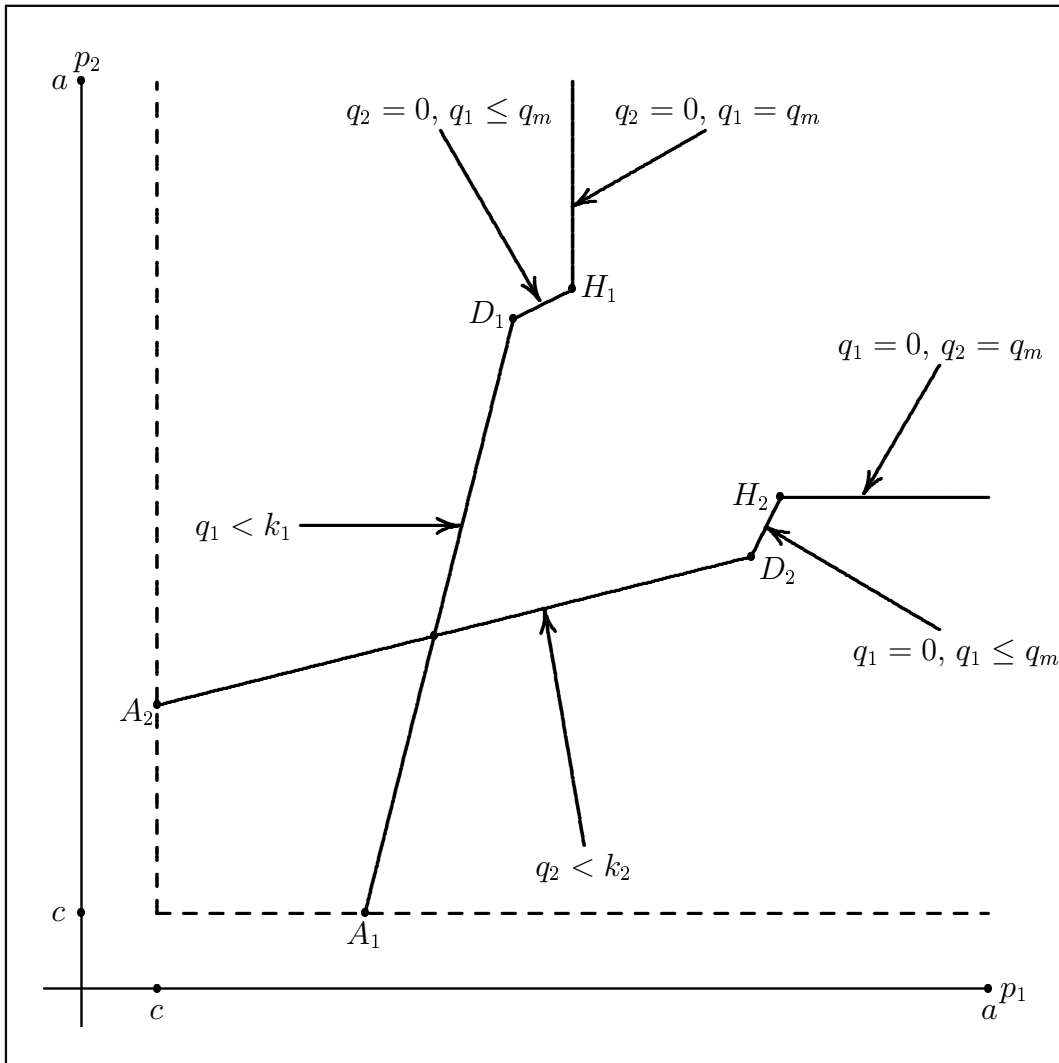
31

Figure 9.1: $(b1, b1)$ second-stage equilibrium, both firms $(b1, b4, b5)$ price best response functions

$$-\theta\frac{1+\theta}{2-\theta^2} + 2\frac{2+\theta}{2-\theta^2} \geq 1$$

$$\frac{4+\theta-\theta^2}{2-\theta^2} \geq 1$$

$$4+\theta-\theta^2 \geq 2-\theta^2$$

$$2+\theta \geq 0,$$

which is always the case. In the same way, point D_2 is always to the right of firm one's branch two in cell (4,4). When both firms have (b1,b4,b5) price best response functions, equilibrium always occurs at the intersection of the branch one segments.

9.2. Cell (3,4)

Cell (3,4) — row 4, column 3 of Figure 7.1 — is defined by the inequalities

$$q_m \leq k_1 \leq k_D, \ k_D \leq k_2.$$

In this region of capacity space, firm 1's price best response function is of form (b1,b2,b4,b5). Firm 2's price best response function is of form (b1,b4,b5).

This configuration is shown in Figure 9.2. One condition for (b1,b1) equilibrium in cell (4,3) is that point D_2 be to the right of firm 1's branch 1; by the argument of Section 9.1, this condition is always met. The second condition is that point G_1 be above firm two's branch one.

The coordinates of point $G_1(k_1)$ are

$$(p_{1G}, p_{2G}) = \left(c + b(1-\theta^2)k_1, c + \frac{2b(1-\theta^2)k_1 - (1-\theta)(a-c)}{\theta}\right)$$

From (9.1), the condition for point G_1 to lie above firm 2's branch one is

$$-\theta(p_{1G} - c) + 2(p_{2G} - c) \geq (1-\theta)(a-c); \tag{9.3}$$

or

$$-\theta(1-\theta^2)bk_1 + 2\frac{2(1-\theta^2)bk_1 - (1-\theta)(a-c)}{\theta} \geq (1-\theta)(a-c)$$

or

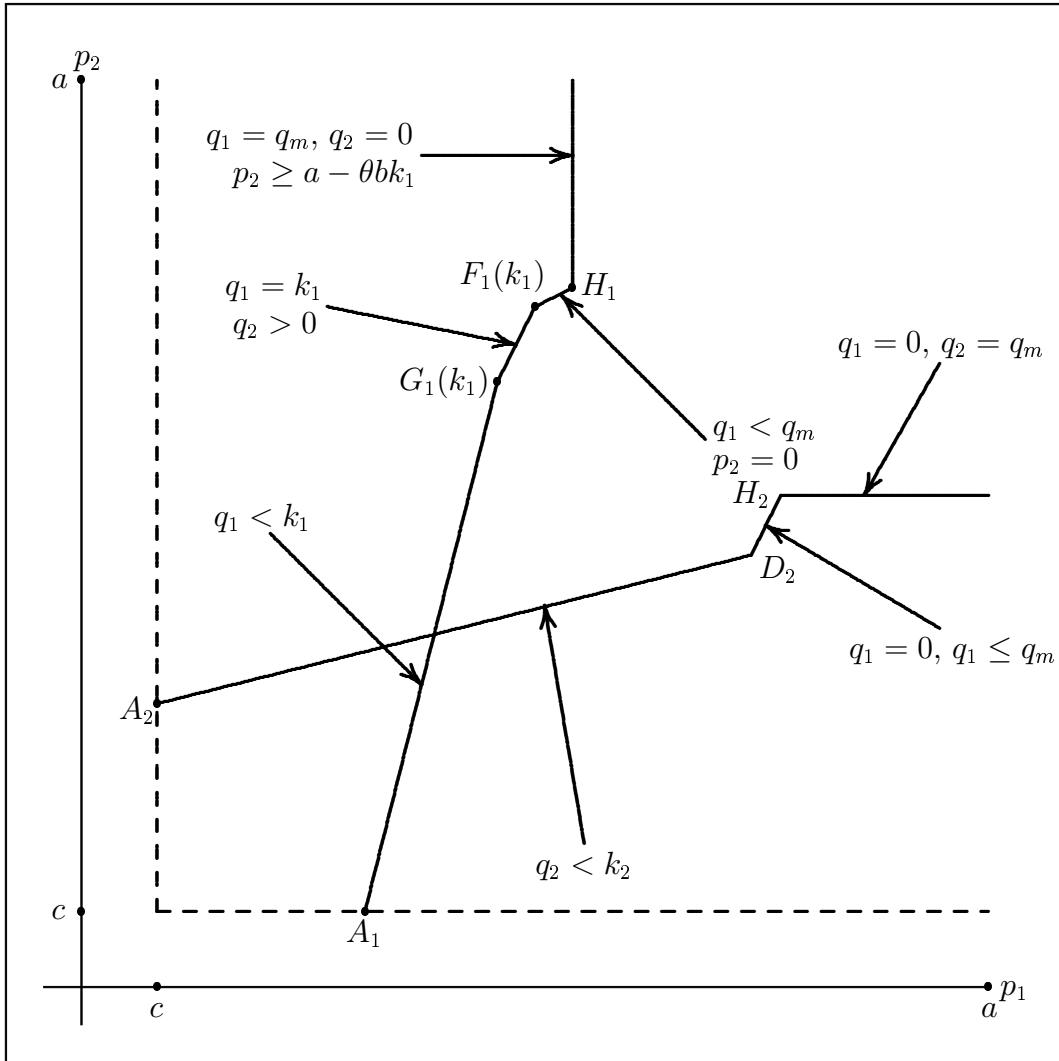$$k_1 \geq \frac{1}{(1+\theta)(2-\theta)}\frac{a-c}{b} \equiv k_{B(c)}^*, \tag{9.4}$$

Figure 9.2: $(b1, b1)$ equilibrium, cell (4,3): $q_m \leq k_1 \leq k_D$, $k_D \leq k_2$

34

where, from (4.4), the capacity level $k^*_{B(c)}$ is the capacity level that is just sufficient to allow the firm to produce Bertrand equilibrium output when both firms have marginal cost c.

The range of k_1 in cell (3,4) is $q_m \leq k_1 \leq k_D$. Since

$$q_m - k^*_{B(c)} = \frac{a-c}{2b} - \frac{1}{(1+\theta)(2-\theta)}\frac{a-c}{b}$$

$$\left(\frac{1}{2} - \frac{1}{(1+\theta)(2-\theta)}\right)\frac{a-c}{b} =$$

$$= \frac{\theta(1-\theta)}{2(1+\theta)(2-\theta)}\frac{a-c}{b} > 0,$$

condition (9.4) is always met in cell (3,4).

By similar arguments, in cell (4,3), equilibrium is of type (b1,b1) for $k_2 \geq k^*_{B(c)}$ and the condition is always met.

9.3. Cell (2,4)

Cell (2,4) — row 4, column 2 of Figure 7.1 — is defined by the inequalities

$$k_A \leq k_1 \leq q_m, \ k_D \leq k_2.$$

In this region of capacity space, firm 1's price best response function is of form (b1,b2,b5). Firm 2's price best response function is of form (b1,b4,b5).

From the discussion of cell (3,4), $q_m > k^*_{B(c)}$. We also have

$$k^*_{B(c)} - k_A =$$

$$\frac{1}{(1+\theta)(2-\theta)}\frac{a-c}{b} - \frac{1}{1+\theta}\frac{a-c}{2b} =$$

$$\frac{1}{1+\theta}\left(\frac{1}{2-\theta} - \frac{1}{2}\right)\frac{a-c}{b} =$$

$$\frac{\theta}{(1+\theta)(2-\theta)}\frac{a-c}{2b} > 0.$$

Hence

$$k_A \leq k^*_{B(c)} \leq q_m$$

and the cases , $k_1 \geq k^*_{B(c)}$, $k_1 \geq k^*_{B(c)}$ can both occur in cell (2,4).
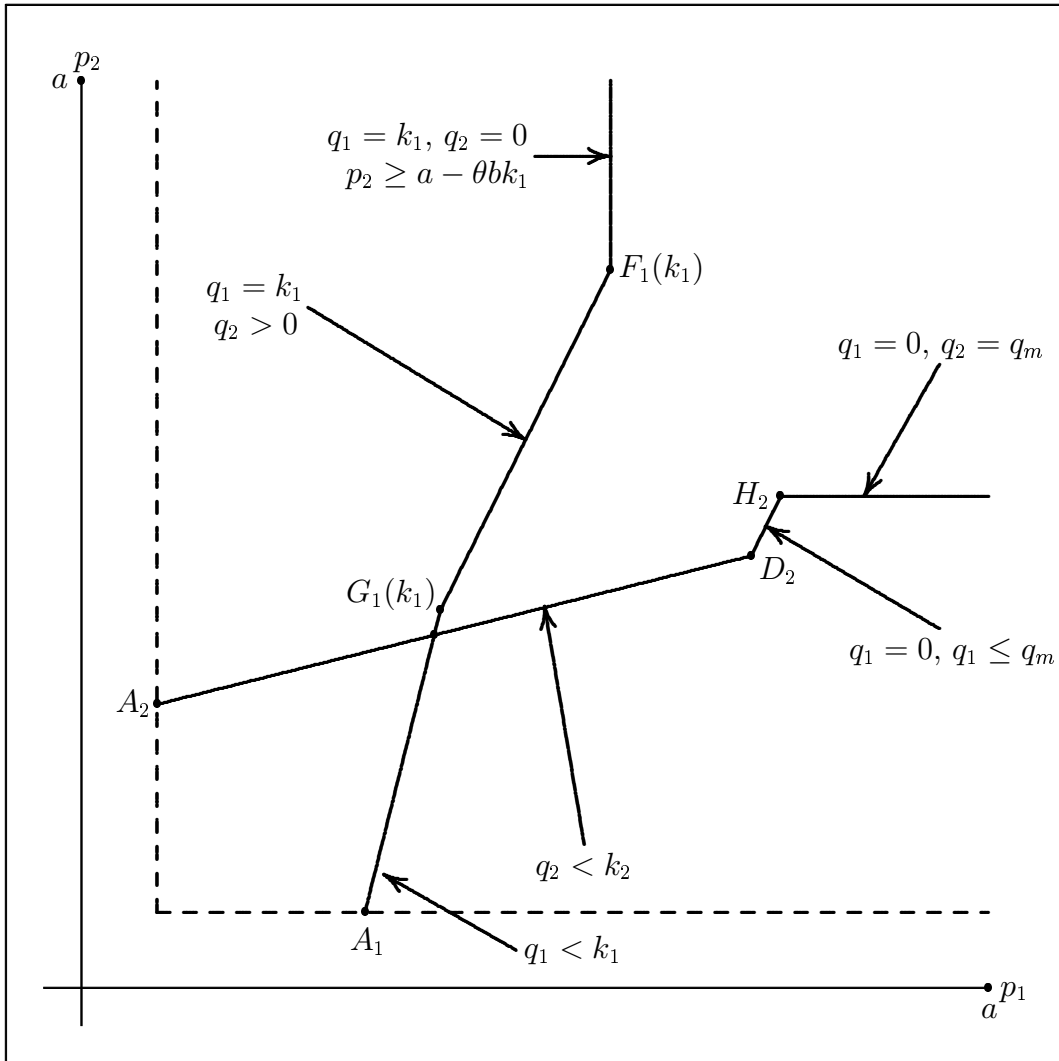
35

Figure 9.3: (b1,b1) equilibrium, firm 1 (b1,b2,b5) price best response function, firm 2 (b1,b4,b5) price best response function

By the argument of Section 9.2, for $k_1 \geq k_{B(c)}^*$ second-stage equilibrium is of type (b1,b1), as shown in Figure 9.3.

