Sunk Cost and Entry: Appendix

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1 No depreciation, no resale

First (and highly stylized) example:

Linear inverse demand curve:

$$p = a - Q$$

Stone-Geary production function:

$$q = \min\left(\frac{K - \overline{K}}{a_K}, \frac{L - \overline{L}}{a_L}\right)$$

Capital does not depreciate and cannot be resold.

Present-discounted value of a monopoly supplier (who, facing unchanging demand conditions, will produce the same output in each period):

$$V_m = -p^k\left(\overline{K} + a_K q\right) + \left[(a-q)\,q - w\left(\overline{L} + a_L q\right)\right]\left[\frac{1}{1+r} + \frac{1}{(1+r)^2} + \right]\ldots$$

$$= \frac{\left(a - wa_L - rp^k a_K - q\right) q - \left(rp^k\overline{K} + w\overline{L}\right)}{r}$$

$$= \frac{(a - c_H - q)\,q - F_H}{r}$$

Monopoly output:

$$q_m = \frac{1}{2}(a - c_H) = \frac{1}{2}\left(a - wa_L - rp^k a_K\right)$$

If entry occurs, the entrant maximizes

$$V_2 = \frac{(a - c_H - q_1 - q_2)\,q_2 - F_H}{r}$$

Taking into account the fact the equilibrium duopoly output of a single firm is less than monopoly output, the incumbent, with surplus and non-depreciating capital, maximizes

$$V_1 = \frac{(a - c_L - q_1 - q_2)\,q_2 - F_L}{r},$$

for

$$c_L = wa_L$$

$$F_L = w\overline{L}$$

First-order conditions are

$$2q_1 + q_2 = a - c_L$$

$$q_1 + 2q_2 = a - c_H$$

and in the usual way, we find equilibrium outputs

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} a - c_L \\ a - c_H \end{pmatrix}$$

$$3 \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} a - c_L \\ a - c_H \end{pmatrix}$$

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} a - 2c_L + c_H \\ a + c_L - 2c_H \end{pmatrix}$$

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} a - 2wa_L + rp^k a_K + wa_L \\ a - 2rp^k a_K - 2wa_L + wa_L \end{pmatrix}$$

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} a - wa_L + rp^k a_K \\ a - wa_L - 2rp^k a_K \end{pmatrix}.$$

The entrant's equilibrium value is

$$V_2 = \frac{q_2^2 - F_H}{r}$$

$$= \frac{1}{r} \left[\frac{1}{9} \left(a - wa_L - 2rp^k a_K \right)^2 - F_H \right]$$

By analogy with the innovation literature, where a drastic innovation is one that makes the post-innovation equilibrium option of the successful innovator's rivals to shut down, we can call sunk costs drastic if

$$\frac{1}{9} \left(a - wa_L - 2rp^k a_K \right)^2 < F_H = rp^k \overline{K} + w\overline{L}.$$

In this case, entry is blocked: not because of the impact the need to make its own investment in sunk cost has on the decision of the entrant, but because of the impact entry would have on the economic costs of the incumbent. Further, if

$$\frac{1}{9} \left(a - rp^k a_K - wa_L \right)^2 \geq F_H > \frac{1}{9} \left(a - 2rp^k a_K - wa_L \right)^2,$$

entry is blocked because of the sunk nature of the investments made by the incumbent: if investments were not sunk, entry would be profitable.

2 Depreciation, no resale

Second (and less stylized) example: let capital depreciate at rate δ per period, $0 \leq \delta \leq 1$.

Monopoly:

$$V_m = -p^k \left(\overline{K} + a_K q\right) + \frac{1}{1+r}\left[(a-q)\,q - w\left(\overline{L} + a_L q\right)\right]$$

$$+\frac{1}{1+r}\left\{-\delta p^k \left(\overline{K} + a_K q\right) + \frac{1}{1+r}\left[(a-q)\,q - w\left(\overline{L} + a_L q\right)\right]\right\}$$

$$+\frac{1}{(1+r)^2}\left\{-\delta p^k \left(\overline{K} + a_K q\right) + \frac{1}{1+r}\left[(a-q)\,q - w\left(\overline{L} + a_L q\right)\right]\right\} + \ldots$$

(collecting terms that are multiplied by $1/(1+r)$, by $1/(1+r)^2$, etc.)

$$= -p^k \left(\overline{K} + a_K q\right) + \frac{1}{1+r}\left[(a-q)\,q - w\left(\overline{L} + a_L q\right) - \delta p^k \left(\overline{K} + a_K q\right)\right]$$

$$+\frac{1}{(1+r)^2}\left[(a-q) - w\left(\overline{L} + a_L q\right) - \delta p^k \left(\overline{K} + a_K q\right)\right] + \ldots$$

(adding up the infinite series)

$$= -p^k \left(\overline{K} + a_K q\right) + \frac{\left[a - q - w\left(\overline{L} + a_L q\right) - \delta p^k \left(\overline{K} + a_K q\right)\right]}{r}$$

$$= \frac{(a-q)\,q - w\left(\overline{L} + a_L q\right) - (r+\delta)\,p^k \left(\overline{K} + a_K q\right)}{r}$$

(separating out fixed costs)

$$= \frac{\left[a - w a_L - (r+\delta)\,p^k a_K - q\right] q - \left[w\overline{L} + (r+\delta)\,p^k \overline{K}\right]}{r}$$

$$= \frac{(a - c_H - q)\,q - F_H}{r},$$

redefining

$$c_H = w a_L + (r+\delta)\,p^k a_K$$

and

$$F_H = w\overline{L} + (r+\delta)\,p^k \overline{K}.$$

Monopoly output is

$$q_m = \frac{1}{2}(a - c_H) = \frac{1}{2}\left[a - wa_L - (r+\delta) p^k a_K\right]$$

and the incumbent purchases capital stock

$$K_m = \overline{K} + \frac{1}{2} a_K (a - c_H)$$

at the start of the first period.

Entry occurs.

Let μ_{1t} be the Lagrangian multiplier associated with the identify that defines the incumbent's period t capital stock,

$$K_{it} = (1-\delta) K_{1,t-1} + I_{1t}$$

(where $K_{10} = K_m$).

Let λ_{1t} be the Lagrangian multiplier associated with the period t capital input constraint,

$$K_{1t} \geq \overline{K} + a_K q_{1t}.$$

The Lagrangian that describes the incumbent's constrained optimization problem, from the moment of entry, is

$$V_1 =$$

$$\mu_{11}\left[(1-\delta) K_m + I_{11} - K_{11}\right] - p^k I_{11}$$

$$+\frac{1}{1+r}\left[(a - q_{11} - q_{21}) q_{11} - \left(wa_L q_{11} + w\overline{L}\right) + \lambda_{11}\left(K_{11} - \overline{K} - a_K q_{11}\right)\right]$$

$$+\frac{1}{1+r}\left\{\mu_{12}\left[(1-\delta) K_{11} + I_{12} - K_{12}\right] - p^k I_{12}\right.$$

$$\left.+\frac{1}{1+r}\left[(a - q_{12} - q_{22}) q_{12} - \left(wa_L q_{12} + w\overline{L}\right) + \lambda_{12}\left(K_{12} - \overline{K} - a_K q_{12}\right)\right]\right\}$$

$$+\frac{1}{(1+r)^2}\left\{\mu_{13}\left[(1-\delta) K_{12} + I_{13} - K_{13}\right] - p^k I_{13}\right.$$

$$\left.+\frac{1}{1+r}\left[(a - q_{13} - q_{23}) q_{13} - \left(wa_L q_{13} + w\overline{L}\right) + \lambda_{13}\left(K_{13} - \overline{K} - a_K q_{13}\right)\right]\right\} + \ldots$$

or, more compactly and writing $K_{10} = K_m$,

$$V_1 = \sum_{t=1}^{\infty} \frac{1}{(1+r)^{t-1}} \left\{ \mu_{1t} \left[(1-\delta) K_{1,t-1} + I_{1t} - K_{1t} \right] - p^k I_{1t} \right.$$

$$\left. + \frac{1}{1+r} \left[(a - q_{1t} - q_{2t}) q_{1t} - \left(w a_L q_{1t} + w \overline{L} \right) \right] + \lambda_{1t} \left(K_{12} - \overline{K} - a_K q_{1t} \right) \right\}.$$

Kuhn-Tucker first-order conditions for this constrained optimization problem are

$t = 1$:

$$\frac{\partial V_1}{\partial \mu_{11}} = (1-\delta) K_m + I_{11} - K_{11} = 0.$$

$$\frac{\partial V_1}{\partial I_{11}} = \mu_{11} - p^k \leq 0 \qquad I_{11} \left(\mu_{11} - p^k \right) \equiv 0 \qquad I_{11} \geq 0.$$

$$\frac{\partial V_1}{\partial K_{11}} = -\mu_{11} + \frac{\lambda_{11} + (1-\delta) \mu_{12}}{1+r} = 0.$$

$$(1+r) \frac{\partial V_1}{\partial q_{11}} = a - w a_L - \lambda_{11} a_K - 2 q_{11} - q_{21} = 0.$$

$$(1+r) \frac{\partial V_1}{\partial \lambda_{11}} = K_{11} - \overline{K} - a_K q_{11} \geq 0 \qquad \lambda_{11} \left(K_{11} - \overline{K} - a_K q_{11} \right) \qquad \lambda_{11} \geq 0.$$

$t = 2$:

$$(1+r) \frac{\partial V_1}{\partial \mu_{12}} = (1-\delta) K_{11} + I_{12} - K_{12} = 0.$$

$$(1+r) \frac{\partial V_1}{\partial I_{12}} = \mu_{12} - p^k \leq 0 \qquad I_{12} \left(\mu_{12} - p^k \right) \equiv 0 \qquad I_{12} \geq 0.$$

$$(1+r) \frac{\partial V_1}{\partial K_{12}} = -\mu_{12} + \frac{\lambda_{12} + (1-\delta) \mu_{13}}{1+r} = 0.$$

$$(1+r)^2 \frac{\partial V_1}{\partial q_{12}} = a - w a_L - \lambda_{12} a_K - 2 q_{12} - q_{22} = 0.$$

$$(1+r)^2 \frac{\partial V_1}{\partial \lambda_{12}} = K_{12} - \overline{K} - a_K q_{12} \geq 0 \qquad \lambda_{12} \left(K_{12} - \overline{K} - a_K q_{12} \right) \qquad \lambda_{12} \geq 0.$$

$t = 3$:

$$(1+r)^2 \frac{\partial V_1}{\partial \mu_{13}} = (1-\delta) K_{12} + I_{13} - K_{13} = 0.$$

$$(1+r)^2 \frac{\partial V_1}{\partial I_{13}} = \mu_{13} - p^k \leq 0 \qquad I_{13}\left(\mu_{13} - p^k\right) \equiv 0 \qquad I_{13} \geq 0.$$

$$(1+r)^2 \frac{\partial V_1}{\partial K_{13}} = -\mu_{13} + \frac{\lambda_{13} + (1-\delta)\,\mu_{14}}{1+r} = 0.$$

$$(1+r)^3 \frac{\partial V_1}{\partial q_{13}} = a - wa_L - \lambda_{13} a_K - 2q_{13} - q_{23} = 0.$$

$$(1+r)^3 \frac{\partial V_1}{\partial \lambda_{13}} = K_{13} - \overline{K} - a_K q_{13} \geq 0 \qquad \lambda_{13}\left(K_{13} - \overline{K} - a_K q_{13}\right) \qquad \lambda_{13} \geq 0.$$

(And so on.)

The entrant maximizes

$$V_2 = \mu_{21}\left(I_{21} - K_{21}\right) - p^k I_{21}$$

$$+\frac{1}{1+r}\left[\left(a - q_{11} - q_{12}\right) q_{21} - \left(wa_L q_{21} + w\overline{L}\right) + \lambda_{21}\left(K_{21} - \overline{K} - a_K q_{21}\right)\right]$$

$$+\frac{1}{1+r}\left\{\mu_{22}\left[(1-\delta) K_{21} + I_{22} - K_{22}\right] - p^k I_{22} +\right.$$

$$\left.\frac{1}{1+r}\left[\left(a - q_{12} - q_{22}\right) q_{22} - \left(wa_L q_{22} + w\overline{L}\right) + \lambda_{22}\left(K_{22} - \overline{K} - a_K q_{22}\right)\right]\right\}$$

$$+\frac{1}{(1+r)^2}\left\{\mu_{23}\left[(1-\delta) K_{22} + I_{23} - K_{23}\right] - p^k I_{23}\right.$$

$$\left.+\frac{1}{1+r}\left[\left(a - q_{13} - q_{23}\right) q_{23} - \left(wa_L q_{23} + w\overline{L}\right) + \lambda_{23}\left(K_{23} - \overline{K} - a_K q_{23}\right)\right]\right\} + \ldots$$

or, more compactly, with $K_{20} = 0$,

$$V_2 =$$

$$\sum_{t=1}^{\infty} \frac{1}{(1+r)^{t-1}} \left\{\mu_{2t}\left[(1-\delta) K_{2,t-1} + I_{2t} - K_{2t}\right] - p^k I_{2t}\right.$$

$$\left.+\frac{1}{1+r}\left[\left(a - q_{1t} - q_{2t}\right) q_{2t} - \left(wa_L q_{2t} + w\overline{L}\right)\right] + \lambda_{2t}\left(K_{2t} - \overline{K} - a_K q_{2t}\right)\right\}.$$

μ_{2t} is the Lagrangian multiplier for the identify that defines the capital stock in period t. λ_{2t} is the Lagrangian multiplier for the capital input constraint in period t.

Kuhn-Tucker first-order conditions:

$t = 1$:

$$(1+r)\frac{\partial V_2}{\partial \mu_{21}} = I_{21} - K_{21} = 0.$$

$$\frac{\partial V_2}{\partial I_{21}} = \mu_{21} - p^k \leq 0 \qquad I_{21}\left(\mu_{21} - p^k\right) = 0 \qquad I_{21} \geq 0.$$

$$\frac{\partial V_2}{\partial K_{21}} = -\mu_{21} + \frac{\lambda_{21} + (1-\delta)\,\mu_{22}}{1+r} = 0.$$

$$(1+r)\frac{\partial V_2}{\partial q_{21}} = a - wa_L - \lambda_{21}a_K - q_{11} - 2q_{21} = 0.$$

$$(1+r)\frac{\partial V_2}{\partial \lambda_{21}} = K_{21} - \overline{K} - a_K q_{21} \geq 0 \qquad \lambda_{21}\left(K_{21} - \overline{K} - a_K q_{21}\right) = 0 \qquad \lambda_{21} \geq 0.$$

$t = 2$:

$$(1+r)\frac{\partial V_2}{\partial \mu_{22}} = (1-\delta)\,K_{21} + I_{22} - K_{22} = 0.$$

$$(1+r)\frac{\partial V_2}{\partial I_{22}} = \mu_{22} - p^k \leq 0 \qquad I_{22}\left(\mu_{22} - p^k\right) = 0 \qquad I_{22} \geq 0.$$

$$(1+r)\frac{\partial V_2}{\partial K_{22}} = -\mu_{22} + \frac{\lambda_{22} + (1-\delta)\,\mu_{23}}{1+r} = 0..$$

$$(1+r)^2\frac{\partial V_2}{\partial q_{22}} = a - wa_L - \lambda_{22}a_K - q_{12} - 2q_{22} = 0.$$

$$(1+r)^2\frac{\partial V_2}{\partial \lambda_{22}} = K_{22} - \overline{K} - a_K q_{22} \geq 0 \qquad \lambda_{22}\left(K_{22} - \overline{K} - a_K q_{22}\right) = 0 \qquad \lambda_{22} \geq 0.$$

$t = 3$:

$$(1+r)^2\frac{\partial V_2}{\partial \mu_{23}} = (1-\delta)\,K_{22} + I_{23} - K_{23} \geq 0 \qquad \mu_{23}\left(I_{23} - K_{23}\right) = 0 \qquad \mu_{23} \geq 0.$$

$$(1+r)^2\frac{\partial V_2}{\partial I_{23}} = \mu_{23} - p^k \leq 0 \qquad I_{23}\left(\mu_{23} - p^k\right) = 0 \qquad I_{23} \geq 0.$$

$$(1+r)^2\frac{\partial V_2}{\partial K_{23}} = -\mu_{23} + \frac{\lambda_{23} + (1-\delta)\,\mu_{24}}{1+r} = 0.$$

$$(1+r)^3\frac{\partial V_2}{\partial q_{23}} = a - wa_L - \lambda_{23}a_K - q_{13} - 2q_{23} = 0.$$

$$(1+r)^3\frac{\partial V_2}{\partial \lambda_{23}} = K_{23} - \overline{K} - a_K q_{23} \geq 0 \qquad \lambda_{23}\left(K_{23} - \overline{K} - a_K q_{23}\right) = 0 \qquad \lambda_{23} \geq 0.$$

(And so on.)

We need to distinguish up to three types of time periods

- the first n_A periods, when incumbent can produce its equilibrium output $q_H = \frac{1}{3}\left[a - wa_L + (r+\delta)\, p^k a_K\right]$ without purchasing capital;
- periods n_B and after, when the incumbent must purchase capital to produce its equilibrium output $q_D = \frac{1}{3}\left[a - wa_L - (r+\delta)\, p^k a_K\right]$;
- if $n_B > n_A+1$, then for periods n_A+1, n_A+2,..., n_B-1 the incumbent produces just enough output to fully utilize the existing capital stock, but does not purchase capital.

How many such capital-constrained periods exist, if any, depends on δ. n_A is the greatest integer in the value n_1 that satisfies

$$(1-\delta)^{n_1}\left(\overline{K} + a_K q_m\right) = \overline{K} + a_K q_H$$

$$n_1 = \frac{\ln\left(\overline{K} + a_K q_H\right) - \ln\left[\overline{K} + a_K q_m\right]}{\ln(1-\delta)}.$$

n_B is the least integer that is greater than the value n_2 that satisfies

$$(1-\delta)^{n_2}\left(\overline{K} + a_K q_m\right) = \overline{K} + a_K q_D$$

$$n_2 = \frac{\ln\left[\overline{K} + a_K q_D\right] - \ln\left(\overline{K} + a_K q_m\right)}{\ln(1-\delta)}.$$

Remark:

$$n_2 - n_1 =$$

$$\frac{\ln\left[\overline{K} + a_K q_D\right] - \ln\left(\overline{K} + a_K q_m\right)}{\ln(1-\delta)} - \frac{\ln\left(\overline{K} + a_K q_H\right) - \ln\left[\overline{K} + a_K q_m\right]}{\ln(1-\delta)} =$$

$$\frac{\ln\left(\overline{K} + a_K q_H\right) - \ln\left[\overline{K} + a_K q_D\right]}{-\ln(1-\delta)} > 0,$$

noting that $\ln(1-\delta) < 0$.

2.1 Example

Let $w = a_L = a_K = p^k = 1$, $r = \delta = 1/10$, $\overline{K} = 1$, $a = 100$.

$$c_H = 1 + \left(\frac{1}{10} + \frac{1}{10}\right) = \frac{6}{5}$$

$$\widehat{n}_A = \frac{\ln\left(\overline{K} + \frac{1}{3}a_K\left(a - wa_L + (r+\delta)\,p^k a_K\right)\right) - \ln\left(\overline{K} + \frac{1}{2}a_K\,(a - c_H)\right)}{\ln(1-\delta)} =$$

$$\frac{\ln\left(1 + \frac{1}{3}(1)\left(100 - (1)(1) + \left(\frac{1}{10} + \frac{1}{10}\right)(1)(1)\right)\right) - \ln\left(1 + \frac{1}{2}(1)\left(100 - \frac{6}{5}\right)\right)}{\ln\left(1 - \frac{1}{10}\right)} = 3.7174$$

$$\widehat{n}_B = \frac{\ln\left[\overline{K} + \frac{1}{3}a_K\,(a - c_H)\right] - \ln\left[\overline{K} + \frac{1}{2}a_K\,(a - c_H)\right]}{\ln(1-\delta)} =$$

$$\frac{\ln\left(1 + \frac{1}{3}(1)\left(100 - \frac{6}{5}\right)\right) - \ln\left(1 + \frac{1}{2}(1)\left(100 - \frac{6}{5}\right)\right)}{\ln\left(1 - \frac{1}{10}\right)} = 3.7547$$

Monopoly output and capital stock:

$$q_m = \frac{1}{2}\left(100 - \frac{6}{5}\right) = \frac{247}{5}$$

$$K_m = 1 + \frac{247}{5} = \frac{252}{5} = 50.4.$$

Duopoly outputs in first phase:

$$\begin{pmatrix} q_{1t} \\ q_{2t} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} a - wa_L + (r+\delta)\,p^k a_K \\ a - wa_L - 2\,(r+\delta)\,p^k a_K \end{pmatrix}$$

$$\begin{pmatrix} q_{1t} \\ q_{2t} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 100 - 1 + \left(\frac{1}{10} + \frac{1}{10}\right)(1)(1) \\ 100 - 1 - 2\left(\frac{1}{10} + \frac{1}{10}\right)(1)(1) \end{pmatrix} = \begin{pmatrix} \frac{496}{15} \\ \frac{493}{15} \end{pmatrix} = \begin{pmatrix} 33.067 \\ 32.867 \end{pmatrix}$$

$$\begin{pmatrix} \frac{496}{15} \\ \frac{493}{15} \end{pmatrix}$$

Capital stocks:

$$\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{496}{15} \\ \frac{493}{15} \end{pmatrix} = \begin{pmatrix} \frac{511}{15} \\ \frac{508}{15} \end{pmatrix} = \begin{pmatrix} 34.067 \\ 33.867 \end{pmatrix}.$$

For firm 1, this is the minimum capital stock required to produce its equilibrium output.

Duopoly outputs in second phase:

$$\frac{1}{3}\left(100 - 1 - \left(\frac{1}{10} + \frac{1}{10}\right)(1)\right) = \frac{494}{15} = 32.933.$$

Capital stocks in second phase:

$$1 + \frac{494}{15} = \frac{509}{15} = 33.933$$

Incumbent's capital stock

$$\begin{array}{lr} 0 & \frac{252}{5} = 50.4 \\ 1 & \left(1 - \frac{1}{10}\right) \frac{252}{5} = 45.36 \\ 2 & \left(1 - \frac{1}{10}\right)^2 \frac{252}{5} = 40.824 \\ 3 & \left(1 - \frac{1}{10}\right)^3 \frac{252}{5} = 36.742 \\ 4 & \left(1 - \frac{1}{10}\right)^4 \frac{252}{5} = 33.067 \end{array}$$

For these parameters, there is no "capital constrained" transition phase. The incumbent has excess capital for periods 1, 2, and 3, and purchases capital in period 4.