For $k_1 \leq k_{B(c)}^*$, second-stage equilibrium is of type (b2,b1) as shown in Figure 9.4.

In the same way, second-stage equilibrium in cell (4,2) is of type (b1,b1) for $k_2 \geq k_{B(c)}^*$, and second-stage equilibrium is of type (b1,b2) in cell (4,2) for $k_2 \leq k_{B(c)}^*$.

9.4. Cell (1,4)

Cell (1,4) — row 4, column 1 of Figure 7.1 — is defined by the inequalities

$$0 \leq k_1 \leq k_A, \ k_D \leq k_2.$$

In this region of capacity space, firm 1's price best response function is of form (b2,b5). Firm 2's price best response function is of form (b1,b4,b5).

The conditions for (b2,b1) second-stage equilibrium in cell (1,4), as shown in Figure 9.5, are that point F_1 be above firm 2's branch one and that point D_2 be to the right of firm 1's branch 2.

The coordinates of point F_1 are

$$(p_{1F}, p_{2F}) = (a - bk_1, a - \theta b k_1),$$

and the condition for point F_1 to be above firm 2's branch one is

$$-\theta(p_{1F} - c) + 2(p_{2F} - c) \geq (1 - \theta)(a - c)$$

$$-\theta(a - c - bk_1) + 2(a - c - \theta b k_1) \geq (1 - \theta)(a - c)$$

$$-\theta(a - c) + \theta b k_1 + 2(a - c) - 2\theta b k_1 \geq (1 - \theta)(a - c)$$

$$-(1 - \theta)(a - c) - \theta(a - c) + 2(a - c) \geq \theta b k_1$$

$$k_1 \leq \frac{a - c}{\theta b}. \tag{9.5}$$

$(a-c)/b$ is the long-run equilibrium output of a perfectly competitive industry; (9.5) will be met for all k_1 of interest.

The coordinates of point D_2 are

$$(p_{1D}, p_{2D}) = \left(c + \frac{(1 - \theta)(2 + \theta)}{2 - \theta^2}(a - c), c + \frac{1 - \theta^2}{2 - \theta^2}(a - c) \right).$$
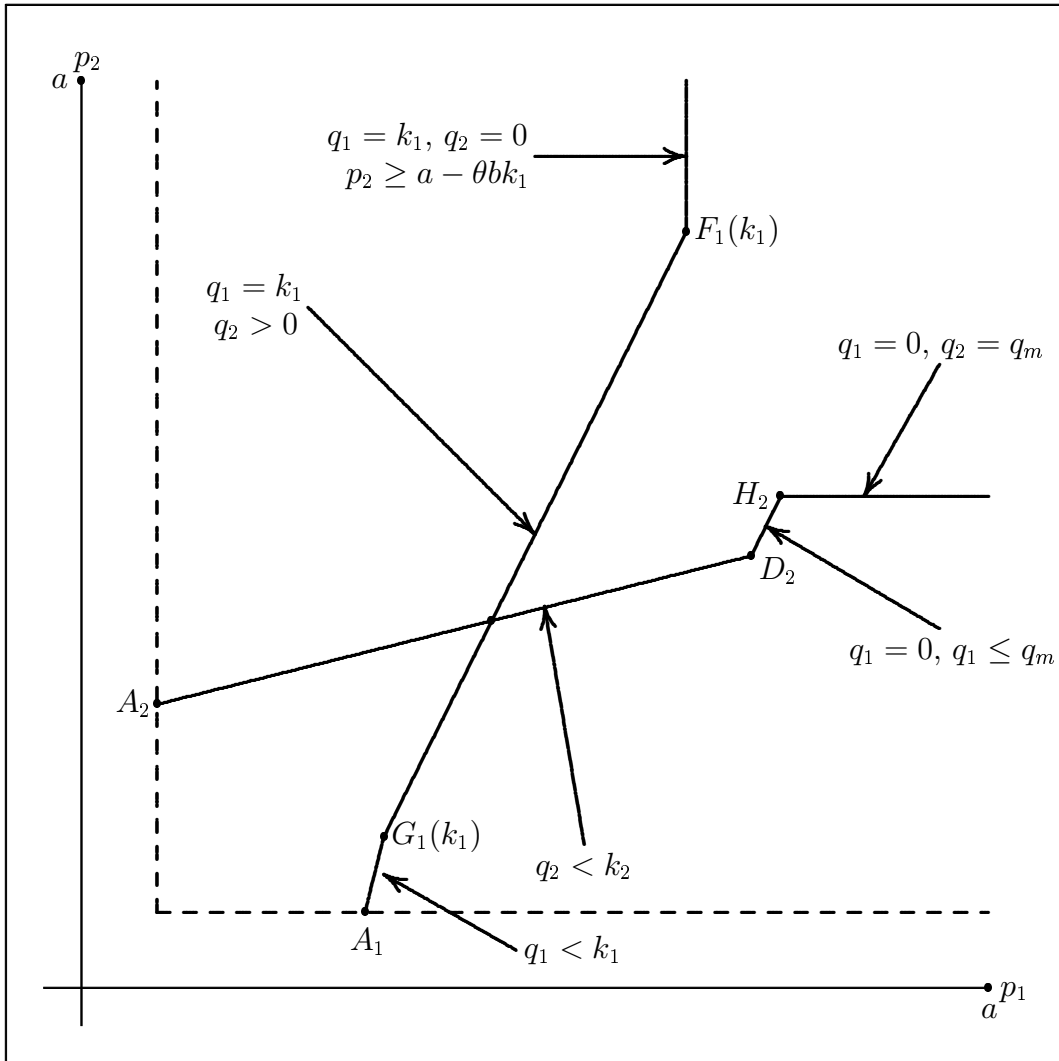
37

Figure 9.4: (b2,b1) equilibrium, firm 1 (b1,b2,b5) price best response function, firm 2 (b1,b4,b5) price best response function
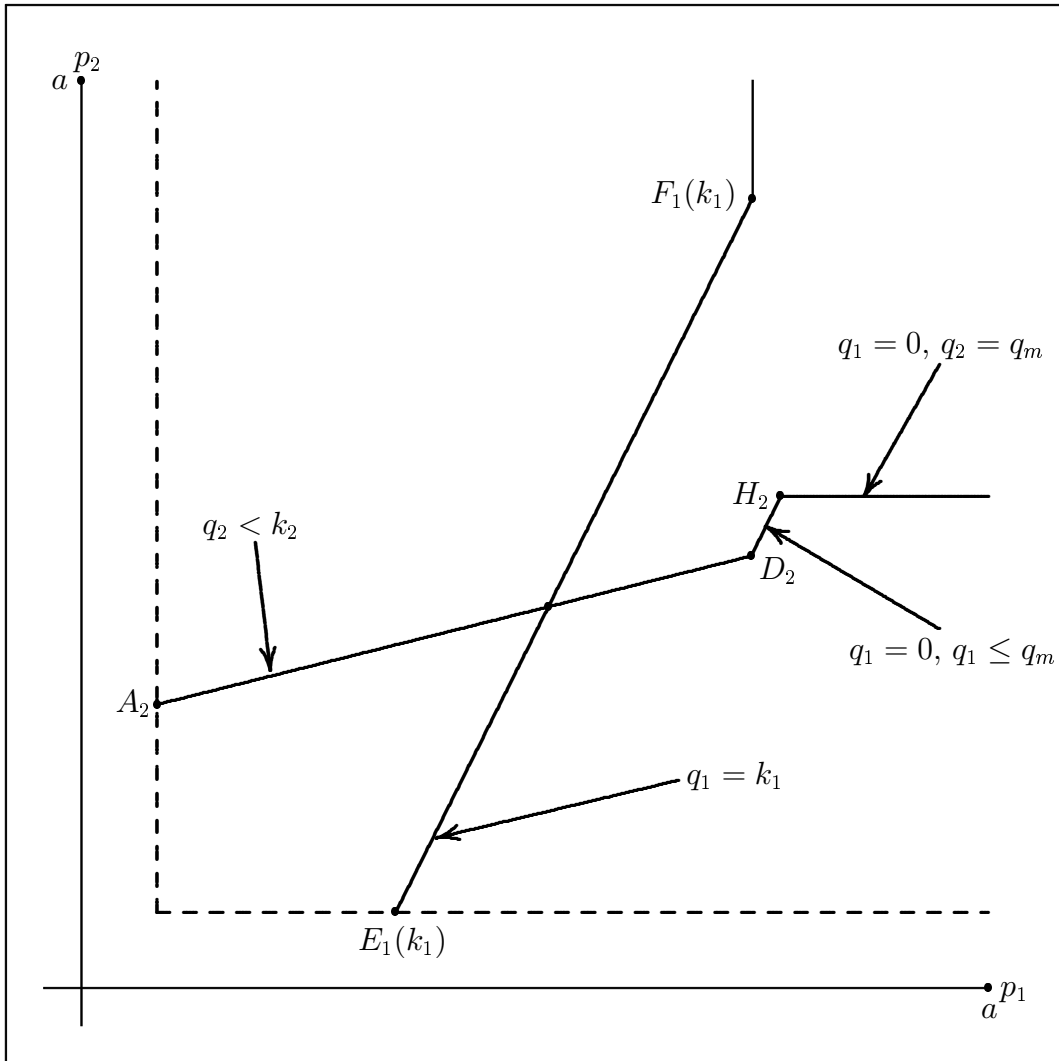
Figure 9.5: Firm 1's price best response function, $k_1 \leq k_A$

The condition for point D_2 to be to the right of firm 1's branch two is

$$-\theta(p_{2D} - c) + p_{1D} - c \geq +(1 - \theta)(a - c) - b(1 - \theta^2)k_1$$

$$-\theta\frac{1 - \theta^2}{2 - \theta^2}(a - c) + \frac{(1 - \theta)(2 + \theta)}{2 - \theta^2}(a - c) \geq (1 - \theta)(a - c) - b(1 - \theta^2)k_1$$

$$b(1 - \theta^2)k_1 \geq (1 - \theta)\left[1 + \theta\frac{1 + \theta}{2 - \theta^2} - \frac{2 + \theta}{2 - \theta^2}\right](a - c)$$

$$b(1 + \theta)k_1 \geq 0$$

and this condition is always met.

In cell (1,4), second-stage equilibrium is of type (b2,b1). In cell (4,1), second-stage equilibrium is of type (b1,b2).

9.5. Cell (3,3)

Cell (3,3) — row 3, column 3 of Figure 7.1 — is defined by the inequalities $q_{m(c)} \leq k_1 \leq k_D$, $q_{m(c)} \leq k_2 \leq k_D$.

Figure 9.6 shows a second-stage equilibrium with both firms on branch one of their (b1,b2,b4,b5) best response functions, producing less than capacity. This combination of price best response functions occurs in the (3,3) cell of Figure 7.1.

The conditions for second-stage equilibrium to have this configuration are that point $G_1(k_1)$ be above firm 2's branch one and point $G_2(k_2)$ be to the right of firm 1's branch one. From Section 9.2, the conditions for this are $k_1 \geq k_{B(c)}^*$, $k_2 \geq k_{B(c)}^*$.

Since

$$q_{m(c)} - k_{B(c)}^* = \frac{a - c}{2b} - \frac{1}{(1 + \theta)(2 - \theta)}\frac{a - c}{b} =$$

$$\frac{\theta(1 - \theta)}{2(1 + \theta)(2 - \theta)}\frac{a - c}{b} \geq 0,$$

this condition is always satisfied, and in cell (3,3) the second-stage equilibrium is always of type is (b2,b2).

9.6. Cell (2,3)

Cell (2,3) — row 3, column 2 of Figure 7.1 — is defined by the inequalities $k_A \leq k_1 \leq q_m$, $q_{m(c)} \leq k_2 \leq k_D$. Firm 1's best response function is of form (b1,b2,b5). Firm 2's best response function is of form (b1,b2,b4,b5).
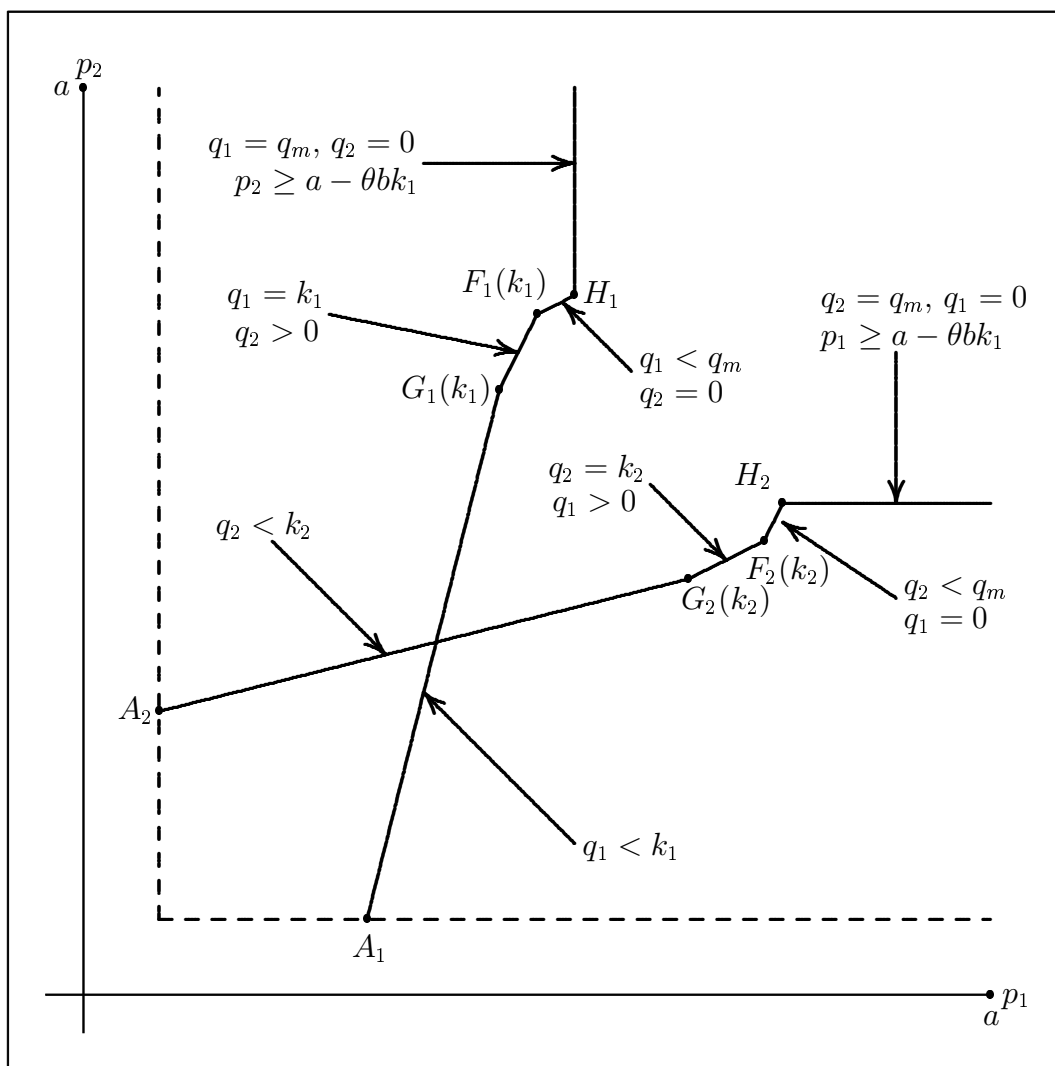
Figure 9.6: (b1,b1) equilibrium, both firms (b1,b2,b4,b5) price best response functions
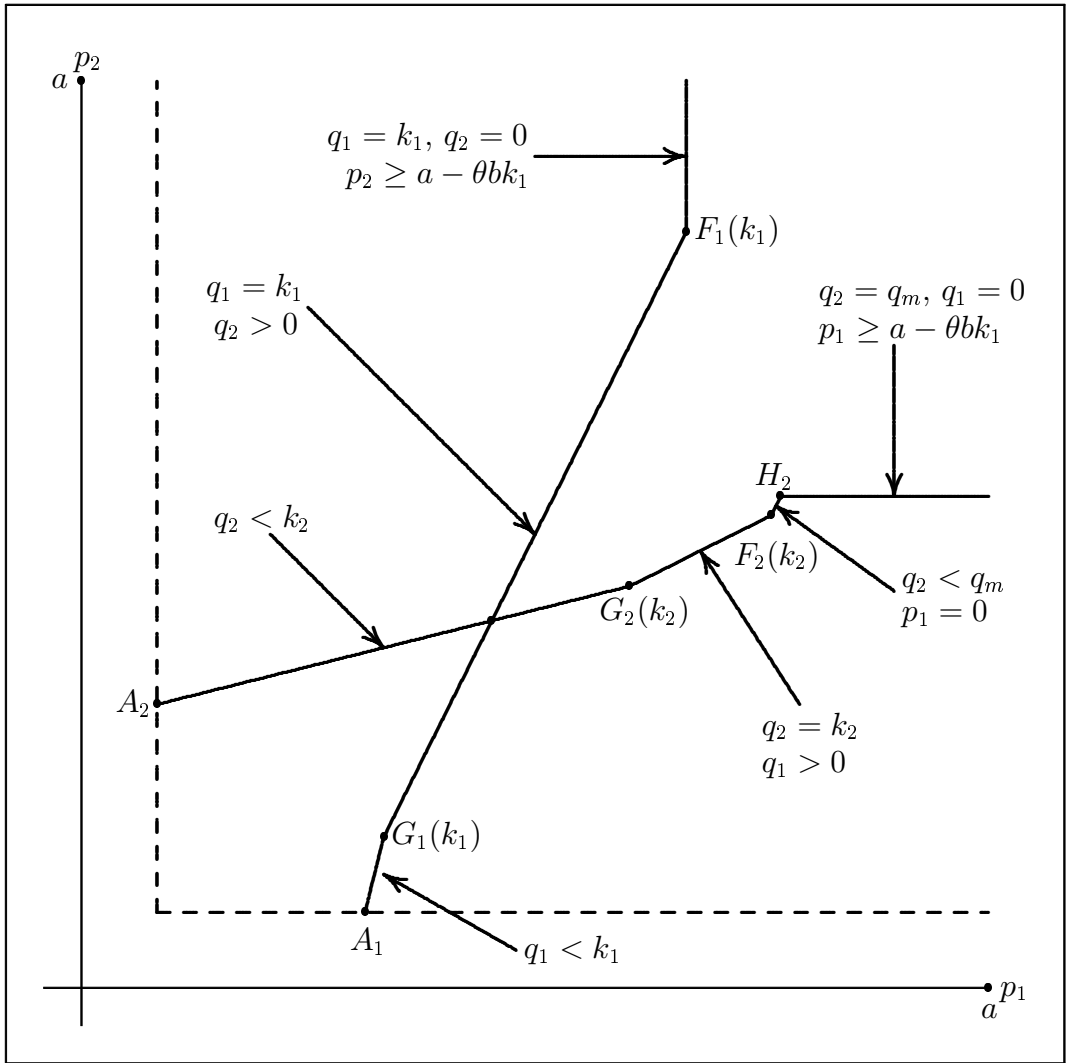
Figure 9.7: (b2,b1) second-stage equilibrium, firm 1 (b1,b2,b5) price best response function, firm 2 (b1,b2,b4,b5) price best response function

Figure 9.7 shows a (b2,b1) second-stage equilibrium when firm 1 has a (b1,b2,b5) price best response function and firm 2 has a (b1,b2,b4,b5) price best response function.

The conditions for such an equilibrium are first that point $F_1(k_1)$ be above firm 2's branch one while point $G_1(k_1)$ is below firm 2's branch one, and second that point $G_2(k_2)$ be to the right of firm 1's branch two.

From Section 9.4, point $F_1(k_1)$ is above firm 2's branch one for all k_1 of interest. From Section 9.2, the condition for point $G_1(k_1)$ to be below firm 2's branch one is that $k_1 \leq k^*_{B(c)}$.

The coordinates of point $G_2(k_2)$ are

$$\left(c + \frac{2(1-\theta^2)bk_2 - (1-\theta)(a-c)}{\theta}, c + (1-\theta^2)bk_2 \right).$$

The condition for point $G_2(k_2)$ to be to the right of firm 1's branch two is

$$p_1 - c - \theta(p_2 - c) \geq (1-\theta)(a-c) - (1-\theta^2)bk_1$$

$$\frac{2(1-\theta^2)bk_2 - (1-\theta)(a-c)}{\theta} - \theta(1-\theta^2)bk_2 \geq (1-\theta)(a-c) - (1-\theta^2)bk_1$$

$$2(1-\theta^2)bk_2 - (1-\theta)(a-c) - \theta^2(1-\theta^2)bk_2 \geq \theta(1-\theta)(a-c) - \theta(1-\theta^2)bk_1$$

$$\theta(1-\theta^2)bk_1 + (1-\theta^2)(2-\theta^2)bk_2 \geq \theta(1-\theta)(a-c) + (1-\theta)(a-c)$$

$$\theta(1-\theta^2)bk_1 + (1-\theta^2)(2-\theta^2)bk_2 \geq (1-\theta^2)(a-c)$$

$$\theta k_1 + (2-\theta^2)k_2 \geq \frac{a-c}{b}$$

$$k_2 \geq \frac{1}{2-\theta^2}\left(\frac{a-c}{b} - \theta k_1 \right) \equiv k^{br}_{B(c)}(k_1), \tag{9.6}$$

where $k^{br}_{B(c)}(k_1)$ is the capacity level that is just sufficient to allow firm 2 to produce its Bertrand best response output if firm 2's marginal cost is c and firm 1's output is k_1.

If (9.6) is met, second-stage equilibrium in cell (2,3) is of type (b2,b1). If $k_2 \leq k^{br}_{B(c)}(k_1)$, second-stage equilibrium in cell (2,3) is of type (b1,b1).
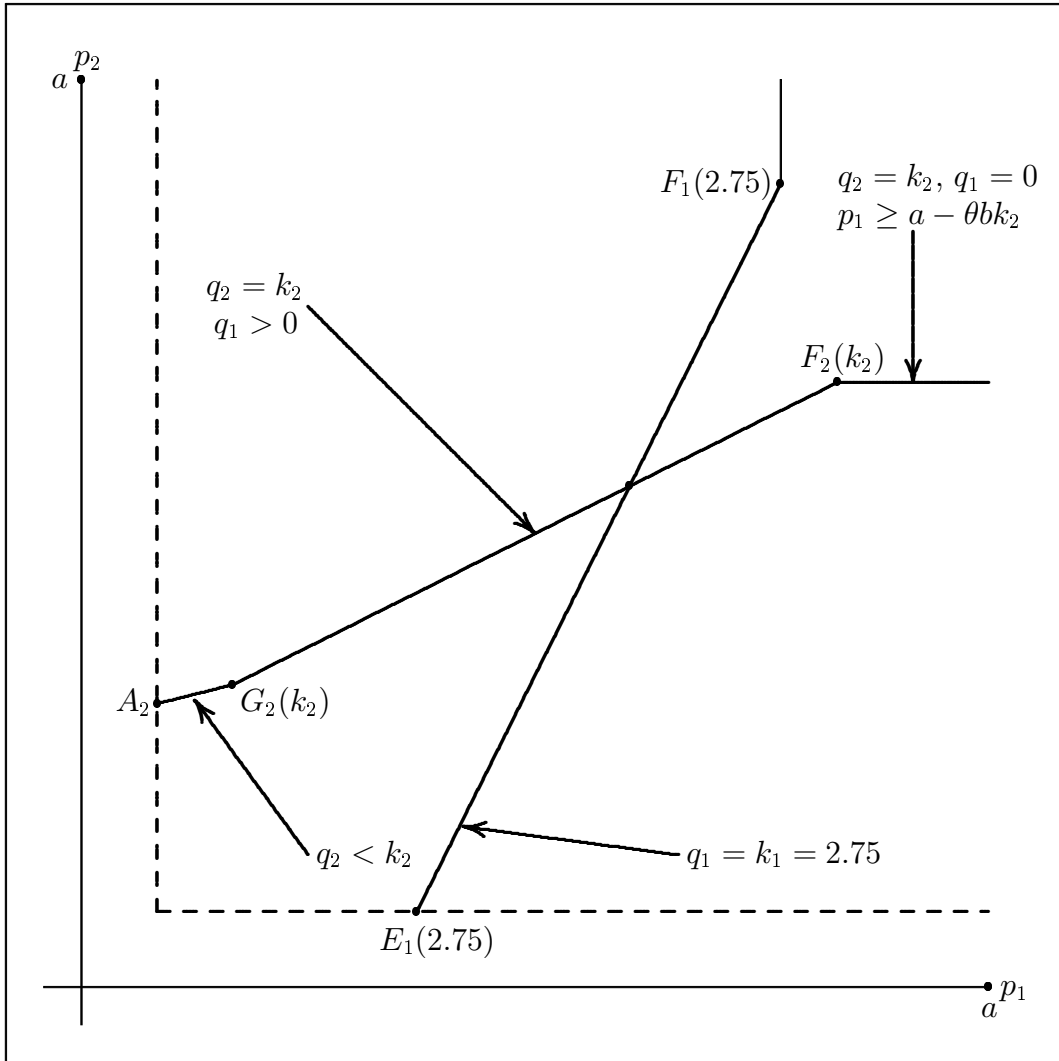
Figure 9.8: (b2,b2) second-stage equilibrium, cell(1,3)

9.7. Cell (1,3)

Cell (1,3) - row 3, column 1 in Figure 7.1 — is defined for $q_{m(c)} \leq k_2 \leq k_D$, $0 \leq k_1 \leq k_A$. Firm 1's best response function is of form (b2,b5). Firm 2's best response function is of form (b1,b2,b4,b5).

The conditions for equilibrium to occur at the intersection of the branch two segments of the price best response functions in cell (1,3) are that point $F_1(k_1)$ be above firm 2's branch two, that point $F_2(k_2)$ be to the right of firm 1's branch two, and that point $G_2(k_2)$ be to the left of firm 1's branch two.

The coordinates of point $F_1(k_1)$ are

$$(p_{1F}, p_{2F}) = (a - bk_1, a - \theta bk_1).$$

The equation of firm 2's branch two is

$$-\theta(p_1 - c) + p_2 - c = (1 - \theta)(a - c) - (1 - \theta^2)bk_2.$$

The condition for point $F_1(k_1)$ to be above firm 2's branch two is

$$-\theta(a - c - bk_1) + a - c - \theta bk_1 \geq (1 - \theta)(a - c) - (1 - \theta^2)bk_2$$

$$-\theta(a - c) + \theta bk_1 + a - c - \theta bk_1 \geq (1 - \theta)(a - c) - (1 - \theta^2)bk_2$$

$$k_2 \geq 0,$$

and this is satisfied for all k_2.

In the same way, point $F_2(k_2)$ is always to the right of firm 1's branch two.

From Section 9.6, the condition for point $G_2(k_2)$ to be to the left of firm 1's branch two is

$$k_2 \leq k_{B(c)}^{br}(k_1),$$

and if this condition is met, the second-stage equilibrium in cell (1,3) is of type (b2,b2). See Figure 9.8. If instead $k_2 \geq k_{B(c)}^{br}(k_1)$, second-stage equilibrium in cell (1,3) is of type (b2,b1).

In the same way, in cell (3,2), second-stage equilibrium is of type (b1,b2) if $k_1 \leq k_{B(c)}^{br}(k_2)$ and of type (b2,b2) if $k_1 \geq k_{B(c)}^{br}(k_2)$.
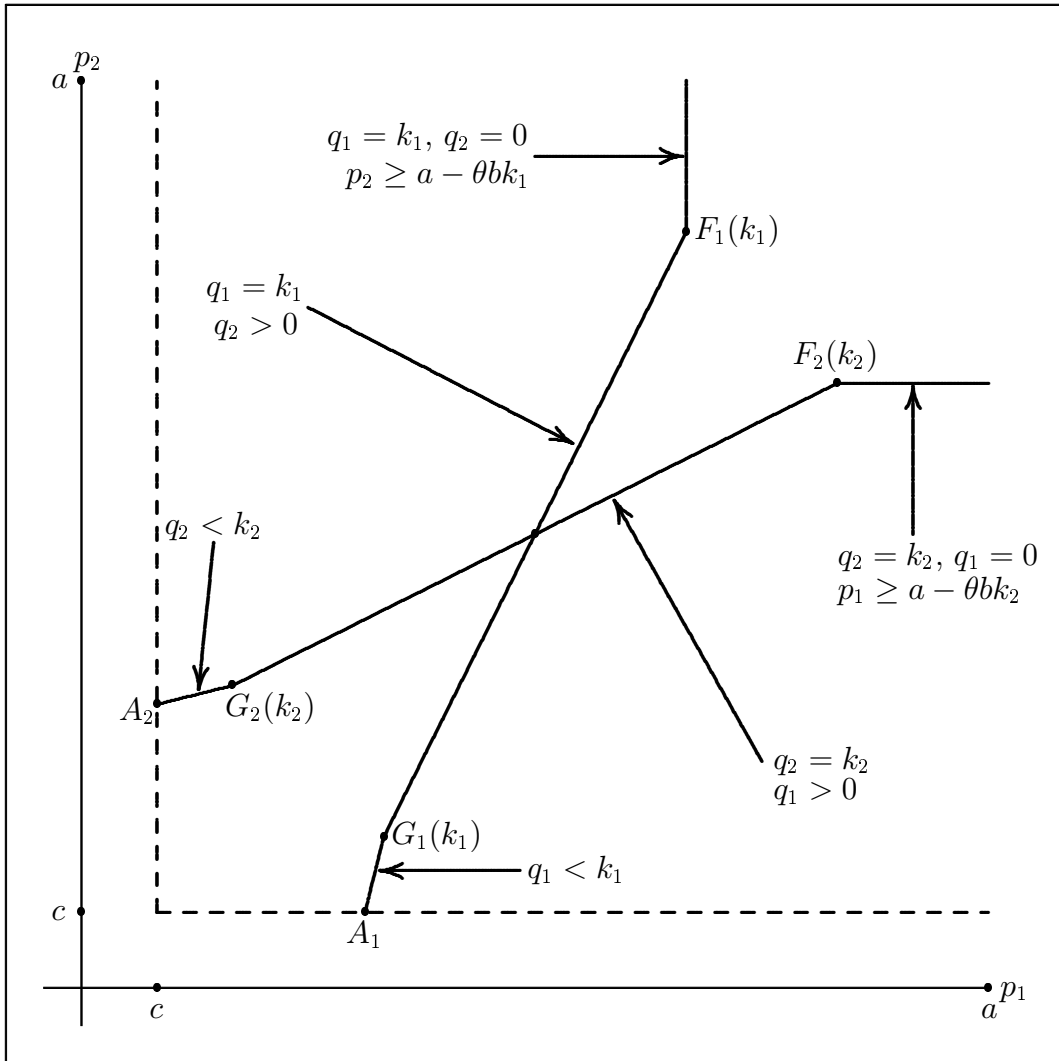
Figure 9.9: Second-stage $(b2, b2)$ equilibrium for equilibrium capacities; both firms $(b1, b2, b5)$ price best response functions

9.8. Cell (2,2)

Cell (2,2) is defined by the inequalities $k_A \leq k_1 \leq q_{m(c)}$, $k_A \leq k_2 \leq q_{m(c)}$.