If there is a capital constrained case, then during those periods, the shadow value of a unit of capital is positive, but less than the cost of a unit of capital.

3 $n_B = n_A + 1$

Consider the case $n_B = n_A + 1$.

Write out Kuhn-Tucker for periods n_A and n_B.

Incumbent:

n_A:

$$(1+r)^{n_A-1} \frac{\partial V_1}{\partial \mu_{1n_A}} = (1-\delta) K_{1,n_A-1} + I_{1n_A} - K_{1n_A} = 0.$$

$$(1+r)^{n_A-1} \frac{\partial V_1}{\partial I_{1n_A}} = \mu_{1n_A} - p^k \leq 0 \qquad I_{1n_A} \left(\mu_{1n_A} - p^k\right) \equiv 0 \qquad I_{1n_A} \geq 0.$$

$$(1+r)^{n_A-1} \frac{\partial V_1}{\partial K_{1n_A}} = -\mu_{1n_A} + \frac{\lambda_{1n_A} + (1-\delta) \mu_{1n_B}}{1+r} = 0.$$

$$(1+r)^{n_A} \frac{\partial V_1}{\partial q_{1n_A}} = a - wa_L - \lambda_{1n_A} a_K - 2q_{1n_A} - q_{2n_A} = 0.$$

$$(1+r)^{n_A} \frac{\partial V_1}{\partial \lambda_{1n_A}} = K_{1n_A} - \overline{K} - a_K q_{1n_A} \geq 0 \qquad \lambda_{1n_A} \left(K_{1n_A} - \overline{K} - a_K q_{1n_A}\right) \qquad \lambda_{1n_A} \geq 0.$$

n_B:

$$(1+r)^{n_A}\frac{\partial V_1}{\partial \mu_{1n_B}} = (1-\delta)\,K_{1n_A} + I_{1n_B} - K_{1n_B} = 0.$$

$$(1+r)^{n_A}\frac{\partial V_1}{\partial I_{1n_B}} = \mu_{1n_B} - p^k \leq 0 \qquad I_{1n_B}\left(\mu_{1n_B} - p^k\right) \equiv 0 \qquad I_{1n_B} \geq 0.$$

$$(1+r)^{n_A-1}\frac{\partial V_1}{\partial K_{1n_B}} = -\mu_{1n_B} + \frac{\lambda_{1n_B} + (1-\delta)\,\mu_{1,n_B+1}}{1+r} = 0.$$

$$(1+r)^{n_A}\frac{\partial V_1}{\partial q_{1n_B}} = a - wa_L - \lambda_{1n_B}a_K - 2q_{1n_B} - q_{2n_A} = 0.$$

$$(1{+}r)^{n_A}\frac{\partial V_1}{\partial \lambda_{1n_B}} = K_{1n_B} - \overline{K} - a_K q_{1n_B} \geq 0 \qquad \lambda_{1n_B}\left(K_{1n_B} - \overline{K} - a_K q_{1n_B}\right) \qquad \lambda_{1n_B} \geq 0.$$

That firm 1's investment in periods n_B and n_B+1 is positive allows us to determine μ_{1n_B} and μ_{1n_B+1}:

$$I_{1n_B} > 0 \Rightarrow \mu_{1n_B} = p^k \qquad I_{1,n_B+1} > 0 \Rightarrow \mu_{1,n_B+1} = p^k.$$

Knowing μ_{1n_B} and μ_{1,n_B+1} gives λ_{1n_B}, and by the same sort of argument this is the value of λ_{1t} for all $t \geq n_B$:

$$-\mu_{1n_B} + \frac{\lambda_{1n_B} + (1-\delta)\,\mu_{1,n_B+1}}{1+r} = 0$$

$$-(1+r)\,p^k + \lambda_{1n_B} + (1-\delta)\,p^k = 0$$

$$\lambda_{1n_B} = (r+\delta)\,p^k.$$

In period n_A, capital is still in excess supply,

$$K_{1n_A} - \overline{K} - a_K q_{1n_A} > 0 \Rightarrow \lambda_{1n_A} = 0;$$

by the same argument,

$$\lambda_{1t} = 0 \;\forall t \leq n_A.$$

We then have the first-order condition for the incumbent's output for period n_A, with corresponding expressions for all periods before n_A:

$$2q_{1n_A} + q_{2n_A} = a - wa_L.$$

We also have the first-order condition for the incumbent's output for period n_B, with corresponding expressions for all periods after period n_B:

$$2q_{1n_B} + q_{2n_B} = a - wa_L - (r+\delta)\, p^k a_K.$$

Knowing that $\lambda_{1n_A} = 0$ and $\mu_{1n_B} = p^k$, we are able to find μ_{1n_A}:

$$\mu_{1n_A} = \frac{\lambda_{1n_A} + (1-\delta)\,\mu_{1n_B}}{1+r} = \frac{0 + (1-\delta)\, p^k}{1+r} = \frac{1-\delta}{1+r} p^k.$$

A consistency condition for this solution to be valid is

$$\frac{1-\delta}{1+r} p^k - p^k \leq 0$$

$$1 \geq \frac{1-\delta}{1+r}$$

and this condition evidently is met.

The Kuhn-Tucker condition for firm 1's capital stock for period $n_A - 1$ is

$$(1+r)^{n_A-2} \frac{\partial V_1}{\partial K_{1,n_A-1}} = -\mu_{1,n_A-1} + \frac{\lambda_{1,n_A-1} + (1-\delta)\,\mu_{1n_A}}{1+r} = 0.$$

We know that $\lambda_{1,n_A-1} = 0$ and that $\mu_{1n_A} = (1-\delta)\, p^k$; hence

$$-\mu_{1,n_A-1} + \frac{0 + (1-\delta)\, \frac{1-\delta}{1+r} p^k}{1+r} = 0$$

$$\mu_{1,n_A-1} = \left(\frac{1-\delta}{1+r}\right)^2 p^k.$$

In the same way,

$$\mu_{1,n_A-2} = \frac{0 + (1-\delta)\left(\frac{1-\delta}{1+r}\right)^2 p^k}{1+r} = \left(\frac{1-\delta}{1+r}\right)^3 p^k ...$$

$$\mu_{11} = \mu_{1,n_A-(n_A-1)} = \left(\frac{1-\delta}{1+r}\right)^{n_A} p^k.$$

For periods $t = 1, 2, ..., n_A - 1,\ n_A$.

$$\mu_{1t} = \left(\frac{1-\delta}{1+r}\right)^{n_A+1-t} p^k.$$

Entrant:

n_A:

$$(1+r)^{n_A-1}\frac{\partial V_2}{\partial \mu_{2n_A}} = (1-\delta)\,K_{2,n_A-1} + I_{2n_A} - K_{2n_A} = 0.$$

$$(1+r)^{n_A-1}\frac{\partial V_2}{\partial I_{2n_A}} = \mu_{2n_A} - p^k \leq 0 \qquad I_{2n_A}\left(\mu_{2n_A} - p^k\right) = 0 \qquad I_{2n_A} \geq 0.$$

$$(1+r)^{n_A-1}\frac{\partial V_2}{\partial K_{2n_A}} = -\mu_{2n_A} + \frac{\lambda_{2n_A} + (1-\delta)\,\mu_{2n_B}}{1+r} = 0.$$

$$(1+r)^{n_A}\frac{\partial V_2}{\partial q_{2n_A}} = a - wa_L - \lambda_{2n_A}a_K - q_{1n_A} - 2q_{2n_A} = 0.$$

$$(1+r)^{n_A}\frac{\partial V_2}{\partial \lambda_{2n_A}} = K_{2n_A} - \overline{K} - a_K q_{2n_A} \geq 0 \qquad \lambda_{2n_A}\left(K_{2n_A} - \overline{K} - a_K q_{2n_A}\right) = 0 \qquad \lambda_{2n_A} \geq 0.$$

n_B:

$$(1+r)^{n_A}\frac{\partial V_2}{\partial \mu_{2n_B}} = (1-\delta)\,K_{2,n_B-1} + I_{2n_B} - K_{2n_B} = 0.$$

$$(1+r)^{n_A}\frac{\partial V_2}{\partial I_{2n_B}} = \mu_{2n_B} - p^k \leq 0 \qquad I_{2n_B}\left(\mu_{2n_B} - p^k\right) = 0 \qquad I_{2n_B} \geq 0.$$

$$(1+r)^{n_A}\frac{\partial V_2}{\partial K_{2n_B}} = -\mu_{2n_B} + \frac{\lambda_{2n_B} + (1-\delta)\,\mu_{2,n_B+1}}{1+r} = 0.$$

$$(1+r)^{n_B}\frac{\partial V_2}{\partial q_{2n_B}} = a - wa_L - \lambda_{2n_B}a_K - q_{1n_A} - 2q_{2n_B} = 0.$$

$$(1+r)^{n_B}\frac{\partial V_2}{\partial \lambda_{2n_B}} = K_{2n_B} - \overline{K} - a_K q_{2n_B} \geq 0 \qquad \lambda_{2n_B}\left(K_{2n_B} - \overline{K} - a_K q_{2n_B}\right) = 0 \qquad \lambda_{2n_B} \geq 0.$$

The solution for the entrant is more straightforward, since the entrant makes positive investment in every period.

$$\mu_{2t} = p^k \;\forall t.$$

$$\lambda_{2t} = (r+\delta)\,p^k \;\forall t.$$

$$q_{1t} + 2q_{2n_B} = a - wa_L - \lambda_{2t}a_K = a - wa_L - (r+\delta)\,p^k a_K.$$

3.1 $n_B = n_A + 2$

Consider briefly the case $n_B = n_A + 2$, so there is one period in which the incumbent is capital constrained.

n_A:

$$(1+r)^{n_A-1}\frac{\partial V_1}{\partial \mu_{1n_A}} = (1-\delta)K_{1,n_A-1} + I_{1n_A} - K_{1n_A} = 0.$$

$$(1+r)^{n_A-1}\frac{\partial V_1}{\partial I_{1n_A}} = \mu_{1n_A} - p^k \leq 0 \qquad I_{1n_A}\left(\mu_{1n_A} - p^k\right) \equiv 0 \qquad I_{1n_A} \geq 0.$$

$$(1+r)^{n_A-1}\frac{\partial V_1}{\partial K_{1n_A}} = -\mu_{1n_A} + \frac{\lambda_{1n_A} + (1-\delta)\mu_{1n_B}}{1+r} = 0.$$

$$(1+r)^{n_A}\frac{\partial V_1}{\partial q_{1n_A}} = a - wa_L - \lambda_{1n_A}a_K - 2q_{1n_A} - q_{2n_A} = 0.$$

$$(1+r)^{n_A}\frac{\partial V_1}{\partial \lambda_{1n_A}} = K_{1n_A} - \overline{K} - a_K q_{1n_A} \geq 0 \qquad \lambda_{1n_A}\left(K_{1n_A} - \overline{K} - a_K q_{1n_A}\right) \qquad \lambda_{1n_A} \geq 0.$$