Figure 9.9, which is Figure 7.4 reproduced here for convenience, shows a second-stage (b2,b2) equilibrium in cell (2,2). The condition for this to occur is that point $F_1(k_1)$ be above firm 2's branch two, $G_1(k_1)$ below firm 2's branch two, $F_2(k_2)$ to the right of firm 1's branch two, and $G_2(k_2)$ to the left of firm 1's branch two.

From Section 9.7, $F_1(k_1)$ and $F_2(k_2)$ always have the required positions, while $G_1(k_1)$ and $G_2(k_2)$ have the required positions if

$$k_1 \leq k_{B(c)}^*(k_2), \; k_2 \leq k_{B(c)}^*(k_1). \tag{9.7}$$

If (9.7) is not met,

$$k_1 \geq k_{B(c)}^*(k_2), \; k_2 \geq k_{B(c)}^*(k_1), \tag{9.8}$$

cell (2,2) second-stage equilibrium is of type (b1,b1).

If point $F_1(k_1)$ is above firm 2's branch two, point $G_1(k_1)$ below firm 2's branch two, and point $G_2(k_2)$ to the right of firm 1's branch two, then second-stage equilibrium is of type (b2,b1). From Section 9.6, the condition for point $G_2(k_2)$ to be to the right of firm 1's branch two is

$$k_2 \geq k_{B(c)}^{br}(k_1).$$

Thus the conditions for (b2,b1) second-stage equilibrium in cell (2,2) are

$$k_1 \leq k_{B(c)}^*(k_2), \; k_2 \geq k_{B(c)}^{br}(k_1). \tag{9.9}$$

In the same way, the conditions for (b1,b2) second-stage equilibrium in cell (2,2) are

$$k_1 \geq k_{B(c)}^{br}(k_2), \; k_2 \leq k_{B(c)}^*(k_1). \tag{9.10}$$

9.9. Cell (1,2)

Cell (1,2) — row 2, column 1 in Figure 7.1 — is defined by the inequalities $0 \leq k_1 \leq k_A$, $k_A \leq k_2 \leq q_{m(c)}$. Firm 1's price best response function is of form (b2,b5). Firm 2's best response function is of form (b1,b2,b5).

Figure 9.11 shows a (b2,b2) second-stage equilibrium in cell (1,2). The conditions for the type of equilibrium to occur are that point $F_1(k_1)$ be above firm 2's
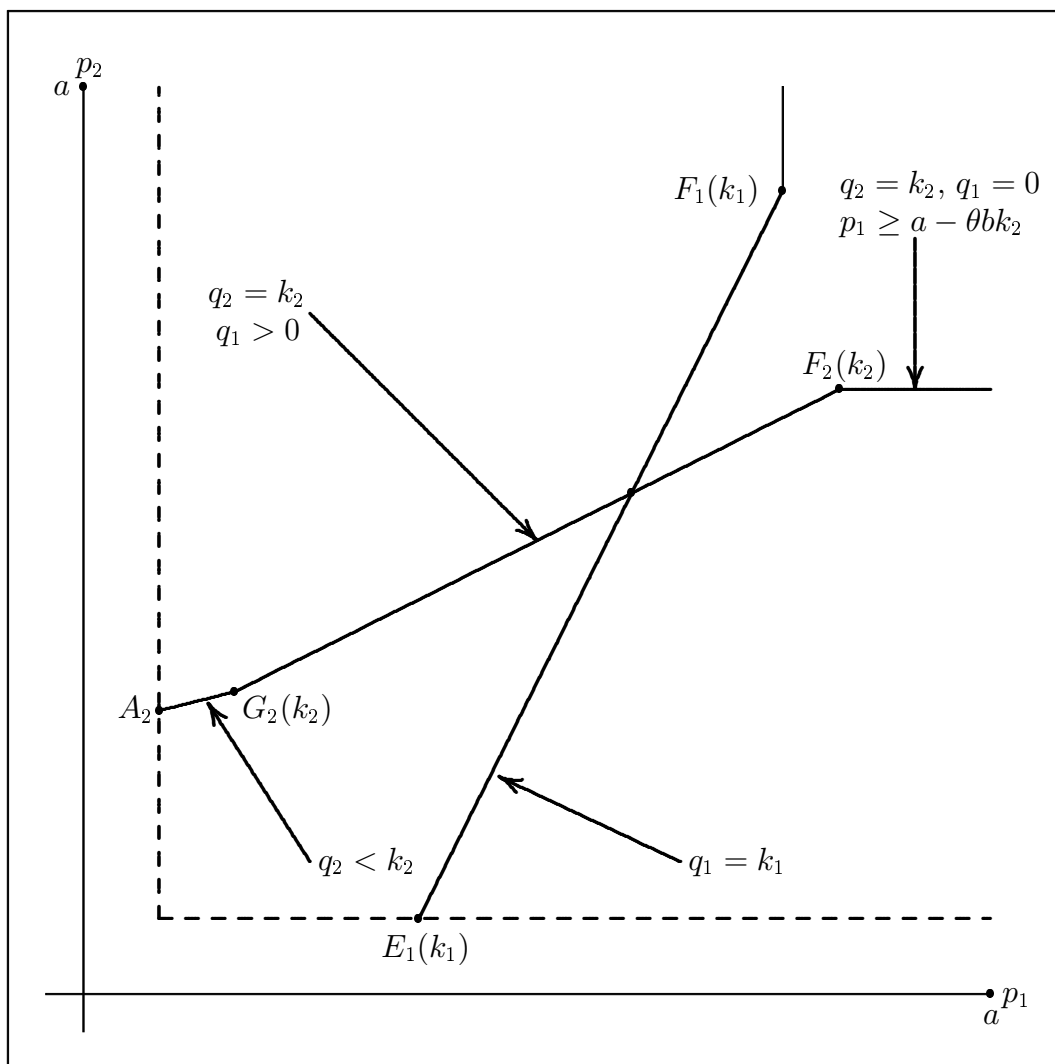
Figure 9.10: cell 12: firm 1 (b2,b5), firm 2 (b1,b2,b5)

branch two, point $F_2(k_2)$ be to the right of firm 1's branch (which conditions are always satisfied, and that point $E_1(k_1)$ be below firm 2's branch two, while point $G_2(k_2)$ be to the right of firm 1's branch two.

Point $E_1(k_1)$, which is on the horizontal axis, is always below firm 2's branch two. From Section 9.6, the condition for point $G_2(k_2)$ to be to the right of firm 1's branch two is

$$k_2 \geq k_{B(c)}^{br}(k_1). \tag{9.11}$$

If on the other hand

$$k_2 \leq k_{B(c)}^{br}(k_1), \tag{9.12}$$

second stage equilibrium in cell (1,2) is of type (b2,b2).

In the same way, if

$$k_1 \geq k_{B(c)}^{br}(k_2), \tag{9.13}$$

then second-stage equilibrium in cell (2,1) is of type (b2,b2), while if

$$k_1 \leq k_{B(c)}^{br}(k_2), \tag{9.14}$$

then second-stage equilibrium in cell (2,1) is of type (b1,b2).

9.10. Cell (1,1)

Cell (1,1) is defined by the inequalities $0 \leq k_1 \leq k_A$, $0 \leq k_2 \leq k_A$.

Figure 9.11 shows a second-stage (b2,b2) equilibrium in cell (1,1). The conditions for cell (1,2) second-stage equilibrium to have this form are that point $F_1(k_1)$ be above firm 2's branch two while point $F_2(k_2)$ is to the right of firm 1's branch two. These conditions are always satisfied.

10. Proof of *Lemma 3*

10.1. $(b1, b1)$

In $(b1, b1)$ equilibrium, equilibrium prices are the Bertrand equilibrium prices with marginal cost equal to c:

$$p_{b1b1} = c + \frac{1-\theta}{2-\theta}(a - c); \tag{10.1}$$

(see (4.3)); equilibrium quantities are

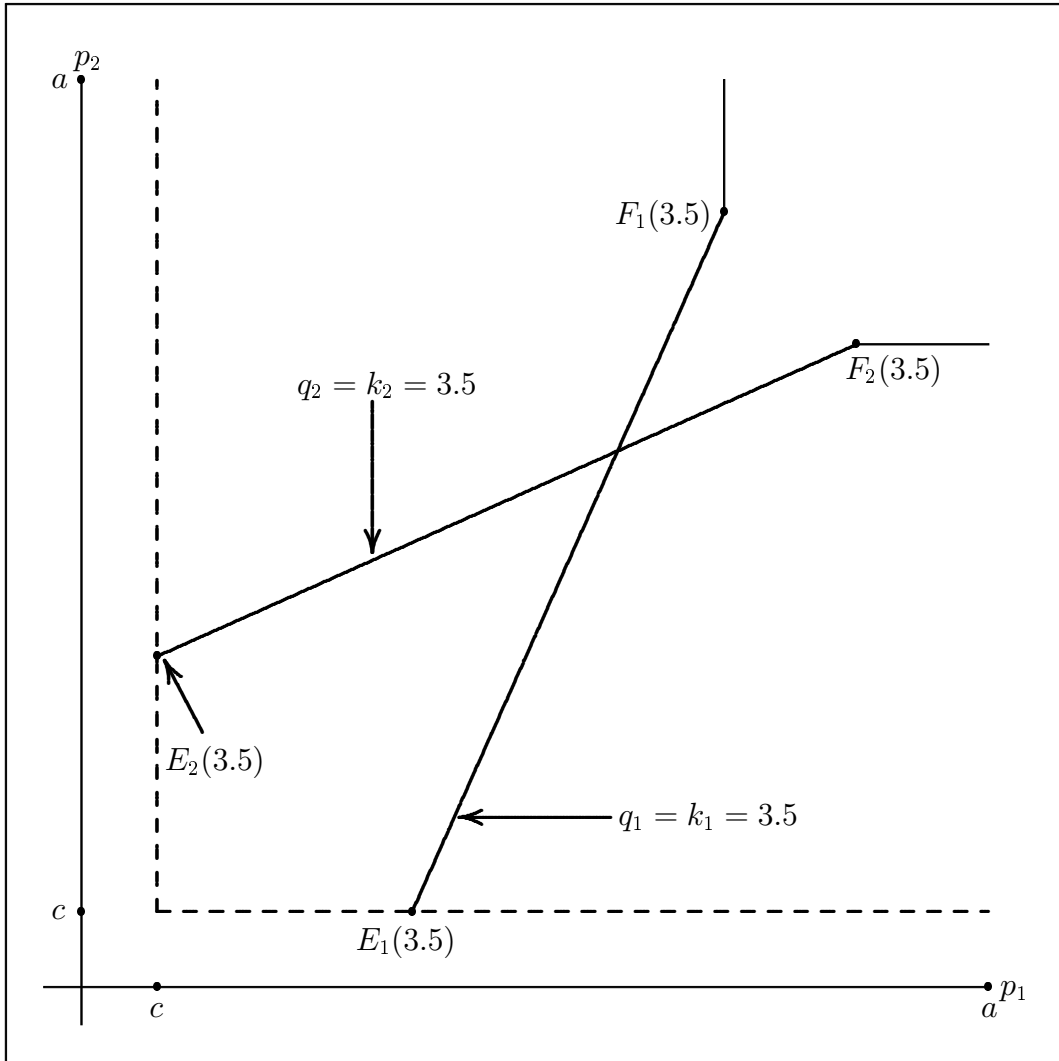$$q_{b1b1} = \frac{1}{(1+\theta)(2-\theta)}\frac{a-c}{b}; \tag{10.2}$$

Figure 9.11: $(b2, b2)$ equilibrium, both firms $(b2, b5)$ price best response functions

equilibrium payoffs are

$$\pi_{ib1b1} = \frac{1-\theta}{(1+\theta)(2-\theta)^2}\frac{(a-c)^2}{b} - \rho k_i. \tag{10.3}$$

10.2. $(b2, b2)$

If both firms are on their quantity constraint lines, equilibrium prices are

$$\begin{pmatrix} 1 & -\theta \\ -\theta & 1 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = (1-\theta)\begin{pmatrix} 1 \\ 1 \end{pmatrix}a - b(1-\theta^2)\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \tag{10.4}$$

with solution

$$p_{1b2b2} = a - b(k_1 + \theta k_2) \tag{10.5}$$

$$p_{2b2b2} = a - b(\theta k_1 + k_2), \tag{10.6}$$

which simply recovers the equations of the inverse demand curves (2.11), (2.12).

Equilibrium outputs are k_1 and k_2. Second-stage payoff functions are

$$\pi_{1b2b2} = [a - c - \rho - b(k_1 + \theta k_2)]\,k_1 \tag{10.7}$$

$$\pi_{2b2b2} = [a - c - \rho - b(\theta k_1 + k_2)]\,k_2. \tag{10.8}$$

10.3. $(b2, b1)$

If point $C_1(k_1)$ is below the $q_2 < k_2$ line,

$$k_1 \le k^*_{B(c)}, \tag{10.9}$$

and point $C_2(k_2)$ is to the right of the $q_1 = k_1$ line,

$$k_2 \ge k^{br}_{B(c)}(k_1), \tag{10.10}$$

then equilibrium occurs where firm 1's branch two intersects firm 2's branch one. This case is symmetric with the $(b1, b2)$ equilibrium.

Firm 1 is capacity constrained; the equation of firm 1's price best response function is (5.6)

$$p_1 = \theta p_2 + (1-\theta)a - b(1-\theta^2)k_1,$$

or, rewritten in terms of deviations from c,

$$p_1 - c - \theta(p_2 - c) = (1-\theta)(a-c) - b(1-\theta^2)k_1.$$

51

Firm is not capacity constrained; the equation of firm 2's price best response function is (9.1)

$$-\theta(p_1 - c) + 2(p_2 - c) = (1 - \theta)(a - c).$$

Equilibrium prices solve

$$\begin{pmatrix} 1 & -\theta \\ -\theta & 2 \end{pmatrix} \begin{pmatrix} p_1 - c \\ p_2 - c \end{pmatrix} = (1 - \theta)(a - c) \begin{pmatrix} 1 \\ 1 \end{pmatrix} - b(1 - \theta^2)k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

leading to

$$p_{1b2b1} = c + \frac{(1 - \theta)(2 + \theta)(a - c) - 2(1 - \theta^2)bk_1}{2 - \theta^2} \tag{10.11}$$

$$p_{2b2b1} = c + \frac{1 - \theta^2}{2 - \theta^2} (a - c - \theta bk_1). \tag{10.12}$$

Equilibrium quantities demanded are

$$q_{1b2b1} = k_1$$

$$q_{2b2b1} = \frac{1}{2 - \theta^2} \frac{a - c - \theta bk_1}{b}. \tag{10.13}$$

Second-stage payoffs are

$$\pi_{1b2b1} = \left[\frac{(1 - \theta)(2 + \theta)(a - c) - 2(1 - \theta^2)bk_1}{2 - \theta^2} - \rho \right] k_1 \tag{10.14}$$

$$\pi_{2b2b1} = \frac{1 - \theta^2}{(2 - \theta^2)^2} \frac{(a - c - \theta bk_1)^2}{b} - \rho k_2. \tag{10.15}$$

10.4. $(b1, b2)$

The equation of the branch one segment of firm 1's price best response function is

$$2(p_1 - c) - \theta(p_2 - c) = (1 - \theta)(a - c). \tag{10.16}$$

The equation of the branch two segment of firm 2's best response function is

$$p_2 = \theta p_1 + (1 - \theta)a - b(1 - \theta^2)k_2. \tag{10.17}$$

Rewrite (10.17) in terms of deviations from c:

$$-\theta(p_1 - c) + p_2 - c = (1 - \theta)(a - c) - b(1 - \theta^2)k_2.$$

The system of equations is

$$\begin{pmatrix} 2 & -\theta \\ -\theta & 1 \end{pmatrix} \begin{pmatrix} p_1 - c \\ p_2 - c \end{pmatrix} = (1 - \theta)(a - c) \begin{pmatrix} 1 \\ 1 \end{pmatrix} - (1 - \theta^2)bk_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (10.18)$$

Prices in $(b1, b2)$ equilibrium are

$$p_{1b1b2} = c + \frac{1 - \theta^2}{2 - \theta^2} (a - c - \theta bk_2) \quad (10.19)$$

$$p_{2b1b2} = c + \frac{(1 - \theta)(2 + \theta)(a - c) - 2(1 - \theta^2)bk_2}{2 - \theta^2} \quad (10.20)$$

Firm 1 is on branch one of its best response function; from (8.3), the quantity demanded of firm 1 is

$$q_{1b1b2} = \frac{p_1 - c}{b(1 - \theta^2)} = \frac{1}{b(1 - \theta^2)} \frac{1 - \theta^2}{2 - \theta^2} (a - c - \theta bk_2)$$

$$= \frac{1}{2 - \theta^2} \frac{a - c - \theta bk_2}{b}. \quad (10.21)$$

Firm 1's second-stage payoff is

$$\pi_{1b1b2} = \frac{1 - \theta^2}{(2 - \theta^2)^2} \frac{(a - c - \theta bk_2)^2}{b} - \rho k_1. \quad (10.22)$$

Firm 2's second-stage payoff is

$$\pi_{2b1b2} = (p_{2b1b2} - c - \rho)k_2 =$$

$$\left[\frac{(1 - \theta)(2 + \theta)(a - c) - 2(1 - \theta^2)bk_2}{2 - \theta^2} - \rho \right] k_2 \quad (10.23)$$

11. Proof of *Lemma 4*

By the way in which the first-stage payoff functions are derived, they must be continuous in capacities. I have verified this, but omit the details of these parts of the proof.

11.1. Lower region: $k_2 \leq k^*_{B(c)}$

Let $k_2 \leq k^*_{B(c)}$. Firm 1's payoff function is

$$\pi_1(k_1, k_2) =$$

$$
\begin{cases}
[a - c - \rho - b(k_1 + \theta k_2)]\, k_1 & 0 \leq k_1 \leq k^{br}_{B(c)}(k_2) \quad (b2, b2) \\
\frac{1-\theta^2}{(2-\theta^2)^2} \frac{(a-c-\theta b k_2)^2}{b} - \rho k_1. & k^{br}_{B(c)}(k_2) \leq k_1 \qquad (b1, b2)
\end{cases}
\tag{11.1}
$$

If $k_1 \leq k^{br}_{B(c)}(k_2)$, second-stage equilibrium occurs where both firms are on branch two, producing at capacity.

By (10.5), firm 1's equilibrium price is given by the equation of its inverse demand curve, writing capacities in place of quantities demanded:

$$p_1 = a - b(k_1 + \theta k_2).$$

Its payoff is

$$\pi_1(k_1, k_2) = (p_1 - c - \rho)k_1 = (a - c - \rho - b(k_1 + \theta k_2))k_1. \tag{11.2}$$

If $k_1 \geq k^{br}_{B(c)}(k_2)$, then in second-stage equilibrium, firm 1 is on its branch one $(q_1 \leq k_1)$, while firm 2 is on its branch two $(q_2 = k_2)$. From (10.22), firm 1's equilibrium payoff is

$$\pi_1(k_1, k_2) = \frac{1 - \theta^2}{(2 - \theta^2)^2} \frac{(a - c - \theta b k_2)^2}{b} - \rho k_1. \tag{11.3}$$

For the numerical example, firm 1's payoff function for $k_2 = 4$ is shown in Figure 11.1.

Comparing (11.2) and (4.6), for $k_1 \leq k^{br}_{B(c)}(k_2)$, firm 1's payoff function has the same functional form as firm 1's payoff function in the standard Cournot duopoly model with product differentiation and marginal cost $c+\rho$, (4.6), and the capacity that maximizes firm 1's payoff in the left-hand region $k_1 \leq k^{br}_{B(c)}(k_2)$, which with a certain abuse of notation we denote as

$$k^{br}_{1Cour}(k_2) = \frac{1}{2}\left(\frac{a - c - \rho}{b} - \theta k_2\right), \tag{11.4}$$

has the same functional form as the Cournot best response output with marginal cost $c + \rho$, (4.7).
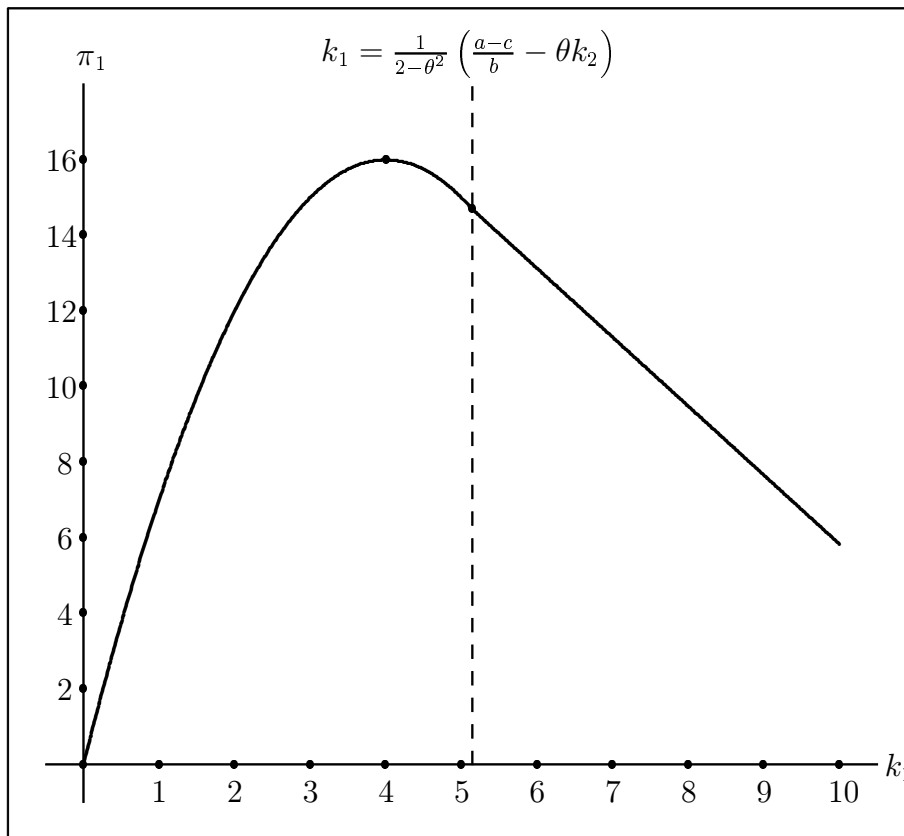
Figure 11.1: Firm 1's profit function, $k_2 = 4$

In order for (11.4) to give the value that maximizes firm 1's equilibrium profit in the $(b2, b2)$ region, the capacity level identified by (11.4) must be in the $(b2, b2)$ region. The condition for this is

$$k_1(k_2) = \frac{1}{2}\left(\frac{a - c - \rho}{b} - \theta k_2\right) \le k_{B(c)}^{br}(k_2) \tag{11.5}$$

$$k_2 \le \frac{\theta^2(a - c) + (2 - \theta^2)\rho}{\theta^3 b} \equiv k_{LLint}. \tag{11.6}$$

We know that in the region now under analysis

$$k_2 \le k_{B(c)}^* = \frac{1}{(1 + \theta)(2 - \theta)}\frac{a - c}{b}.$$

Comparing k_{LLint} and $k_{B(c)}^*$,
$$k_{LLint} - k_{B(c)}^* =$$

$$\frac{2 - \theta^2}{\theta}\left[\frac{1}{(1 + \theta)(2 - \theta)}\frac{a - c}{b} + \frac{1}{\theta^2}\frac{\rho}{b}\right] > 0.$$

Hence in the case we are now considering

$$k_2 \le k_{B(c)}^* \le k_{LLint},$$

and the global maximum of (11.2) occurs within the $(b2, b2)$ region.

To the right of the line $k_1 = k_{B(c)}^{br}(k_2)$ firm 1's payoff, (11.3),

$$\frac{1 - \theta^2}{(2 - \theta^2)^2}\frac{(a - c - \theta b k_2)^2}{b} - \rho k_1,$$

is its payoff for (b1,b2) equilibrium. This is maximized by making k_1 as small as possible within the relevant region, that is, by setting $k_1 = k_{B(c)}^*$. Since the payoff function is continuous, and rising moving to the left from $k_1 = k_{B(c)}^{br}(k_2)$, the global maximum of the payoff function for the left-hand segment occurs within the left-hand segment, and firm 1 maximizes its payoff for $k_2 \le k_{B(c)}^*$ by setting the capacity (11.4),

$$k_{1Cour}^{br} = \frac{1}{2}\left(\frac{a - c - \rho}{b} - \theta k_2\right),$$

which is the equation of the segment of firm 1's capacity best response function $0 \le k_2 \le k_{B(c)}^*$.

11.2. Upper region: $\frac{1}{2-\theta^2}\frac{a-c}{b} \leq k_2$

$$\pi_1(k_1, k_2) =$$

$$\begin{cases} \left[\frac{(1-\theta)(2+\theta)}{2-\theta^2}(a-c) - \rho - 2\frac{1-\theta^2}{2-\theta^2}bk_1\right]k_1 & 0 \leq k_1 \leq k^*_{B(c)} & (b2, b1) \\ \frac{1-\theta}{(1+\theta)(2-\theta)^2}\frac{(a-c)^2}{b} - \rho k_1 & k^*_{B(c)} \leq k_1 & (b1, b1) \end{cases} \qquad (11.7)$$

Firm 1's payoff function in the left-hand region is its payoff in $(b2, b1)$ equilibrium,

$$\pi_{1b2b1} = \left[\frac{(1-\theta)(2+\theta)}{2-\theta^2}(a-c) - \rho - 2\frac{1-\theta^2}{2-\theta^2}bk_1\right]k_1. \qquad (11.8)$$

The global maximum of (11.8) is at

$$k^{br}_{1b2b1} = \frac{(1-\theta)(2+\theta)(a-c) - (2-\theta^2)\rho}{4(1-\theta^2)b}. \qquad (11.9)$$

This is interior to the left-hand region, for which $k_1 \leq k^*_{B(c)} = \frac{1}{(1+\theta)(2-\theta)}\frac{a-c}{b}$:

$$k^*_{B(c)} - k^{br}_{1b2b1} =$$

$$\frac{1}{4(1+\theta)}\left[\frac{\theta^2}{2-\theta}\frac{a-c}{b} + \frac{2-\theta^2}{1+\theta}\frac{\rho}{b}\right] > 0. \qquad (11.10)$$

k^{br}_{1b2b1} may be negative, if ρ is sufficiently large. We will assume that $k^{br}_{1b2b1} > 0$. This assumption will be used several times below to determine the signs of various expressions.