$n_A + 1$:

$$(1+r)^{n_A}\frac{\partial V_1}{\partial \mu_{1,n_A+1}} = (1-\delta)K_{1,n_A} + I_{1,n_A+1} - K_{1,n_A+1} = 0.$$

$$(1+r)^{n_A}\frac{\partial V_1}{\partial I_{1,n_A+1}} = \mu_{1,n_A+1} - p^k \leq 0 \qquad I_{1,n_A+1}\left(\mu_{1,n_A+1} - p^k\right) \equiv 0 \qquad I_{1,n_A+1} \geq 0.$$

$$(1+r)^{n_A}\frac{\partial V_1}{\partial K_{1,n_A+1}} = -\mu_{1,n_A+1} + \frac{\lambda_{1,n_A+1} + (1-\delta)\mu_{1,n_A+2}}{1+r} = 0.$$

$$(1+r)^{n_A+1}\frac{\partial V_1}{\partial q_{1,n_A+1}} = a - wa_L - \lambda_{1,n_A+1}a_K - 2q_{1,n_A+1} - q_{2n_A} = 0.$$

$$(1+r)^{n_A+1}\frac{\partial V_1}{\partial \lambda_{1,n_A+1}} = K_{1,n_A+1} - \overline{K} - a_K q_{1,n_A+1} \geq 0.$$

$$\lambda_{1,n_A+1}\left(K_{1,n_A+1} - \overline{K} - a_K q_{1,n_A+1}\right) \qquad \lambda_{1,n_A+1} \geq 0.$$

In period $n_A + 1$, the incumbent's output is determined by its capital stock, i.e.

$$q_{1,n_A+1} = \frac{(1-\delta)^{n_A+1}K_m - \overline{K}}{a_K}.$$

The first-order condition for the entrant's output in period $n_A + 1$ is

$$q_{1,n_A+1} + 2q_{2,n_A+1} = a - wa_L - (r + \delta)\, a_K.$$

Since we know the incumbent's output, we can solve for the entrant's output:

$$q_{2,n_A+1} = \frac{1}{2}\left[a - wa_L - (r + \delta)\, a_K - \frac{(1-\delta)^{n_A} K_m - \overline{K}}{a_K}\right].$$

Knowing the entrant's output, we can solve the first-order condition for the incumbent's output for λ_{1,n_A+1}. First evaluate

$$2q_{1,n_A+1} + q_{2,n_A+1} =$$

$$2q_{1,n_A+1} + \frac{1}{2}\left[a - wa_L - (r + \delta)\, a_K - q_{1,n_A+1}\right] =$$

$$\frac{3}{2}q_{1,n_A+1} + \frac{1}{2}\left[a - wa_L - (r + \delta)\, a_K\right] =$$

$$\frac{3}{2}\frac{(1-\delta)^{n_A} K_m - \overline{K}}{a_K} + \frac{1}{2}\left[a - wa_L - (r + \delta)\, a_K\right]$$

Then substitute in

$$a - wa_L - \lambda_{1,n_A+1} a_K - 2q_{1,n_A+1} - q_{2,n_A+1} = 0.$$

$$\lambda_{1,n_A+1} = a - wa_L - (2q_{1,n_A+1} + q_{2,n_A+1})$$

$$= a - wa_L - \left\{\frac{3}{2}\frac{(1-\delta)^{n_A} K_m - \overline{K}}{a_K} + \frac{1}{2}\left[a - wa_L - (r + \delta)\, a_K\right]\right\}$$

$$= \frac{3}{2}(a - wa_L) - \left\{\frac{3}{2}\frac{(1-\delta)^{n_A} K_m - \overline{K}}{a_K} - \frac{1}{2}(r + \delta)\, a_K\right\}$$

$$= \frac{3}{2}\left[a - wa_L - \frac{(1-\delta)^{n_A} K_m - \overline{K}}{a_K}\right] + \frac{1}{2}(r + \delta)\, a_K$$

A consistency condition for this solution to be valid is that this value of λ_{1,n_A+1} must be positive.

Knowing λ_{1,n_A+1} and knowing that

$$\mu_{1n_B} = p^k$$

we can solve the Kuhn-Tucker condition for K_{1,n_A+1}

$$(1+r)^{n_A}\frac{\partial V_1}{\partial K_{1,n_A+1}}=-\mu_{1,n_A+1}+\frac{\lambda_{1,n_A+1}+(1-\delta)\,\mu_{1,n_A+2}}{1+r}=0$$

for μ_{1,n_A+1}:

$$\begin{aligned}\mu_{1,n_A+1}&=\frac{\lambda_{1,n_A+1}+(1-\delta)\,\mu_{1,n_A+2}}{1+r}\\&=\frac{\lambda_{1,n_A+1}+(1-\delta)\,p^k}{1+r}.\end{aligned}$$

Knowing equilibrium values for period n_A+1, we can solve backward and forward in time for equilibrium values in other periods. For the first n_A periods, the incumbent does not purchase capital and is not capital constrained; its rental cost of capital services is zero. In and after period n_B, the incumbent purchases capital and its rental cost of capital services is $(r+\delta)\,p^k$.

If there is more than one period between periods n_A and n_B, solve first for equilibrium values in the capital constrained periods, beginning with period n_B-1 and working backward to period n_A+1. This will yield consistency conditions that must be satisfied for the solution to be valid, and it will permit finding the equilibrium values for earlier and later periods.

4 Entrant's value, $n_B=n_A+1$

Return to the case $n_B=n_A+1$. The solution is that for the first n_A periods,

$$\begin{pmatrix} q_{1t}\\ q_{2t}\end{pmatrix}=\frac{1}{3}\begin{pmatrix} a-wa_L+(r+\delta)\,p^k a_K\\ a-wa_L-2\,(r+\delta)\,p^k a_K\end{pmatrix}$$

$$\begin{pmatrix} \lambda_{1t}\\ \lambda_{2t}\end{pmatrix}=\begin{pmatrix} 0\\ (r+\delta)\,p^k\end{pmatrix}$$

$$\begin{pmatrix} \mu_{1t}\\ \mu_{2t}\end{pmatrix}=\begin{pmatrix} \left(\frac{1-\delta}{1+r}\right)^{n_A+1-t}\\ 1\end{pmatrix}p^k$$

$$\begin{pmatrix} I_{11}\\ I_{21}\end{pmatrix}=\begin{pmatrix} 0\\ \overline{K}+a_K q_{20}\end{pmatrix}$$

$$\begin{pmatrix} I_{1t}\\ I_{2t}\end{pmatrix}=\begin{pmatrix} 0\\ \delta\left(\overline{K}+a_K q_{2t}\right)\end{pmatrix},t=2,3,...,n_A;$$

after which

$$q_{1t} = q_{2t} = \frac{1}{3}\left(a - c_H\right)$$

$$\lambda_{1t} = \lambda_{2t} = \left(r + \delta\right) p^k$$

$$\mu_{1t} = \mu_{2t} = p^k$$

$$\begin{pmatrix} I_{1n} \\ I_{2n} \end{pmatrix} =$$

$$\begin{pmatrix} \overline{K} + \frac{1}{3}a_K\left[a - wa_L - \left(r+\delta\right)a_K\right] - \left(1-\delta\right)^n\left[\overline{K} + \frac{1}{2}a_K\left(a - c_H\right)\right] \\ \overline{K} + \frac{1}{3}a_K\left[a - wa_L - \left(r+\delta\right)a_K\right] - \left(1-\delta\right)\left[\overline{K} + \frac{1}{3}a_K\left(a - wa_L - 2\left(r+\delta\right)p^k a_K\right)\right] \end{pmatrix}$$

$$I_{1t} = I_{2t} = \delta\left(\overline{K} + a_K q_{2t}\right),\ t = n_A + 2, n+3, ...$$

Now find the entrant's equilibrium value, noting that the expressions multiplied by Lagrangian multipliers drop out by the Kuhn-Tucker conditions:

$$V_2 = -p^k I_{21} + \frac{1}{1+r}\left[\left(a - wa_L - q_{11} - q_{21}\right)q_{21} - w\overline{L}\right]$$

$$+\frac{1}{1+r}\left\{-p^k I_{22} + \frac{1}{1+r}\left[\left(a - wa_L - q_{12} - q_{22}\right)q_{22} - w\overline{L}\right]\right\}$$

$$+\frac{1}{\left(1+r\right)^2}\left\{-p^k I_{23} + \frac{1}{1+r}\left[\left(a - wa_L - q_{13} - q_{23}\right)q_{23} - w\overline{L}\right]\right\} + ...$$

$$+\frac{1}{\left(1+r\right)^{n_A}}\left\{-p^k I_{2,n_A} + \frac{1}{1+r}\left[\left(a - wa_L - q_{1,n_A} - q_{2,n_A}\right)q_{2,n_A} - w\overline{L}\right]\right\} +$$

$$+\frac{1}{\left(1+r\right)^{n_A+1}}\left\{-p^k I_{2,n_A+1} + \frac{1}{1+r}\left[\left(a - wa_L - q_{1,n_A+1} - q_{2,n_A+1}\right)q_{2,n_A+1} - w\overline{L}\right]\right\} +$$

$$+\frac{1}{\left(1+r\right)^{n_A+2}}\left\{-p^k I_{2,n_A+2} + \frac{1}{1+r}\left[\left(a - wa_L - q_{1,n_A+2} - q_{2,n_A+2}\right)q_{2,n_A+2} - w\overline{L}\right]\right\}+...$$

(expressing investment in terms of the capital stock)

$$= -p^k\left[\overline{K} + a_K q_L\right] + \frac{1}{1+r}\left[\left(a - wa_L - q_{11} - q_{21}\right)q_{21} - w\overline{L}\right]$$

$$+\frac{1}{1+r}\left\{-\delta p^k\left[\overline{K} + a_K q_L\right] + \frac{1}{1+r}\left[\left(a - wa_L - q_{12} - q_{22}\right)q_{22} - w\overline{L}\right]\right\}$$

$$+\frac{1}{(1+r)^2}\left\{-\delta p^k\left[\overline{K}+a_K q_L\right]+\frac{1}{1+r}\left\{\left[(a-wa_L-q_{13}-q_{23})\,q_{23}-w\overline{L}\right]\right\}\right\}+...$$

$$+\frac{1}{(1+r)^{n_A}}\left\{-\delta p^k\left[\overline{K}+a_K q_L\right]+\frac{1}{1+r}\left\{\left[(a-wa_L-q_{1,n_A}-q_{2,n_A})\,q_{2,n_A}-w\overline{L}\right]\right\}\right\}$$

$$+\frac{1}{(1+r)^{n_A+1}}\left\{-p^k\left[\overline{K}+a_K q_H-(1-\delta)\left(\overline{K}+a_K q_L\right)\right]\right.$$

$$\left.+\frac{1}{1+r}\left[(a-wa_L-q_{1,n_A+1}-q_{2,n_A+1})\,q_{2,n_A+1}-w\overline{L}\right]\right\}$$

$$+\frac{1}{(1+r)^{n_A+2}}\left\{-\delta p^k\left[\overline{K}+a_K q_H\right]+\frac{1}{1+r}\left\{\left[(a-wa_L-q_{1,n_A+2}-q_{2,n_A+2})\,q_{2,n_A+2}-w\overline{L}\right]\right\}\right\}+...$$