Firm 1's payoff at (11.9) is

$$\pi_{1TL} = 2\frac{1-\theta^2}{2-\theta^2}b\left(k^{br}_{1b2b1}\right)^2 =$$

$$\frac{b}{8}\frac{2-\theta^2}{1-\theta^2}\left[\frac{(1-\theta)(2+\theta)(a-c) - (2-\theta^2)\rho}{(2-\theta^2)b}\right]^2. \qquad (11.11)$$

Firm 1's payoff in the right-hand region, (10.3), is its payoff in $(b1, b1)$ equilibrium,

$$\frac{1-\theta}{(1+\theta)(2-\theta)^2}\frac{(a-c)^2}{b} - \rho k_1,$$
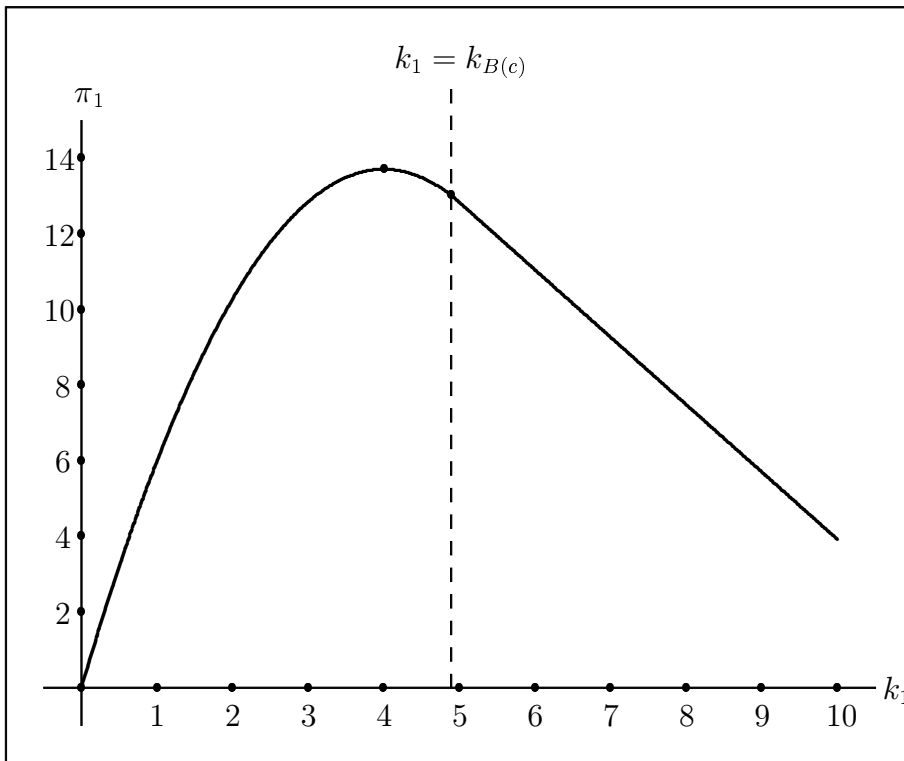
Figure 11.2: Firm 1's profit function, $k_2 \geq \frac{1}{2-\theta^2} \frac{a-c}{b}$

and this is maximized by making k_1 as small as possible within the relevant range, that is, by setting $k_1 = k^*_{B(c)}$. By continuity of the payoff function and the fact that the payoff function rises moving left from the boundary $k_1 = k^*_{B(c)}$, the global maximum in the top region occurs at (11.9).

Figure 11.2 shows firm 1's payoff function for $k_2 \geq \frac{1}{2-\theta^2}\frac{a-c}{b}$.

Figure 11.3 shows the lower ($0 \leq k_2 \leq k^*_{B(c)}$) and upper ($\frac{1}{2-\theta^2}\frac{a-c}{b} \leq k_2 \leq \frac{a-c-\rho}{b}$) segments of the capacity best response functions.

Figure 11.3: Capacity best response functions, upper and lower segments, Kreps & Scheinkman model with product differentiation

59

11.3. Middle region: $k_{B(c)}^* \leq k_2 \leq \frac{1}{2-\theta^2}\frac{a-c}{b}$
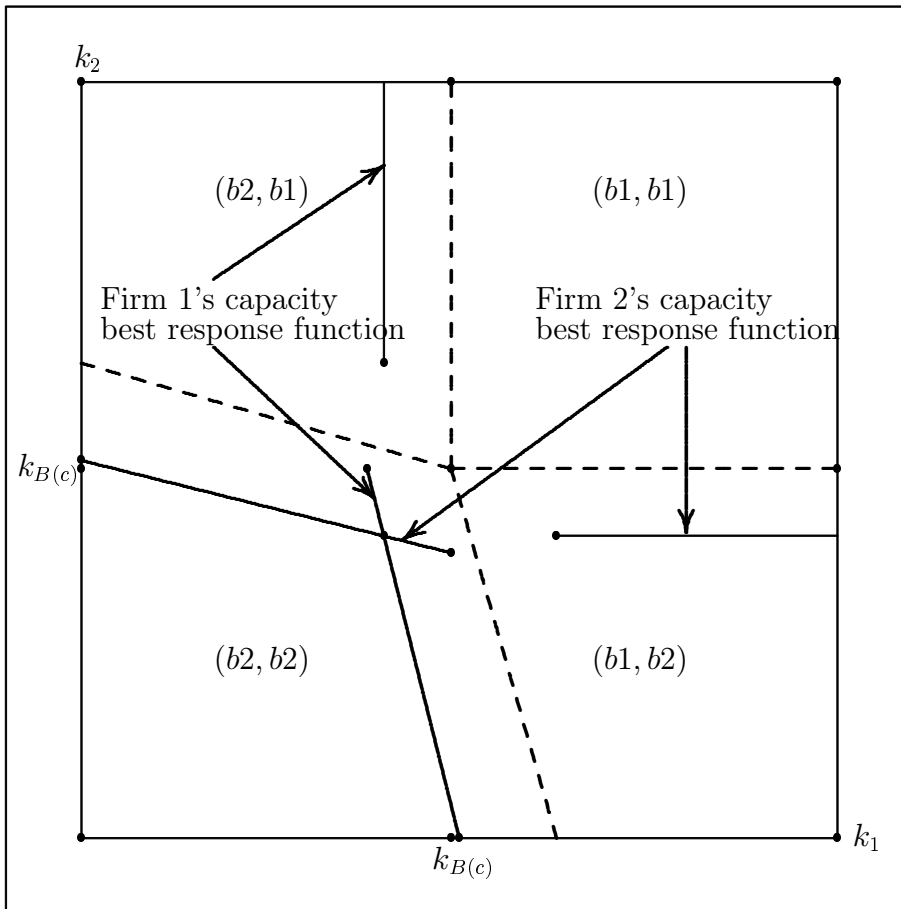
The equation of the boundary between the $(b2,b2)$ region and the $(b2,b1)$ region is

$$\theta k_1 + (2 - \theta^2)k_2 = \frac{a-c}{b} \qquad (11.12)$$

or

$$k_2 = k_{B(c)}^{br}(k_1)$$

or

$$k_1 = \left[k_{B(c)}^{br}\right]^{-1}(k_2).$$

If $k_1 = 0$ along this line, then $k_2 = \frac{1}{2-\theta^2}\frac{a-c}{b}$. For $k_{B(c)}^* \leq k_2 \leq \frac{1}{2-\theta^2}\frac{a-c}{b}$, firm 1's payoff function has three segments:

$$\pi_1(k_1, k_2) = \qquad (11.13)$$

$$
\begin{cases}
(a - c - \rho - b(k_1 + \theta k_2))k_1 & 0 \leq k_1 \leq \left[k_{B(c)}^{br}\right]^{-1}(k_2) & (b2, b2) \\
\left(\frac{(1-\theta)(2+\theta)}{2-\theta^2}(a-c) - \rho - 2\frac{1-\theta^2}{2-\theta^2}bk_1\right)k_1 & \left[k_{B(c)}^{br}\right]^{-1}(k_2) \leq k_1 \leq k_{B(c)}^* & (b2, b1) \\
\frac{1-\theta}{(1+\theta)(2-\theta)^2}\frac{(a-c)^2}{b} - \rho k_1 & k_{B(c)}^* \leq k_1 & (b1, b1)
\end{cases}.
$$

11.3.1. Left segment: $0 \leq k_1 \leq \left[k_{B(c)}^{br}\right]^{-1}(k_2)$

In this region, firm 1's payoff is that of $(b2, b2)$ equilibrium; this case has been analyzed in Section 11.1.

Firm 1's profit-maximizing capacity is (11.4),

$$k_{1Cour}^{br} = \frac{1}{2}\left(\frac{a-c-\rho}{b} - \theta k_2\right),$$

if k_{1Cour}^{br} lies with the left-hand range of the middle region $0 \leq k_1 \leq \frac{1}{\theta}\left[\frac{a-c}{b} - (2-\theta^2)k_2\right]$. The condition for k_{1Cour}^{br} to be within the left-hand range of the middle region is

$$\frac{1}{2}\left(\frac{a-c-\rho}{b} - \theta k_2\right) \leq \frac{1}{\theta}\left[\frac{a-c}{b} - (2-\theta^2)k_2\right],$$

or

$$k_2 \leq \frac{(2-\theta)(a-c)+\theta\rho}{(4-3\theta^2)b} \equiv k_{MLint}. \qquad (11.14)$$

60

In the middle region

$$k^*_{B(c)} \leq k_2 \leq \frac{1}{2-\theta^2}\frac{a-c}{b}.$$

Compare k_{MLint} with the upper end of the range of k_2 that defines the middle region:

$$\frac{1}{2-\theta^2}\frac{a-c}{b} - k_{MLint} = \frac{\theta}{4-3\theta^2}\left[\frac{(1-\theta)(2+\theta)(a-c)-(2-\theta^2)\rho}{2-\theta^2}\right] > 0,$$

where the sign depends on the assumption that (11.4), firm one's best-response capacity in the upper region, is positive.

Now compare k_{MLint} with the lower end of the range of k_2 that defines the middle region:

$$k_{MLint} - k^*_{B(c)} =$$

$$\frac{\theta}{4-3\theta^2}\left[\frac{\theta^2}{(1+\theta)(2-\theta)}\frac{a-c}{b} + \frac{\rho}{b}\right] > 0.$$

Hence

$$k^*_{B(c)} < k_{MLint} < \frac{1}{2-\theta^2}\frac{a-c}{b}.$$

For firm 2 capacity levels falling in the range

$$k^*_{B(c)} \leq k_2 \leq k_{MLint}, \tag{11.15}$$

k^{br}_{1Cour} is interior to the left-hand segment of the middle region, firm 1's payoff function has an interior maximum at k^{br}_{1Cour} on the left-hand segment, and firm 1's payoff at this maximum is

$$\frac{b}{4}\left(\frac{a-c-\rho}{b} - \theta k_2\right)^2.$$

In contrast, for

$$k_{MLint} \leq k_2 \leq \frac{1}{2-\theta^2}\frac{a-c}{b},$$

the local maximum of firm 1's payoff function on the range $0 \leq k_1 \leq \frac{1}{\theta}\left[\frac{a-c}{b} - (2-\theta^2)k_2\right]$ is at the right-hand border of this range, $k_1 = \frac{1}{\theta}\left[\frac{a-c}{b} - (2-\theta^2)k_2\right]$.

11.3.2. Middle segment: $\left[k_{B(c)}^{br}\right]^{-1}(k_2) \leq k_1 \leq k_{B(c)}^*$

Firm 1's payoff function in the middle segment of the middle region is its payoff in $(b2, b1)$ equilibrium,

$$\pi_{1b2b1} = \left[\frac{(1-\theta)(2+\theta)}{2-\theta^2}(a-c) - \rho - 2\frac{1-\theta^2}{2-\theta^2}bk_1\right]k_1.$$

The global maximum occurs for (11.9)

$$k_{1b2b1}^{br} = \frac{(1-\theta)(2+\theta)(a-c) - (2-\theta^2)\rho}{4(1-\theta^2)b}.$$

We know from our discussion of the upper region (see (11.10) and the associated text) that

$$k_{1b2b1}^{br} < k_{B(c)}^*.$$

The condition for

$$\frac{1}{\theta}\left[\frac{a-c}{b} - (2-\theta^2)k_2\right] \leq k_{1b2b1}^{br},$$

so that k_{1b2b1}^{br} lies within the middle range of the region, is

$$k_2 \geq \frac{4+2\theta-\theta^2}{4(1+\theta)(2-\theta^2)}\frac{a-c}{b} + \frac{\theta}{4(1-\theta^2)}\frac{\rho}{b} \equiv k_{MMint}. \tag{11.16}$$

k_{MMint} can also be written

$$k_{MMint} = \frac{1}{2-\theta^2}\left(\frac{a-c}{b} - \theta k_{1b2b1}^{br}\right). \tag{11.17}$$

We know that

$$k_{B(c)}^* \leq k_2 \leq \frac{1}{2-\theta^2}\frac{a-c}{b}.$$

Compare k_{MMint} and $k_{B(c)}^*$:

$$k_{MMint} - k_{B(c)}^* = \frac{\theta}{4(1+\theta)}\left[\frac{\theta^2}{(2-\theta)(2-\theta^2)}\frac{a-c}{b} + \frac{1}{1-\theta}\frac{\rho}{b}\right] > 0.$$

62

Compare k_{MMint} and $\frac{1}{2-\theta^2}\frac{a-c}{b}$:

$$\frac{1}{2-\theta^2}\frac{a-c}{b} - k_{MMint} = \frac{\theta}{4(1+\theta)}\frac{(1-\theta)(2+\theta)(a-c)-(2-\theta^2)\rho}{(1+\theta)(2-\theta^2)} > 0.$$

where once again the sign depends on the assumption that (11.4), firm one's best-response capacity in the upper region, is positive.

Hence

$$k_{B(c)}^* \leq k_{MMint} \leq \frac{1}{2-\theta^2}\frac{a-c}{b}.$$

For

$$k_{B(c)}^* \leq k_2 \leq k_{MMint},$$

the maximum of firm 1's profit function on the range

$$\frac{1}{\theta}\left[\frac{a-c}{b} - (2-\theta^2)k_2\right] \leq k_1 \leq k_{B(c)}^*$$

occurs at the left-hand boundary,

$$k_1 = \frac{1}{\theta}\left[\frac{a-c}{b} - (2-\theta^2)k_2\right],$$

while for

$$k_{MMint} \leq k_2 \leq \frac{1}{2-\theta^2}\frac{a-c}{b},$$

firm 1's profit function on the range

$$\frac{1}{\theta}\left[\frac{a-c}{b} - (2-\theta^2)k_2\right] \leq k_1 \leq k_{B(c)}^*$$

has an interior maximum at k_{1b2b1}^{br}:

$$k_{1b2b1}^{br} = \frac{(1-\theta)(2+\theta)(a-c)-(2-\theta^2)\rho}{4(1-\theta^2)b},$$

and firm 1's payoff at this capacity level is

$$2\frac{1-\theta^2}{2-\theta^2}b\left(k_{1b2b1}^{br}\right)^2.$$

11.3.3. Right segment: $k^*_{B(c)} \leq k_1$

Firm 1's payoff in the right-hand region,

$$\frac{1-\theta}{(1+\theta)(2-\theta)^2}\frac{(a-c)^2}{b} - \rho k_1,$$

is maximized by making k_1 as small as possible while remaining in the region, $k_1 = k^*_{B(c)}$.

11.3.4. Middle region: overall

On the left-hand segment of the middle region, for firm 2 capacity levels falling in the range

$$k^*_{B(c)} \leq k_2 \leq k_{MLint}, \tag{11.18}$$

firm 1's payoff function has an interior maximum on the range $0 \leq k_1 \leq \left[k^{br}_{B(c)}\right]^{-1}(k_2)$, at

$$k^{br}_{1Cour} = \frac{1}{2}\left(\frac{a-c-\rho}{b} - \theta k_2\right),$$

and firm 1's payoff at this maximum is

$$\frac{b}{4}\left(\frac{a-c-\rho}{b} - \theta k_2\right)^2.$$

In contrast, for

$$k_{MLint} \leq k_2 \leq \frac{1}{2-\theta^2}\frac{a-c}{b}, \tag{11.19}$$

the local maximum of firm 1's payoff function on the range $0 \leq k_1 \leq \left[k^{br}_{B(c)}\right]^{-1}(k_2)$ is at the right-hand border of this range, $k_1 = \left[k^{br}_{B(c)}\right]^{-1}(k_2)$.

In the middle segment of the middle region, for

$$k^*_{B(c)}\frac{a-c}{b} \leq k_2 \leq k_{MMint}, \tag{11.20}$$

the maximum of firm 1's profit function on the range

$$\left[k^{br}_{B(c)}\right]^{-1}(k_2) \leq k_1 \leq k^*_{B(c)}$$

	left	middle	global
$k_{B(c)}^* \le k_2 \le k_{MMint}$	interior max	left boundary	left interior max
$k_{MMint} \le k_2 \le k_{MLint}$	interior max	interior max	?
$k_{MLint} \le k_2 \le \frac{1}{2-\theta^2}\frac{a-c}{b}$	right boundary	interior max	middle interior max

Table 11.1: Maxima, middle region

occurs at the left-hand boundary, $k_1 = \frac{1}{\theta}\left(\frac{a-c}{b} - (2-\theta^2)k_2\right)$ while for

$$k_{MMint} \le k_2 \le \frac{1}{2-\theta^2}\frac{a-c}{b}, \tag{11.21}$$

firm 1's profit function on the range

$$\left[k_{B(c)}^{br}\right]^{-1}(k_2) \le k_1 \le k_{B(c)}^*$$

has an interior maximum at

$$k_{1b2b1}^{br} = \frac{(1-\theta)(2+\theta)(a-c) - (2-\theta^2)\rho}{4(1-\theta^2)b},$$

and firm 1's payoff at this capacity level is

$$2\frac{1-\theta^2}{2-\theta^2}b\left[\frac{(1-\theta)(2+\theta)(a-c) - (2-\theta^2)\rho}{4(1-\theta^2)b}\right]^2.$$

Compare k_{MLint} and k_{MMint}:

$$k_{MLint} - k_{MMint} = \frac{\theta^3}{4(1+\theta)(4-3\theta^2)}\left[\frac{(1-\theta)(2+\theta)(a-c) - \left(2-\theta^2\right)\rho}{\left(2-\theta^2\right)}\right] > 0.$$

For

$$k_{B(c)}^* \le k_2 \le k_{MMint},$$

the global maximum of firm 1's payoff function is $k_{1Cour}^{br}(k_2)$, which therefore is the best-response capacity for $0 \le k_2 \le k_{MMint}$.

For

$$k_{MLint} \le k_2 \le \frac{1}{2-\theta^2}\frac{a-c}{b},$$

the global maximum of firm 1's payoff function is k^{br}_{1b2b1},

$$k^{br}_{1b2b1} = \frac{(1-\theta)(2+\theta)(a-c) - (2-\theta^2)\rho}{4(1-\theta^2)b},$$

which therefore is firm 1's best response capacity for $k_{MLint} \leq k_2$.

Figure 11.4 extends the segments of the best response functions shown in Figure 11.3 to include the upper and lower segments of the middle range of the best response functions.
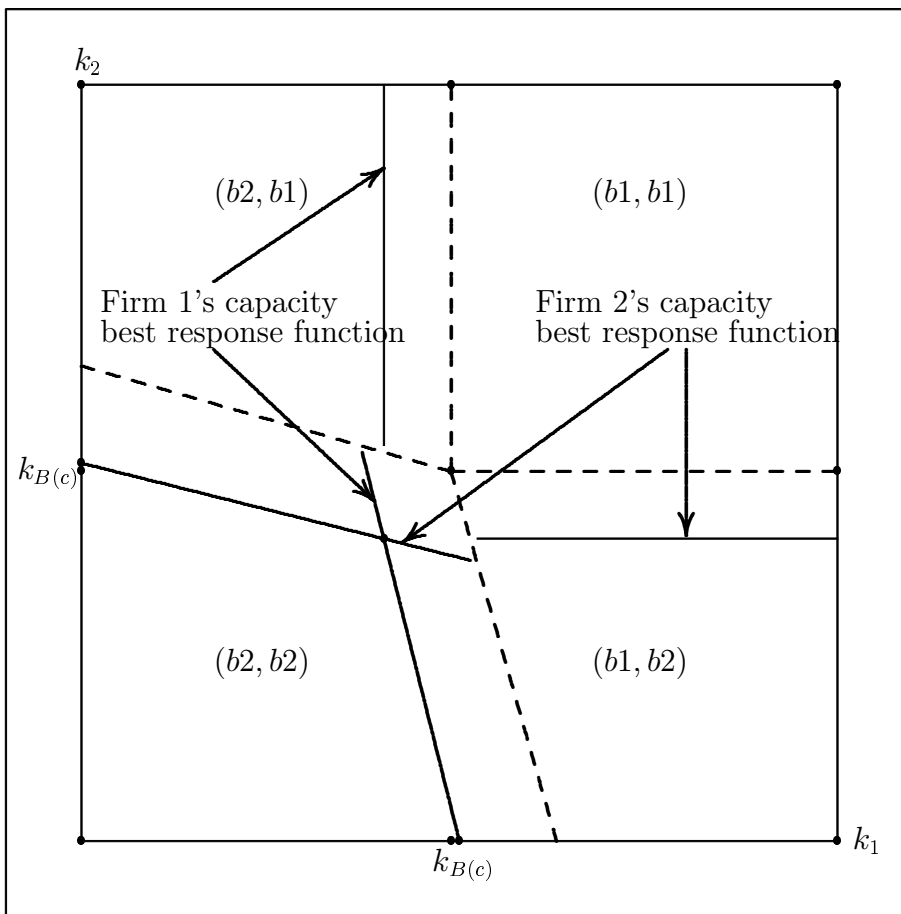

Figure 11.4: Capacity best response functions, upper and lower segments, upper and lower segments of middle range, Kreps & Scheinkman model with product differentiation

For
$$k_{MMint} \le k_2 \le k_{MLint}$$
firm 1's payoff function has two local maxima, $k_{1Cour}^{br}(k_2)$ on the left segment and k_{1b2b1}^{br} on the middle segment.

Firm 1's payoff at $k_{1Cour}^{br}(k_2)$ is
$$\frac{b}{4}\left(\frac{a-c-\rho}{b}-\theta k_2\right)^2.$$

Firm 1's payoff at k_{1b2b1}^{br} is
$$2\frac{1-\theta^2}{2-\theta^2}b\left(k_{1b2b1}^{br}\right)^2 = 2\frac{1-\theta^2}{2-\theta^2}b\left[\frac{(1-\theta)(2+\theta)(a-c)-(2-\theta^2)\rho}{4(1-\theta^2)b}\right]^2$$

We need to compare firm 1's payoffs at the two local maxima to determine the global maximum.

If $k_2 = k_{MMint}$,
$$k_{1Cour}^{br}(k_2) = \frac{1}{2}\left[\frac{a-c-\rho}{b}-\theta\left(\frac{4+2\theta-\theta^2}{4(1+\theta)(2-\theta^2)}\frac{a-c}{b}+\frac{\theta}{4(1-\theta^2)}\frac{\rho}{b}\right)\right] =$$
$$\frac{4-3\theta^2}{2(2-\theta^2)}k_{1b2b1}^{br}$$
and firm 1's payoff at the left-hand local maximum is
$$b\left[\frac{4-3\theta^2}{2(2-\theta^2)}k_{1b2b1}^{br}\right]^2.$$

When $k_2 = k_{MMint}$, the difference between firm 1's payoffs at the left and middle local maxima is
$$b\left[\frac{4-3\theta^2}{2(2-\theta^2)}k_{1b2b1}^{br}\right]^2 - 2\frac{1-\theta^2}{2-\theta^2}b\left(k_{1b2b1}^{br}\right)^2 = \frac{\theta^4}{4(2-\theta^2)^2}b\left(k_{1b2b1}^{br}\right)^2 > 0.$$

At the smallest level of k_2 for which firm 1's payoff function has two local maxima, the global maxima is in the center segment.

Now compare payoffs at the two local maxima for the largest value of k_2 for which firm 1's payoff function has two local maxima, k_{MLint}. For $k_2 = k_{MLint}$
$$k_{1Cour}^{br}(k_2) = \frac{1}{2}\left[\frac{a-c-\rho}{b}-\theta\left(\frac{(2-\theta)(a-c)+\theta\rho}{(4-3\theta^2)b}\right)\right] =$$

$$4\frac{1-\theta^2}{4-3\theta^2}k_{1b2b1}^{br}$$

and firm 1's payoff is

$$b\left(4\frac{1-\theta^2}{4-3\theta^2}bk_{1b2b1}^{br}\right)^2$$

and the difference between firm 1's payoffs at the middle and left local maxima is

$$b\left[\frac{4-3\theta^2}{2(2-\theta^2)}k_{1b2b1}^{br}\right]^2 - b\left(4\frac{1-\theta^2}{4-3\theta^2}k_{1b2b1}^{br}\right)^2 =$$

$$b\left\{\left[\frac{4-3\theta^2}{2(2-\theta^2)}\right]^2 - \left(4\frac{1-\theta^2}{4-3\theta^2}\right)^2\right\}\left(k_{1b2b1}^{br}\right)^2 =$$

$$b\left[\frac{4-3\theta^2}{2(2-\theta^2)}+4\frac{1-\theta^2}{4-3\theta^2}\right]\left[\frac{4-3\theta^2}{2(2-\theta^2)}-4\frac{1-\theta^2}{4-3\theta^2}\right]\left(k_{1b2b1}^{br}\right)^2 =$$

$$b\left[\frac{4-3\theta^2}{2(2-\theta^2)}+4\frac{1-\theta^2}{4-3\theta^2}\right]\left[\frac{1}{2}\frac{\theta^4}{\left(2-\theta^2\right)\left(4-3\theta^2\right)}\right]\left(k_{1b2b1}^{br}\right)^2 > 0.$$

The payoff at the center local maxima falls as k_2 rises. Payoffs at the two local maxima are equal for

$$2\frac{1-\theta^2}{2-\theta^2}b\left(k_{1b2b1}^{br}\right)^2 = \frac{b}{4}\left(\frac{a-c-\rho}{b}-\theta k_2\right)^2$$

$$k_2 = k_S = \frac{1}{\theta}\left(\frac{a-c-\rho}{b}-2\sqrt{2\frac{1-\theta^2}{2-\theta^2}}k_{1b2b1}^{br}\right) =$$

$$= \frac{1}{\theta}\left\{\frac{a-c-\rho}{b}-2\sqrt{2\frac{1-\theta^2}{2-\theta^2}\left[\frac{(1-\theta)(2+\theta)(a-c)-(2-\theta^2)\rho}{4(1-\theta^2)b}\right]}\right\}.$$

Overall in the middle region, firm 1's capacity best response function is

$$k_1 = \begin{cases} k_{1Cour}^{br} = \frac{1}{2}\left(\frac{a-c-\rho}{b}-\theta k_2\right) & \text{for } k_{B(c)}^* \leq k_2 \leq k_S \\ k_{1b2b1}^{br} = \frac{(1-\theta)(2+\theta)(a-c)-(2-\theta^2)\rho}{4(1-\theta^2)b} & \text{for } k_{S)} \leq k_2 \leq \frac{1}{2-\theta^2}\frac{a-c}{b} \end{cases} .$$

Figures 11.5 and 11.6 illustrate the profit function in the middle segment of the middle region for the numerical example.

68

In this instance,

$$k_{MMint} = \frac{36}{7} = 5.1429$$

$$k_S = 5.1869$$

$$k_{MLint} = \frac{68}{13} = 5.2308$$

Figure 11.5 shows firm 1's payoff function for $k_2 = 5.1469$, which is midway

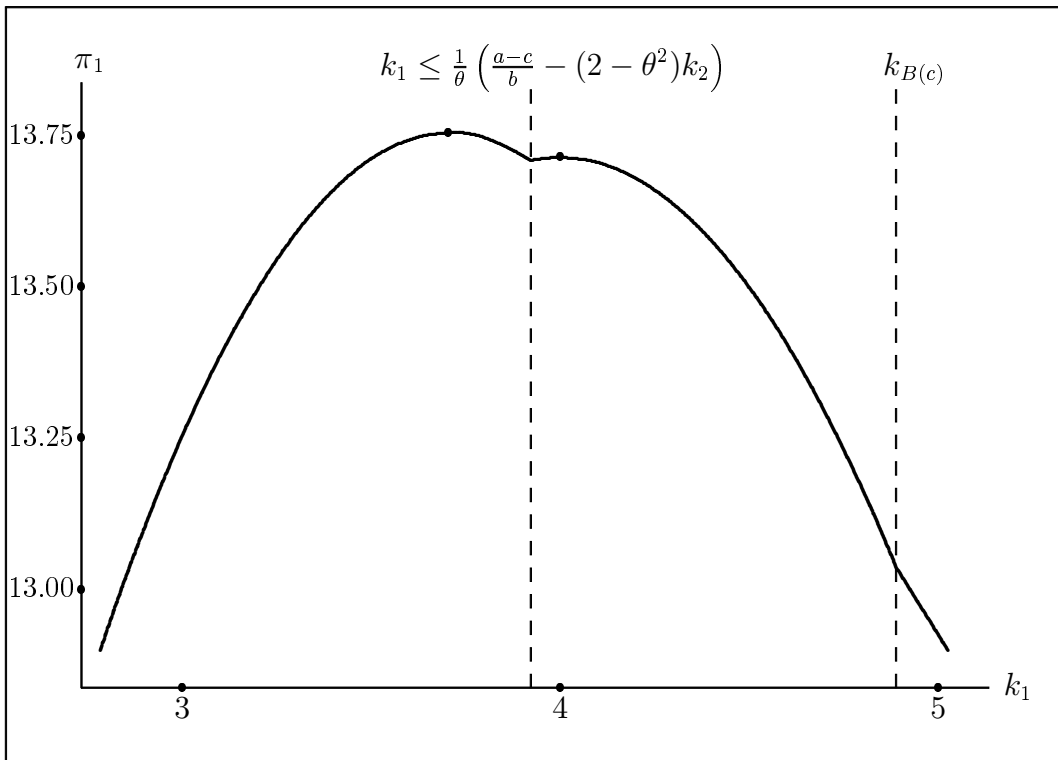Figure 11.5: Firm 1's profit function, $k_2 = 5.1469$

between k_{MMint} and k_S. The payoff function has two local maxima, and the global maximum is on the left.