(collecting terms multiplied by $1/(1+r)$, by $1/(1+r)^2$, and so on)

$$=-p^k\left[\overline{K}+a_K q_L\right]+\frac{1}{1+r}\left[\left(a-wa_L-\delta p^k a_K-q_H-q_L\right)q_L-w\overline{L}-\delta p^k\overline{K}\right]$$

$$+\frac{1}{(1+r)^2}\left[\left(a-wa_L-\delta p^k a_K-q_H-q_L\right)q_L-w\overline{L}-\delta p^k\overline{K}\right]+...$$

$$+\frac{1}{(1+r)^{n_A+1}}\left[\left(a-wa_L-\delta p^k a_K-q_H-q_L\right)q_L-w\overline{L}-\delta p^k\overline{K}\right]$$

$$+\frac{1}{(1+r)^{n_A+1}}p^k a_K\left(q_D-q_L\right)$$

$$+\frac{1}{(1+r)^{n_A+2}}\left[\left(a-wa_L-\delta p^k a_K-q_D-q_D\right)q_D-w\overline{L}-\delta p^k\overline{K}\right]+$$

$$+\frac{1}{(1+r)^{n_A+3}}\left[\left(a-wa_L-\delta p^k a_K-q_D-q_D\right)q_D-w\overline{L}-\delta p^k\overline{K}\right]+...$$

(factoring where possible)

$$=-p^k\left[\overline{K}+a_K q_L\right]$$

$$+\left[\frac{1}{1+r}+...+\frac{1}{(1+r)^{n_A+1}}\right]\left[\left(a-wa_L-\delta p^k a_K-q_H-q_L\right)q_L-w\overline{L}-\delta p^k\overline{K}\right]$$

$$+\frac{1}{(1+r)^{n_A+1}}p^k a_K\left(q_D-q_L\right)$$

$$+\left[\frac{1}{(1+r)^{n_A+2}}+...\right]\left[\left(a-wa_L-\delta p^k a_K-q_D-q_D\right)q_D-w\overline{L}-\delta p^k\overline{K}\right]$$

(making summations)

$$=-\frac{1}{r}rp^k\left[\overline{K}+a_K q_L\right]\left[1-\frac{1}{(1+r)^{n_A+1}}+\frac{1}{(1+r)^{n_A+1}}\right]$$

$$+\frac{1}{r}\left[1-\frac{1}{(1+r)^{n_A+1}}\right]\left[\left(a-wa_L-\delta p^k a_K-q_H-q_L\right)q_L-w\overline{L}-\delta p^k\overline{K}\right]$$

$$+\frac{1}{(1+r)^{n_A+1}}p^k a_K\left(q_D-q_L\right)$$

$$+\frac{1}{(1+r)^{n_A+1}}\frac{1}{r}\left[\left(a-wa_L-\delta p^k a_K-q_D-q_D\right)q_D-w\overline{L}-\delta p^k\overline{K}\right]$$

(rearranging terms involving $\overline{K}$)

$$=\frac{1}{r}\left[1-\frac{1}{(1+r)^{n_A+1}}\right]\left[\left(a-wa_L-(r+\delta)p^k a_K-q_H-q_L\right)q_L-w\overline{L}-(r+\delta)p^k\overline{K}\right]$$

$$+\frac{1}{(1+r)^{n_A+1}}\frac{1}{r}\left[\left(a-wa_L-(r+\delta)p^k a_K-q_D-q_D\right)q_D-w\overline{L}-(r+\delta)p^k\overline{K}\right]$$

(using first-order conditions for output to express gross profit per period as the square of output)

$$=\frac{1}{r}\left\{\left[1-\frac{1}{(1+r)^{n_A+1}}\right]\left[q_L^2-w\overline{L}-(r+\delta)p^k\overline{K}\right]+\frac{1}{(1+r)^{n_A+1}}\left[q_D^2-w\overline{L}-(r+\delta)p^k\overline{K}\right]\right\}$$

(rearranging terms)

$$=\frac{1}{r}\left[q_L^2+\frac{1}{(1+r)^{n_A+1}}\left(q_D^2-q_L^2\right)-w\overline{L}-(r+\delta)p^k\overline{K}\right].$$

Then if

$$q_L^2+\frac{1}{(1+r)^{n_A+1}}\left(q_D^2-q_L^2\right)-w\overline{L}-(r+\delta)p^k\overline{K}<0,$$

entry is blocked.

5 Infinite horizon, partial sunk cost

Linear inverse demand curve:

$$p = a - Q$$

Stone-Geary production function:

$$q = \min\left(\frac{K - \overline{K}}{a_K}, \frac{L - \overline{L}}{a_L}\right)$$

Capital depreciate at rate δ, $0 \leq \delta \leq 1$.
Monopoly:

$$V_m = \frac{(a - c_H)\, q - F_H}{r},$$

For

$$c_H = wa_L + (r + \delta)\, p^k a_K$$

and

$$F_H = w\overline{L} + (r + \delta)\, p^k \overline{K}.$$

Monopoly output is

$$q_m = \frac{1}{2}(a - c_H) = \frac{1}{2}\left[a - wa_L - (r + \delta)\, p^k a_K\right].$$

The incumbent's capital stock is

$$K_m = \overline{K} + \frac{1}{2} a_K (a - c_H),$$

and at the start of the first period after entry the incumbent inherits capital stock

$$(1 - \delta)\, K_m$$

from the past.

Let μ_{1t} be the Lagrangian multiplier associated with the identify that defines the incumbent's period t capital stock,

$$K_{it} = (1 - \delta)\, K_{1,t-1} + I_{1t} - J_{it}$$

(where $K_{10} = K_m$).

Let λ_{1t} be the Lagrangian multiplier associated with the period t capital input constraint,

$$K_{1t} \geq \overline{K} + a_K q_{1t}.$$

The Lagrangian that describes the incumbent's constrained optimization problem, from the moment of entry, is

$$V_1 = \mu_{11}\left[(1-\delta) K_m + I_{11} - J_{11} - K_{11}\right] - p^k I_{11} + \alpha p^k J_{11}$$

$$+\frac{1}{1+r}\left[(a - q_{11} - q_{21}) q_{11} - \left(w a_L q_{11} + w\overline{L}\right) + \lambda_{11}\left(K_{11} - \overline{K} - a_K q_{11}\right)\right]$$

$$+\frac{1}{1+r}\left\{\mu_{12}\left[(1-\delta) K_{11} + I_{12} - J_{12} - K_{12}\right] - p^k I_{12} + \alpha p^k J_{12} + \right.$$

$$\left. +\frac{1}{1+r}\left[(a - q_{12} - q_{22}) q_{12} - \left(w a_L q_{12} + w\overline{L}\right) + \lambda_{12}\left(K_{12} - \overline{K} - a_K q_{12}\right)\right]\right\} +$$

$$+\frac{1}{(1+r)^2}\left\{\mu_{13}\left[(1-\delta) K_{13} + I_{13} - J_{13} - K_{13}\right] - p^k I_{13} + \alpha p^k J_{13} + \right.$$

$$\left. +\frac{1}{1+r}\left[(a - q_{13} - q_{23}) q_{13} - \left(w a_L q_{13} + w\overline{L}\right) + \lambda_{13}\left(K_{13} - \overline{K} - a_K q_{13}\right)\right]\right\} + ...$$

or, with $K_{10} = K_m$,

$$V_1 = \sum_{t=1}^{\infty} \frac{1}{(1+r)^{t-1}}\left\{\mu_{1t}\left[(1-\delta) K_{1,t-1} + I_{1t} - J_{1t} - K_{1t}\right] - p^k I_{1t}\right.$$

$$\left. +\frac{1}{1+r}\left[(a - q_{1t} - q_{2t}) q_{1t} - \left(w a_L q_{1t} + w\overline{L}\right)\right] + \lambda_{1t}\left(K_{12} - \overline{K} - a_K q_{1t}\right)\right\}.$$

Kuhn-Tucker first-order conditions:

Period 1:

$$\frac{\partial V_1}{\partial \mu_{11}} = (1-\delta) K_m + I_{11} - J_{11} - K_{11} = 0.$$

$$\frac{\partial V_1}{\partial I_{11}} = \mu_{11} - p^k \leq 0 \qquad I_{11}\left(\mu_{11} - p^k\right) = 0 \qquad I_{11} \geq 0$$

$$\frac{\partial V_1}{\partial J_{11}} = -\mu_{11} + \alpha p^k \leq 0 \qquad J_{11}\left(-\mu_{11} + \alpha p^k\right) = 0 \qquad J_{11} \geq 0$$

$$(1+r)\frac{\partial V_1}{\partial q_{11}} = a - 2q_{11} - q_{21} - w a_L - \lambda_{11} a_K = 0$$

$$\frac{\partial V_1}{\partial K_{11}} = -\mu_{11} + \frac{\lambda_{11} + (1-\delta)\,\mu_{12}}{1+r} = 0$$

$$(1+r)\,\frac{\partial V_1}{\partial \lambda_{11}} = K_{11} - \overline{K} - a_K q_{11} \geq 0 \qquad \lambda_{11}\left(K_{11} - \overline{K} - a_K q_{11}\right) = 0 \qquad \lambda_{11} \geq 0$$

Period t:

$$(1+r)^{t-1}\,\frac{\partial V_1}{\partial \mu_{1t}} = (1-\delta)\,K_{1,t-1} + I_{1t} - J_{1t} - K_{1t} = 0.$$

$$(1+r)^{t-1}\,\frac{\partial V_1}{\partial I_{1t}} = \mu_{1t} - p^k \leq 0 \qquad I_{1t}\left(\mu_{1t} - p^k\right) = 0 \qquad I_{1t} \geq 0$$

$$(1+r)^{t-1}\,\frac{\partial V_1}{\partial J_{1t}} = -\mu_{1t} + \alpha p^k \leq 0 \qquad J_{11}\left(-\mu_{1t} + \alpha p^k\right) = 0 \qquad J_{1t} \geq 0$$

$$(1+r)^{t}\,\frac{\partial V_1}{\partial q_{1t}} = a - 2q_{11} - q_{21} - wa_L - \lambda_{1t} a_K = 0$$

$$(1+r)^{t}\,\frac{\partial V_1}{\partial K_{1t}} = -\mu_{1t} + \frac{\lambda_{1t} + (1-\delta)\,\mu_{1,t+1}}{1+r} = 0$$

$$(1+r)^{t}\,\frac{\partial V_1}{\partial \lambda_{1t}} = K_{1t} - \overline{K} - a_K q_{1t} \geq 0 \qquad \lambda_{1t}\left(K_{1t} - \overline{K} - a_K q_{1t}\right) = 0 \qquad \lambda_{1t} \geq 0$$

The entrant maximizes

$$V_2 = \mu_{21}\left(I_{21} - J_{21} + K_{21}\right) - p^k I_{21} + \alpha p^k J_{21}$$

$$+\frac{1}{1+r}\left[(a - q_{11} - q_{12})\,q_{21} - \left(wa_L q_{21} + w\overline{L}\right) + \lambda_{21}\left(K_{21} - \overline{K} - a_K q_{21}\right)\right]$$

$$+\frac{1}{1+r}\left\{\mu_{22}\left[(1-\delta)\,K_{21} + I_{22} - J_{22} - K_{22}\right] - p^k I_{22} + \alpha p^k J_{22}\right.$$

$$\left.\frac{1}{1+r}\left[(a - q_{12} - q_{22})\,q_{22} - \left(wa_L q_{22} + w\overline{L}\right) + \lambda_{22}\left(K_{22} - \overline{K} - a_K q_{22}\right)\right]\right\}$$

$$+\frac{1}{(1+r)^2}\left\{\mu_{23}\left[(1-\delta)\,K_{22} + I_{23} + J_{23} - K_{23}\right] - p^k I_{23} + \alpha p^k J_{23}\right.$$

$$\left.+\frac{1}{1+r}\left[(a - q_{13} - q_{23})\,q_{23} - \left(wa_L q_{23} + w\overline{L}\right) + \lambda_{23}\left(K_{23} - \overline{K} - a_K q_{23}\right)\right]\right\} + ...$$

or, more compactly, with $K_{20} = 0$, (and taking note that $J_{21} = 0$)

$$V_2 = \sum_{t=1}^{\infty} \frac{1}{(1+r)^{t-1}} \left\{ \mu_{2t} \left[(1-\delta) K_{2,t-1} + I_{2t} - J_{2t} - K_{tt} \right] - p^k I_{2t} + \alpha p^k J_{2t} \right.$$

$$\left. + \frac{1}{1+r} \left[(a - q_{1t} - q_{2t}) q_{1t} - \left(w a_L q_{1t} + w\overline{L} \right) \right] + \lambda_{1t} \left(K_{12} - \overline{K} - a_K q_{1t} \right) \right\}.$$

Kuhn-Tucker first-order conditions.

Period 1:

$$\frac{\partial V_2}{\partial \mu_{21}} = I_{21} - J_{21} - K_{21} = 0.$$

$$\frac{\partial V_2}{\partial I_{21}} = -p^k + \mu_{21} \leq 0 \qquad I_{21} \left(-p^k + \mu_{21} \right) = 0 \qquad I_{21} \geq 0.$$

$$\frac{\partial V_2}{\partial J_{21}} = \alpha p^k - \mu_{21} \leq 0 \qquad J_{21} \left(\alpha p^k - \mu_{21} \right) = 0 \qquad J_{21} \geq 0.$$

$$(1+r) \frac{\partial V_2}{\partial q_{21}} = a - q_{11} - 2q_{21} - w a_L - \lambda_{21} a_K = 0.$$

$$(1+r) \frac{\partial V_2}{\partial K_{21}} = -\mu_{21} + \frac{\lambda_{21} + (1-\delta) \mu_{22}}{1+r} = 0.$$

$$(1+r) \frac{\partial V_2}{\partial \lambda_{21}} = K_{21} - \overline{K} - a_K q_{21} \geq 0 \qquad \lambda_{21} \left[K_{21} - \overline{K} - a_K q_{21} \right] = 0 \qquad \lambda_{21} \geq 0.$$

Period t:

$$(1+r)^{t-1} \frac{\partial V_2}{\partial \mu_{2t}} = (1-\delta) K_{2,t-1} + I_{2t} - J_{2t} - K_{2t} = 0.$$

$$(1+r)^{t-1} \frac{\partial V_2}{\partial I_{2t}} = \mu_{2t} - p^k \leq 0 \qquad I_{2t} \left(\mu_{2t} - p^k \right) = 0 \qquad I_{2t} \geq 0.$$

$$(1+r)^{t-1} \frac{\partial V_2}{\partial J_{2t}} = \alpha p^k - \mu_{2t} \leq 0 \qquad J_{2t} \left(\alpha p^k - \mu_{2t} \right) = 0 \qquad J_{2t} \geq 0.$$

$$(1+r)^{t} \frac{\partial V_2}{\partial q_{2t}} = a - q_{1t} - 2q_{2t} - w a_L - \lambda_{2t} a_K = 0.$$

$$(1+r)^{t} \frac{\partial V_2}{\partial K_{2t}} = -\mu_{2t} + \frac{\lambda_{2t} + (1-\delta) \mu_{2,t+1}}{1+r} = 0.$$

$$(1+r)^{t} \frac{\partial V_2}{\partial \lambda_{2t}} = K_{2t} - \overline{K} - a_K q_{2t} \geq 0 \qquad \lambda_{2t} \left[K_{2t} - \overline{K} - a_K q_{2t} \right] = 0 \qquad \lambda_{2t} \geq 0.$$

6 First solution

Incumbent sells capital in the first period, thereafter buys.

$$J_{11} > 0 \Rightarrow \mu_{11} = \alpha p^k.$$
$$I_{12} > 0 \Rightarrow \mu_{12} = p^k.$$

Then

$$-\mu_{11} + \frac{\lambda_{11} + (1-\delta)\,\mu_{12}}{1+r} = 0$$
$$-\alpha p^k + \frac{\lambda_{11} + (1-\delta)\,p^k}{1+r} = 0$$
$$-(1+r)\,\alpha p^k + \lambda_{11} + (1-\delta)\,p^k = 0$$
$$\lambda_{11} = [(1+r)\,\alpha - (1-\delta)]\,p^k$$

Note that

$$(1+r)\,\alpha - (1-\delta) = (r+\delta) - (1+r)\,(1-\alpha)\,;$$

thus λ_{11} can be rewritten as

$$\lambda_{11} = [(r+\delta) - (1+r)\,(1-\alpha)]\,p^k < (r+\delta)\,p^k.$$

A consistency condition for this solution to apply is $\lambda_{11} \geq 0$; this requires

$$(1+r)\,\alpha - (1-\delta) \geq 0$$
$$(1+r)\,\alpha \geq (1-\delta)$$
$$\alpha \geq \frac{1-\delta}{1+r}.$$

αp^k is what the firm can sell a unit of excess capital for at the start of the period entry occurs, and after one period invested at interest rate r, this rises to $(1+r)\,\alpha p^k$. If the firm does not sell a unit of excess capital at the start of the period, the value remaining at the end of the period is $(1-\delta)\,p^k$. If

$$(1+r)\,\alpha p^k > (1-\delta)\,p^k,$$

the firm is better off selling excess capital at the start of the period than holding it and allowing it to depreciate.

$\lambda_{11} > 0 \Rightarrow$

$$K_{11} = \overline{K} + a_K q_{11}.$$

Provided $\alpha \geq (1-\delta)/(1+r)$, the incumbent sells unneeded capital at the moment of entry, and never carries excess capacity.

Firm 1's period 1 output first-order condition is

$$2q_{11} + q_{21} = a - [wa_L + \lambda_{11} a_K].$$

Firm 2 buys an initial capital stock in the first period, buys replacement capital each period thereafter.

$$I_{21} > 0 \Rightarrow \mu_{21} = p^k$$
$$I_{22} > 0 \Rightarrow \mu_{22} = p^k$$

Then

$$-\mu_{21} + \frac{\lambda_{21} + (1-\delta)\mu_{22}}{1+r} = 0$$
$$-p^k + \frac{\lambda_{21} + (1-\delta)p^k}{1+r} = 0$$
$$\lambda_{21} = (r+\delta)p^k.$$

Firm 2's period 1 output first-order condition is

$$q_{11} + 2q_{21} = a - \left[wa_L + (r+\delta)p^k a_K\right].$$

We have seen that

$$\lambda_{11} = [(r+\delta) - (1+r)(1-\alpha)]p^k < (r+\delta)p^k;$$

in the first period, the incumbent's rental cost of capital services is less than the entrant's rental cost of capital services, because in the first period the incumbent's opportunity cost of capital is tied to the resale value of capital, while the entrant's rental cost of capital services is tied to the purchase price of capital.

Solve for equilibrium outputs in the first period:

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} q_{11} \\ q_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\left\{a - \left[wa_L + (r+\delta)p^k a_K\right]\right\} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1+r)(1-\alpha)p^k$$

$$3\begin{pmatrix} q_{11} \\ q_{21} \end{pmatrix} =$$

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\left\{a-\left[wa_L+(r+\delta)\,p^k a_K\right]\right\}+\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}(1+r)\,(1-\alpha)\,p^k a_K$$

$$3\begin{pmatrix} q_{11} \\ q_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\left\{a-\left[wa_L+(r+\delta)\,p^k a_K\right]\right\}+\begin{pmatrix} 2 \\ -1 \end{pmatrix}(1+r)\,(1-\alpha)\,p^k a_K$$

$$\begin{aligned} q_{11} &= \frac{1}{3}\left\{a-\left[wa_L+(r+\delta)\,p^k a_K\right]+2\,(1+r)\,(1-\alpha)\,p^k a_K\right\} \\ &= \frac{1}{3}\left\{a-\left\{wa_L+\left[(r+\delta)-2\,(1+r)\,(1-\alpha)\right]p^k a_K\right\}\right\} \\ q_{21} &= \frac{1}{3}\left\{a-\left[wa_L+(r+\delta)\,p^k a_K\right]-(1+r)\,(1-\alpha)\,p^k a_K\right\} \\ &= \frac{1}{3}\left\{a-\left\{wa_L+\left[r+\delta+(1+r)\,(1-\alpha)\right]p^k a_K\right\}\right\} \end{aligned}$$

The amount of capital sold by the incumbent at the start of period 1 is

$$J_{11} = (1-\delta)\,K_m - K_{11} = (1-\delta)\left(\overline{K}+a_K q_m\right)-\left(\overline{K}+a_K q_{11}\right).$$

In the second and later periods, both firms purchase replacement capital in each period: $I_{1t} > 0$, $I_{2t} > 0$. Then

$$\mu_{1t} = \mu_{2t} = p^k,\ t = 2, 3, ...$$

This implies that

$$J_{1t} = J_{2t} = 0,\ t = 2, 3, ...$$

The first-order conditions for K_{1t} and K_{2t} imply

$$\lambda_{1t} = \lambda_{2t} = (r+\delta)\,p^k,\ t = 2, 3, ...$$

In and after period 2, the market is a Cournot duopoly in which firms have identical marginal costs; equilibrium output per period is

$$q_D = \frac{1}{3}\,(a-c_H)\,.$$

Note that

$$q_{11} = \frac{1}{3}\left[a-c_H+2\,(1+r)\,(1-\alpha)\,p^k a_K\right]$$

$$= q_D + \frac{2}{3}(1+r)(1-\alpha)p^k a_K$$

and

$$q_{21} = \frac{1}{3}\left\{a - \left\{wa_L + [r+\delta+(1+r)(1-\alpha)]p^k a_K\right\}\right\}$$

$$= \frac{1}{3}\left\{a - \left[wa_L + (r+\delta)p^k a_K\right] - (1+r)(1-\alpha)p^k a_K\right\}$$

$$= q_D - \frac{1}{3}(1+r)(1-\alpha)p^k a_K.$$

The incumbent produces more, and the entrant less, in the first period than in later periods.