Figure 11.6 shows firm 1's payoff function for $k_2 = 5.2089$, which is midway between k_S and k_{MLint}. The payoff function has two local maxima, and the global maximum is on the right.
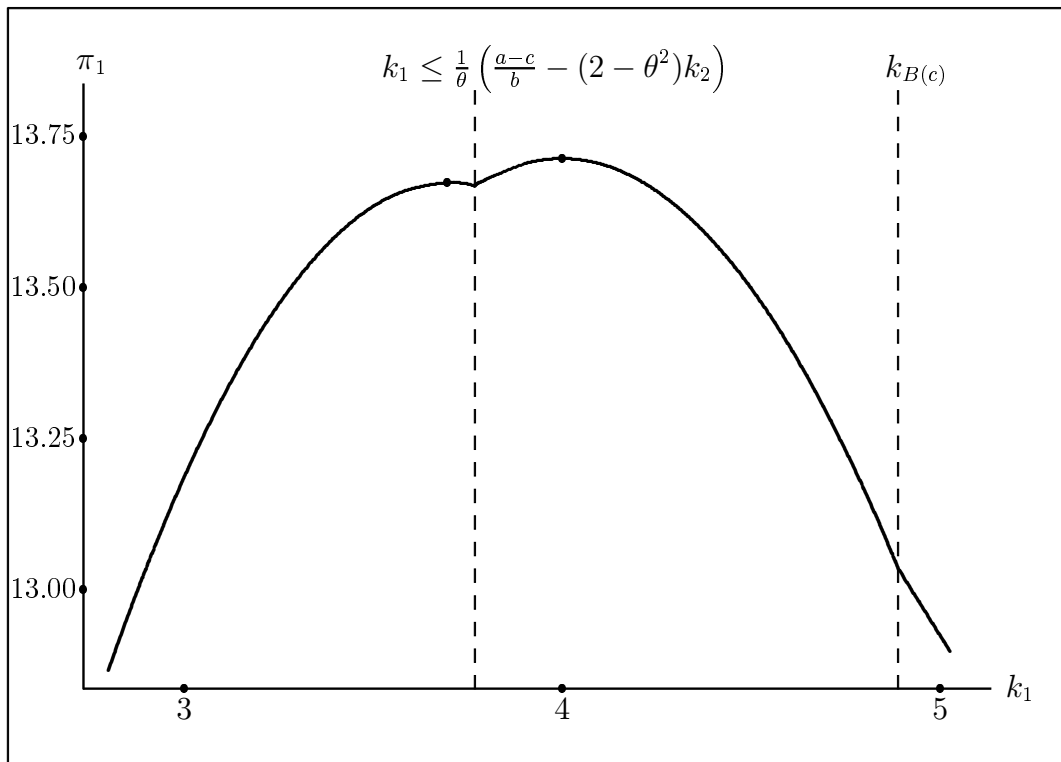
Figure 11.6: Firm 1's profit function, $k_2 = 5.2089$

k_A	$\frac{1}{1+\theta}\frac{a-c}{2b}$	$\frac{1}{1+\frac{1}{2}}\frac{12-1}{2(1)} = \frac{11}{3} = 3.6667$
k_D	$\frac{1}{2-\theta^2}\frac{a-c}{b}$	$\frac{1}{2-\frac{1}{4}}\frac{12-1}{1} = \frac{44}{7} = 6.2857$

Table 12.1: Critical capacity values

For the numerical example, at the switch point between the two branches of the capacity best response function, firm 1's best response capacity switches from

$$k_1 = \frac{1}{2}\left[\frac{12-1-1}{1} - \frac{1}{2}(5.1869)\right]$$

$$k_1 = 3.7033$$

to

$$k_2 = k_{1b2b1}^{br} = 4.$$

The capacity best response functions are shown in Figure 7.3. Figure 7.4 shows the price best response functions for the continuation game when the noncooperative equilibrium capacity levels are chosen in the first stage.

12. Numerical Example

The figures are drawn for a particular set of parameter values,

- $a = 12$ (price-axis intercept of the inverse demand curve)

- $b = 1$ (absolute value of the slope of the inverse demand curve)

- $c = 1$ (marginal production cost)

- $\rho = 1$ (cost per unit of capacity)

- $\theta = 1/2$ (product differentiation parameter).

The units in which capacity is measured are normalized so that one unit of capacity allows production of one unit of output.

The capacity levels k_A and k_D are evaluated for the numerical example in Table 12.1.

Values for the nodes of the segments of the price best response function are given in Table 12.2.

A_1	$\left(c + \frac{(1-\theta)(a-c)}{2}, c\right) = \left(\frac{15}{4}, 1\right)$
D_1	$\left(c + \frac{1-\theta^2}{2-\theta^2}(a-c), c + \frac{(1-\theta)(2+\theta)}{2-\theta^2}(a-c)\right) = \left(\frac{40}{7}, \frac{62}{7}\right)$
E_1	$\left(c + (1-\theta)(a-c) - b(1-\theta^2)k_1, c\right) = \left(\frac{13}{2} - \frac{3}{4}k_1, 1\right)$
F_1	$(a - bk_1, a - \theta bk_1) = \left(12 - k_1, 12 - \frac{1}{2}k_1\right)$
G_1	$\left(c + (1-\theta^2)bk_1, c + \frac{2(1-\theta^2)bk_1 - (1-\theta)(a-c)}{\theta}\right) = \left(1 + \frac{3}{4}k_1, -10 + 3k_1\right)$
H_1	$\left(c + \frac{1}{2}(a-c), c + \frac{1}{2}(2-\theta)(a-c)\right) = \left(\frac{13}{2}, \frac{37}{4}\right)$

Table 12.2: Nodes of firm 1's price best response function, numerical example one, numerical example one

k_1	$E(k_1)$	$F(k_1)$
2.75	$(4.4375, 1)$	$(9.25, 10.625)$
3.125	$(4.15625, 1)$	$(8.875, 10.4375)$
3.3	$(4.025, 1)$	$(8.7, 10.35)$

Table 12.3: Points $E(k_1)$ and $F(k_1)$, alternative values of k_1

The capacity limits for the various configurations of price best response functions are

$$k_i \leq k_A = 3.6667 \qquad \Rightarrow \qquad \text{(b2,b5) best response function}$$
$$3.6667 = k_A \leq k_i \leq q_{m(c)} = 5.5 \qquad \Rightarrow \qquad \text{(b1,b2,b5) best response function}$$
$$5.5 = q_{m(c)} \leq k_i \leq k_D = 6.2857 \qquad \Rightarrow \qquad \text{(b1,b2,b4,b5) best response function}$$
$$6.2857 = k_D \leq k_i \leq \frac{a-c-\rho}{b} = 10 \qquad \Rightarrow \qquad \text{(b1,b4,b5) best response function}$$

The (b2,b5) best response function arises for $k_1 \leq k_A = 3.6667$. The coordinates of the points in Figure 6.1 are given in Table 12.3.

For $3.6667 \leq k_1 \leq q_{m(c)} = 5.5$, firm 1's price best response function is of form (b1,b2,b5). The points used in Figure 6.1 are given in Table 12.4.

For $5.5 = q_{m(c)} \leq k_i \leq k_D = 6.2857$, firm 1's price best response function is of form (b1,b2,b4,b5).

k_1	A_1	$G(k_1)$	$F(k_1)$
4.00	$(3.75, 1)$	$(4, 2)$	$(8, 10)$
5.5	$(3.7, 1)$	$(5.125, 6.5)$	$(6.5, 9.25)$

Table 12.4: Points $F(k_1)$ and $G(k_1)$, alternative values of k_1

k_1	A_1	$G(k_1)$	$F(k_1)$	H_1
6.00	$(3.75, 1)$	$(5.5, 8)$	$(6, 9)$	$(6.5, 9.25)$
6.25	$(3.75, 1)$	$(5.6875, 8.75)$	$(5.75, 8.875)$	$(6.5, 9.25)$

Table 12.5: Points $F(k_1)$, $G(k_1)$, and $H(k_1)$, alternative values of k_1

For $6.2857 = k_D \leq k_i \leq \frac{a-c-\rho}{b} = 10$, firm 1's price best response function is of form (b1,b4,b5). The values for Figure 6.4 are

- $A_1 = (3.75, 1)$

- $D_1 = (5.7143, 8.8571)$

- $H_1 = (6.5, 9.25)$.

For the numerical example

$$k^*_{B(c)} = \frac{1}{(1+\theta)(2-\theta)} \frac{a-c}{b} = \frac{1}{(1+\frac{1}{2})(2-\frac{1}{2})} \frac{12-1}{1} = \frac{44}{9} = 4.8889.$$

(b1,b2) equilibrium
For the numerical example, prices are

$$p_{1b1b2} = c + \frac{1-\theta^2}{2-\theta^2}(a - c - \theta b k_2) = 1 + \frac{1-\frac{1}{4}}{2-\frac{1}{4}}\left((12-1) - \frac{1}{2}(4)\right) = 4.8571$$

$$p_{2b1b2} = c + \frac{(1-\theta)(2+\theta)(a-c) - 2(1-\theta^2)bk_1}{2-\theta^2}$$

$$= 1 + \frac{(1-\frac{1}{2})(2+\frac{1}{2})(12-1) - 2(1-\frac{1}{4})(4)}{2-\frac{1}{4}} = 5.4286.$$

In the numerical example, when $k_2 = 4$, the k_1 coordinate of the border between the $(b2, b2)$ and $(b1, b2)$ regions is

$$k_1 = \frac{1}{2-\frac{1}{4}}\left(\frac{12-1}{1} - \frac{1}{2}(4)\right) = \frac{36}{7} = 5.1429$$

and the value of profit at this point is 14.69388.

For the numerical example, the equation of the best response function over the range $0 \leq k_2 \leq \frac{44}{9} = 4.8889$ is

$$k_1 = \frac{1}{2}\left(\frac{12 - 1 - 1}{1} - \frac{1}{2}k_2\right)$$

$$k_1 = 5 - \frac{1}{4}k_2$$

Endpoints are $(0, 5)$ and $(3.7778, 4.8889)$.

For the numerical example, the top region is for

$$k_2 \geq \frac{1}{2 - \frac{1}{4}}\frac{12 - 1}{1} = \frac{44}{7} = 6.2857$$

and the value of (11.4) is

$$\frac{(1 - \frac{1}{2})(2 + \frac{1}{2})(12 - 1) - (2 - \frac{1}{4})(1)}{4(1 - \frac{1}{4})(1)} = 4.$$

For the numerical example, (11.14) is

$$k_2 \leq \frac{(2 - \frac{1}{2})(12 - 1) + \frac{1}{2}(1)}{(4 - 3\left(\frac{1}{4}\right))(1)} = \frac{68}{13} = 5.2308,$$

while the range of k_2 is

$$\frac{1}{(1 + \theta)(2 - \theta)}\frac{a - c}{b} = k_{B(c)}^* \leq k_2 \leq \frac{1}{2 - \theta^2}\frac{a - c}{b}$$

$$\frac{1}{(1 + \frac{1}{2})(2 - \frac{1}{2})}\frac{12 - 1}{1} \leq k_2 \leq \frac{1}{2 - \frac{1}{4}}\frac{12 - 1}{1}$$

$$\frac{44}{9} \leq k_2 \leq \frac{44}{7}$$

$$4.8889 \leq k_2 \leq 6.2857.$$

For

$$4.8889 \leq k_2 \leq 5.2308,$$

firm 1's payoff function has a local maximum at k_{1Cour}^{br},

$$k_{1Cour}^{br} = \frac{1}{2}\left(\frac{a - c - \rho}{b} - \theta k_2\right)$$

74

$$k_1(k_2) = 5 - \frac{1}{4}k_2$$

and its payoff at this capacity is

$$\frac{b}{4}\left(\frac{a-c-\rho}{b} - \theta k_2\right)^2$$

$$\frac{1}{16}(20 - k_2)^2.$$

For

$$\frac{68}{13} = 5.2308 \leq k_2 \leq 6.2857 = \frac{44}{7},$$

the global maximum on the range $0 \leq k_1 \leq \left[k_{B(c)}^{br}\right]^{-1}(k_2)$ occurs at $k_1 = \left[k_{B(c)}^{br}\right]^{-1}(k_2)$, and firm 1's payoff at this point is

$$\frac{1}{\theta b}\left[-\frac{1-\theta}{\theta}(a-c) - \rho + \frac{2}{\theta}(1-\theta^2)bk_2\right]\left[a-c-(2-\theta^2)bk_2\right] =$$

$$\frac{2(1-\theta^2)(2-\theta^2)}{\theta^2}b\left[k_2 - \frac{(1-\theta)(a-c)+\theta\rho}{2(1-\theta^2)b}\right]\left(\frac{1}{2-\theta^2}\frac{a-c}{b} - k_2\right).$$

For the numerical example, this is

$$\frac{1}{\left(\frac{1}{2}\right)(1)}\left[-\frac{1-\frac{1}{2}}{\frac{1}{2}}(12-1) - 1 + \frac{2}{\left(\frac{1}{2}\right)}(1-\frac{1}{4})(1)k_2\right]\left[12-1-(2-\frac{1}{4})(1)k_2\right]$$

$$= \frac{21}{2}(k_2 - 4)\left(\frac{44}{7} - k_2\right).$$

For the numerical example,

$$k_{MMint} = \frac{4+2\theta-\theta^2}{4(1+\theta)(2-\theta^2)}\frac{a-c}{b} + \frac{\theta}{4(1-\theta^2)}\frac{\rho}{b}$$

$$\frac{4+2\left(\frac{1}{2}\right)-\frac{1}{4}}{4(1+\left(\frac{1}{2}\right))(2-\frac{1}{4})}\frac{11}{1} + \frac{\left(\frac{1}{2}\right)}{4(1-\frac{1}{4})}\frac{1}{1}$$

$$= \frac{36}{7} = 5.1429.$$

$$k_{1Cour}^{br}(k_2) = 5 - \frac{1}{4}k_2$$

is the equation of firm 1's capacity best response function for $(0, 5)$ to $(3.7143, 5.1429)$.

$$k_{MLint} = \frac{(2 - \theta)(a - c) + \theta\rho}{(4 - 3\theta^2)b}$$

$$= \frac{(2 - \frac{1}{2})(12 - 1) + \frac{1}{2}(1)}{(4 - \frac{3}{4})(1)}$$

$$= \frac{68}{13} = 5.2308$$

and

$$k_1 = \frac{(1 - \frac{1}{2})(2 + \frac{1}{2})(12 - 1) - (2 - \frac{1}{4})(1)}{4(1 - \frac{1}{4})(1)} = 4$$

is firm 1's best response capacity for $k_2 \geq 5.2308$.

$$k_S = \frac{1}{\theta} \left\{ \frac{a - c - \rho}{b} - 2\sqrt{2\frac{1 - \theta^2}{2 - \theta^2} \left[\frac{(1 - \theta)(2 + \theta)(a - c) - (2 - \theta^2)\rho}{4(1 - \theta^2)b} \right]} \right\} \geq k_2.$$

Evaluate this for the numerical example:

$$\frac{1}{\left(\frac{1}{2}\right)} \left(\frac{12 - 1 - 1}{1} - 2\sqrt{2\frac{1 - \frac{1}{4}}{2 - \frac{1}{4}} \left(\frac{(1 - \frac{1}{2})(2 + \frac{1}{2})(12 - 1) - (2 - \frac{1}{4})(1)}{4(1 - \frac{1}{4})(1)} \right)} \right) \geq k_2$$

$$20 - \frac{16}{7}\sqrt{42} \geq k_2$$

$$5.1869 \geq k_2.$$

13. References

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