Now evaluate firm 1's equilibrium value. In equilibrium, the capital stock constraint terms in the expression for firm 1's value (those multiplied by the Lagrangian multipliers μ_{1t}) drop out:

$$V_1 =$$

$$-p^k I_{11} + \alpha p^k J_{11} + \frac{1}{1+r}\left\{(a - q_{11} - q_{21})q_{11} - \left(wa_L q_{11} + w\overline{L}\right) + \lambda_{11}\left(K_{11} - \overline{K} - a_K q_{11}\right)\right\}$$

$$+\frac{1}{1+r}\left\{-p^k I_{12} + \alpha p^k J_{12} +\right.$$

$$\left.\frac{1}{1+r}\left[(a - q_{12} - q_{22})q_{12} - \left(wa_L q_{12} + w\overline{L}\right) + \lambda_{12}\left(K_{12} - \overline{K} - a_K q_{12}\right)\right]\right\}$$

$$+\frac{1}{(1+r)^2}\left\{-p^k I_{13} + \alpha p^k J_{13} +\right.$$

$$\left.\frac{1}{1+r}\left[(a - q_{13} - q_{23})q_{13} - \left(wa_L q_{13} + w\overline{L}\right) + \lambda_{13}\left(K_{13} - \overline{K} - a_K q_{13}\right)\right]\right\} + \ldots$$

Substitute $I_{11} = J_{12} = J_{13} = \ldots = 0$.

$$V_1 = \alpha p^k J_{11} + \frac{1}{1+r}\left\{(a - q_{11} - q_{21})q_{11} - \left(wa_L q_{11} + w\overline{L}\right) + \lambda_{11}\left(K_{11} - \overline{K} - a_K q_{11}\right)\right\}$$

$$+\frac{1}{1+r}\left\{-p^k I_{12} + \frac{1}{1+r}\left[(a - q_{12} - q_{22})q_{12} - \left(wa_L q_{12} + w\overline{L}\right) + \lambda_{12}\left(K_{12} - \overline{K} - a_K q_{12}\right)\right]\right\}$$

$$+\frac{1}{(1+r)^2}\left\{-p^k I_{13} + \frac{1}{1+r}\left[(a - q_{13} - q_{23})q_{13} - \left(wa_L q_{13} + w\overline{L}\right) + \lambda_{13}\left(K_{13} - \overline{K} - a_K q_{13}\right)\right]\right\}$$

$$+\ldots$$

Collect terms in q_{1t}:

$$V_1 = \alpha p^k J_{11} + \frac{1}{1+r}\left[(a - wa_L - \lambda_{11} a_K - q_{11} - q_{21})\, q_{11} - w\overline{L} + \lambda_{11}\left(K_{11} - \overline{K}\right)\right]$$

$$+\frac{1}{1+r}\left\{-p^k I_{12} + \frac{1}{1+r}\left[(a - wa_L - \lambda_{12} a_K - q_{12} - q_{22})\, q_{12} - w\overline{L} + \lambda_{12}\left(K_{12} - \overline{K}\right)\right]\right\}$$

$$+\frac{1}{(1+r)^2}\left\{-p^k I_{13} + \frac{1}{1+r}\left[(a - wa_L - \lambda_{13} a_K - q_{13} - q_{23})\, q_{13} - w\overline{L} + \lambda_{13}\left(K_{13} - \overline{K}\right)\right]\right\}$$

$$+...$$

Now substitute

$$J_{11} = (1-\delta)\, K_m - K_{11}$$

$$I_{1t} = K_{1t} - (1-\delta) K_{1,t-1},\ \ t = 2, 3, ...$$

$$V_1 =$$

$$\alpha p^k \left[(1-\delta)\, K_m - K_{11}\right] + \frac{1}{1+r}\left[(a - wa_L - \lambda_{11} a_K - q_{11} - q_{21})\, q_{11} - w\overline{L} + \lambda_{11}\left(K_{11} - \overline{K}\right)\right]$$

$$+\frac{1}{1+r}\left\{-p^k \left[K_{12} - (1-\delta) K_{11}\right] + \right.$$

$$\left.\frac{1}{1+r}\left[(a - wa_L - \lambda_{12} a_K - q_{12} - q_{22})\, q_{12} - w\overline{L} + \lambda_{12}\left(K_{12} - \overline{K}\right)\right]\right\}$$

$$+\frac{1}{(1+r)^2}\left\{-p^k \left[K_{13} - (1-\delta) K_{12}\right] + \right.$$

$$\left.\frac{1}{1+r}\left[(a - wa_L - \lambda_{13} a_K - q_{13} - q_{23})\, q_{13} - w\overline{L} + \lambda_{13}\left(K_{13} - \overline{K}\right)\right]\right\} + ...$$

Now collect terms in K_{1t}:

$$V_1 = (1-\delta)\, \alpha p^k K_m +$$

$$\frac{1}{1+r}\left\{(a - wa_L - \lambda_{11} a_K - q_{11} - q_{21})\, q_{11} - w\overline{L} - \lambda_{11}\overline{K} + \left[\lambda_{11} - (1+r)\alpha p^k + (1-\delta) p^k\right] K_{11}\right\}$$

$$+\frac{1}{(1+r)^2}\left\{(a - wa_L - \lambda_{12} a_K - q_{12} - q_{22})\, q_{12} - w\overline{L} - \lambda_{12}\overline{K}\right.$$

$$\left. + \left[\lambda_{12} - (1+r) p^k + (1-\delta) p^k\right] K_{12}\right\}$$

$$+\frac{1}{(1+r)^3}\left\{(a-wa_L-\lambda_{13}a_K-q_{13}-q_{23})\,q_{13}-w\overline{L}-\lambda_{13}\overline{K}\right.$$

$$\left.+\left[\lambda_{13}-(1+r)p^k+(1-\delta)p^k\right]K_{13}\right\}+...$$

Collect terms in p^k in coefficients of K_{1t}, $t=2,3,4...$:

$$V_1=(1-\delta)\,\alpha p^k K_m+$$

$$\frac{1}{1+r}\left\{(a-wa_L-\lambda_{11}a_K-q_{11}-q_{21})\,q_{11}-w\overline{L}-\lambda_{11}\overline{K}+\left[\lambda_{11}-(1+r)\alpha p^k+(1-\delta)p^k\right]K_{11}\right\}$$

$$+\frac{1}{(1+r)^2}\left\{(a-wa_Lq_{12}-\lambda_{12}a_K-q_{12}-q_{22})\,q_{12}-w\overline{L}-\lambda_{12}\overline{K}+\left[\lambda_{12}-(r+\delta)p^k\right]K_{12}\right\}$$

$$+\frac{1}{(1+r)^3}\left\{(a-wa_L-\lambda_{13}a_K-q_{13}-q_{23})\,q_{13}-w\overline{L}-\lambda_{13}\overline{K}+\left[\lambda_{13}-(r+\delta)p^k\right]K_{13}\right\}+...$$

Using

$$\lambda_{11}=(1+r)\,\alpha p^k-(1-\delta)\,p^k,\ \ \lambda_{1t}=(r+\delta)\,p^k,\ \ t=2,3,...,$$

the terms multiplied by K_{1t} drop out:

$$V_1=(1-\delta)\,\alpha p^k K_m+\frac{1}{1+r}\left[(a-wa_L-\lambda_{11}a_K-q_{11}-q_{21})\,q_{11}-w\overline{L}-\lambda_{11}\overline{K}\right]$$

$$+\frac{1}{(1+r)^2}\left[(a-wa_Lq_{12}-\lambda_{12}a_K-q_{12}-q_{22})\,q_{12}-w\overline{L}-\lambda_{12}\overline{K}\right]$$

$$+\frac{1}{(1+r)^3}\left[(a-wa_L-\lambda_{13}a_K-q_{13}-q_{23})\,q_{13}-w\overline{L}-\lambda_{13}\overline{K}\right]+...$$

Now use the Kuhn-Tucker first-order conditions for output to express gross profit in each period as the square of output:

$$V_1=(1-\delta)\,\alpha p^k K_m+\frac{1}{1+r}\left(q_{11}^2-w\overline{L}-\lambda_{11}\overline{K}\right)$$

$$+\frac{1}{(1+r)^2}\left(q_{12}^2-w\overline{L}-\lambda_{12}\overline{K}\right)+\frac{1}{(1+r)^3}\left(q_{13}^2-w\overline{L}-\lambda_{13}\overline{K}\right)+...$$

Finally substituting equilibrium outputs:

$$V_1=$$

$$(1-\delta)\,\alpha p^k K_m + \frac{1}{1+r}\left(q_{11}^2 - w\overline{L} - \lambda_{11}\overline{K}\right) + \frac{1}{(1+r)^2}\left(q_D^2 - F_H\right) + \frac{1}{(1+r)^3}\left(q_D^2 - F_H\right) + ...$$

$$= (1-\delta)\,\alpha p^k K_m + \frac{1}{1+r}\left(q_{11}^2 - w\overline{L} - \lambda_{11}\overline{K}\right) - \frac{1}{1+r}\left[q_D^2 - w\overline{L} - (r+\delta)\,p^k\overline{K}\right]$$

$$+ \frac{1}{1+r}\left(q_D^2 - F_H\right) + \frac{1}{(1+r)^2}\left(q_D^2 - F_H\right) + \frac{1}{(1+r)^3}\left(q_D^2 - F_H\right) + ...$$

$$= (1-\delta)\,\alpha p^k K_m + \frac{1}{1+r}\left\{q_{11}^2 - q_D^2 + \left[(r+\delta)\,p^k - \lambda_{11}\right]\overline{K}\right\} + \frac{1}{r}\left(q_D^2 - F_H\right)$$

$$= (1-\delta)\,\alpha p^k K_m + \frac{1}{1+r}\left\{q_{11}^2 - q_D^2 + (1+r)\,(1-\alpha)\,p^k\overline{K}\right\} + \frac{1}{r}\left(q_D^2 - F_H\right)$$

$$= (1-\delta)\,\alpha p^k K_m + \left[(1-\alpha)\,p^k\overline{K} + \frac{1}{1+r}\left(q_{11}^2 - q_D^2\right)\right] + \frac{1}{r}\left(q_D^2 - F_H\right).$$

The equilibrium value of the incumbent is the sum of three terms:

- the resale value of capital inherited from the past that becomes excess in the post-entry market;
- the extra economic profit the incumbent earns in the first period because its rental cost of capital services is $\lambda_{11} < (r+\delta)p^k$;
- the value of a firm in a Cournot duopoly where both firms have marginal cost c_H and fixed cost F_H in all periods.

Proceeding in the same general way, find the entrant's equilibrium value. The entrant maximizes

$$V_2 = \mu_{21}\left(I_{21} - J_{21} + K_{21}\right) - p^k I_{21} + \alpha p^k J_{21}$$

$$+ \frac{1}{1+r}\left[(a - q_{11} - q_{12})\,q_{21} - \left(wa_L q_{21} + w\overline{L}\right) + \lambda_{21}\left(K_{21} - \overline{K} - a_K q_{21}\right)\right]$$

$$+ \frac{1}{1+r}\left\{\mu_{22}\left[(1-\delta)\,K_{21} + I_{22} - J_{22} - K_{22}\right] - p^k I_{22} + +\alpha p^k J_{22}\right.$$

$$\left.\frac{1}{1+r}\left[(a - q_{12} - q_{22})\,q_{22} - \left(wa_L q_{22} + w\overline{L}\right) + \lambda_{22}\left(K_{22} - \overline{K} - a_K q_{22}\right)\right]\right\}$$

$$+ \frac{1}{(1+r)^2}\left\{\mu_{23}\left[(1-\delta)\,K_{22} + I_{23} + J_{23} - K_{23}\right] - p^k I_{23} + \alpha p^k J_{23}\right.$$

$$+\frac{1}{1+r}\left[(a-q_{13}-q_{23})\,q_{23}-\left(wa_Lq_{23}+w\overline{L}\right)+\lambda_{23}\left(K_{23}-\overline{K}-a_Kq_{23}\right)\right]\Big\}+...$$

Terms multiplied by the μ_{2t} drop out in equilibrium:

$$V_2=-p^kI_{21}+\alpha p^kJ_{21}$$

$$+\frac{1}{1+r}\left[(a-q_{11}-q_{12})\,q_{21}-\left(wa_Lq_{21}+w\overline{L}\right)+\lambda_{21}\left(K_{21}-\overline{K}-a_Kq_{21}\right)\right]$$

$$+\frac{1}{1+r}\left\{-p^kI_{22}+\alpha p^kJ_{22}+\right.$$

$$\left.\frac{1}{1+r}\left[(a-q_{12}-q_{22})\,q_{22}-\left(wa_Lq_{22}+w\overline{L}\right)+\lambda_{22}\left(K_{22}-\overline{K}-a_Kq_{22}\right)\right]\right\}$$

$$+\frac{1}{(1+r)^2}\left\{-p^kI_{23}+\alpha p^kJ_{23}+\right.$$

$$\left.\frac{1}{1+r}\left[(a-q_{13}-q_{23})\,q_{23}-\left(wa_Lq_{23}+w\overline{L}\right)+\lambda_{23}\left(K_{23}-\overline{K}-a_Kq_{23}\right)\right]\right\}+...$$

Substitute $J_{21}=J_{22}=J_{23}=...=0$.

$$V_2=-p^kI_{21}+\frac{1}{1+r}\left[(a-q_{11}-q_{12})\,q_{21}-\left(wa_Lq_{21}+w\overline{L}\right)+\lambda_{21}\left(K_{21}-\overline{K}-a_Kq_{21}\right)\right]$$

$$+\frac{1}{1+r}\left\{-p^kI_{22}+\frac{1}{1+r}\left[(a-q_{12}-q_{22})\,q_{22}-\left(wa_Lq_{22}+w\overline{L}\right)+\lambda_{22}\left(K_{22}-\overline{K}-a_Kq_{22}\right)\right]\right\}$$

$$+\frac{1}{(1+r)^2}\left\{-p^kI_{23}+\frac{1}{1+r}\left[(a-q_{13}-q_{23})\,q_{23}-\left(wa_Lq_{23}+w\overline{L}\right)+\lambda_{23}\left(K_{23}-\overline{K}-a_Kq_{23}\right)\right]\right\}$$

$$+...$$

Collect terms in q_{2t}:

$$V_2=-p^kI_{21}+\frac{1}{1+r}\left[(a-wa_L-\lambda_{21}a_K-q_{11}-q_{12})\,q_{21}-w\overline{L}+\lambda_{21}\left(K_{21}-\overline{K}\right)\right]$$

$$+\frac{1}{1+r}\left\{-p^kI_{22}+\frac{1}{1+r}\left[(a-wa_L-\lambda_{22}a_K-q_{12}-q_{22})\,q_{22}-w\overline{L}+\lambda_{22}\left(K_{22}-\overline{K}\right)\right]\right\}$$

$$+\frac{1}{(1+r)^2}\left\{-p^kI_{23}+\frac{1}{1+r}\left[(a-wa_L-\lambda_{23}a_K-q_{13}-q_{23})\,q_{23}-w\overline{L}+\lambda_{23}\left(K_{23}-\overline{K}\right)\right]\right\}$$

$$+...$$

Now substitute

$$I_{21} = K_{21}$$

$$I_{2t} = K_{2t} - (1-\delta)K_{2,t-1}, \; t = 2, 3, ...$$

$$V_2 = -p^k K_{21} + \frac{1}{1+r}\left[(a - wa_L - \lambda_{21}a_K - q_{11} - q_{12})\, q_{21} - w\overline{L} + \lambda_{21}\left(K_{21} - \overline{K}\right)\right]$$

$$+\frac{1}{1+r}\left\{-p^k\left[K_{22} - (1-\delta)K_{21}\right] + \right.$$

$$\left.\frac{1}{1+r}\left[(a - wa_L - \lambda_{22}a_K - q_{12} - q_{22})\, q_{22} - w\overline{L} + \lambda_{22}\left(K_{22} - \overline{K}\right)\right]\right\}$$

$$+\frac{1}{(1+r)^2}\left\{-p^k\left[K_{23} - (1-\delta)K_{22}\right] + \right.$$

$$\left.\frac{1}{1+r}\left[(a - wa_L - \lambda_{23}a_K - q_{13} - q_{23})\, q_{23} - w\overline{L} + \lambda_{23}\left(K_{23} - \overline{K}\right)\right]\right\} + ...$$

Now collect terms in K_{2t}:

$$V_2 = \frac{1}{1+r}\left[(a - wa_L - \lambda_{21}a_K - q_{11} - q_{12})\, q_{21} - w\overline{L} - \lambda_{21}\overline{K} + \left[\lambda_{21} - (r+\delta)p^k\right] K_{21}\right]$$

$$+\frac{1}{(1+r)^2}\left\{(a - wa_L - \lambda_{22}a_K - q_{12} - q_{22})\, q_{22} - w\overline{L} - \lambda_{22}\overline{K} + \left[\lambda_{22} - (r+\delta)p^k\right] K_{22}\right\}$$

$$+\frac{1}{(1+r)^3}\left[(a - wa_L - \lambda_{23}a_K - q_{13} - q_{23})\, q_{23} - w\overline{L} - \lambda_{23}\overline{K} + \left[\lambda_{23} - (r+\delta)p^k\right] K_{23}\right]$$

$$+...$$

Use the definition of λ_{2t} to eliminate the terms in K_{2t} and use the Kuhn-Tucker first-order conditions for output to express the entrant's gross profit in each period as the square of its output:

$$V_2 = \frac{1}{1+r}\left(q_{21}^2 - F_H\right) + \frac{1}{(1+r)^2}\left(q_{22}^2 - F_H\right) + \frac{1}{(1+r)^3}\left(q_{23}^2 - F_H\right) + ...$$

$$V_2 = -\frac{1}{1+r}\left(q_D^2 - q_{21}^2\right) + \frac{q_D^2 - F_H}{r}.$$

The entrant's equilibrium value is the value it would have in a Cournot duopoly where both firms have marginal cost c_H and fixed cost F_H in all periods, minus the present value of the profit it does not earn at the end of the first period because the incumbent's rental cost of capital services is less than $(r+\delta)p^k$.

If

$$-\frac{1}{1+r}\left(q_D^2 - q_{21}^2\right) + \frac{q_D^2 - F_H}{r} < 0,$$

entry is blocked. If

$$\frac{q_D^2 - F_H}{r} - \frac{1}{1+r}\left(q_D^2 - q_{21}^2\right) < 0 \leq \frac{q_D^2 - F_H}{r},$$

entry is blocked that would be profitable if the incumbent's costs were not sunk.

7 Second solution

The incumbent does not sell capital in the first period. Since capital is in excess supply, the incumbent lets capital depreciate until it is optimal to purchase capital.

For convenience, repeat the incumbent's first-order conditions:

Period 1:

$$\frac{\partial V_1}{\partial \mu_{11}} = (1-\delta) K_m + I_{11} - J_{11} - K_{11} = 0.$$

$$\frac{\partial V_1}{\partial I_{11}} = \mu_{11} - p^k \leq 0 \qquad I_{11}\left(\mu_{11} - p^k\right) = 0 \qquad I_{11} \geq 0$$

$$\frac{\partial V_1}{\partial J_{11}} = -\mu_{11} + \alpha p^k \leq 0 \qquad J_{11}\left(-\mu_{11} + \alpha p^k\right) = 0 \qquad J_{11} \geq 0$$

$$(1+r)\frac{\partial V_1}{\partial q_{11}} = a - 2q_{11} - q_{21} - wa_L - \lambda_{11} a_K = 0$$

$$\frac{\partial V_1}{\partial K_{11}} = -\mu_{11} + \frac{\lambda_{11} + (1-\delta)\mu_{12}}{1+r} = 0$$

$$(1+r)\frac{\partial V_1}{\partial \lambda_{11}} = K_{11} - \overline{K} - a_K q_{11} \geq 0 \qquad \lambda_{11}\left[K_{11} - \overline{K} - a_K q_{11}\right] = 0 \qquad \lambda_{11} \geq 0$$

Period t:

$$(1+r)^{t-1}\frac{\partial V_1}{\partial \mu_{1t}} = (1-\delta) K_{1,t-1} + I_{1t} - J_{1t} - K_{1t} = 0.$$

$$(1+r)^{t-1}\frac{\partial V_1}{\partial I_{1t}} = \mu_{1t} - p^k \le 0 \qquad I_{1t}\left(\mu_{1t} - p^k\right) = 0 \qquad I_{1t} \ge 0$$

$$(1+r)^{t-1}\frac{\partial V_1}{\partial J_{1t}} = -\mu_{1t} + \alpha p^k \le 0 \qquad J_{11}\left(-\mu_{1t} + \alpha p^k\right) = 0 \qquad J_{1t} \ge 0$$

$$(1+r)^{t}\frac{\partial V_1}{\partial q_{1t}} = a - 2q_{11} - q_{21} - wa_L - \lambda_{1t}a_K = 0$$

$$(1+r)^{t}\frac{\partial V_1}{\partial K_{1t}} = -\mu_{1t} + \frac{\lambda_{1t} + (1-\delta)\,\mu_{1,t+1}}{1+r} = 0$$

$I_{11} = J_{11} = 0$; hence

$$K_{11} = (1-\delta)\,K_m.$$

Capital is in excess supply,

$$K_{11} = (1-\delta)\,K_m \ge \overline{K} + a_K q_{11}, \ \Rightarrow \lambda_{11} = 0.$$

The first-order condition for firm 1's output is

$$2q_{11} + q_{21} = a - wa_L$$

Suppose that the incumbent buys capital in period 2. Then

$$\mu_{12} = p^k.$$

Then from

$$\frac{\partial V_1}{\partial K_{11}} = -\mu_{11} + \frac{\lambda_{11} + (1-\delta)\,\mu_{12}}{1+r} = 0$$

we find

$$\mu_{11} = \frac{\lambda_{11} + (1-\delta)\,\mu_{12}}{1+r} = \frac{0 + (1-\delta)\,p^k}{1+r} = \frac{1-\delta}{1+r}p^k;$$

consistency conditions are

$$\frac{\partial V_1}{\partial I_{11}} = \mu_{11} - p^k \le 0 \text{ or } \mu_{11} \le p^k$$

$$\frac{\partial V_1}{\partial J_{11}} = \alpha p^k - \mu_{11} \le 0 \text{ or } \mu_{11} \ge \alpha p^k$$

Thus for the incumbent to stand pat and let its capital depreciate in the first period, we must have

$$\alpha p^k \le \mu_{11} = \frac{1-\delta}{1+r}p^k \le p^k$$

$$\alpha \le \frac{1-\delta}{1+r} \le 1.$$

The right-hand inequality always holds, and also $\alpha \le 1$. The left-hand inequality may or may not hold. If it does not, it is optimal for the incumbent to sell excess capital immediately if entry occurs; this is the first solution.

If the incumbent buys capital in period 2, it buys capital in every period thereafter.

$$\lambda_{12} = (r+\delta)\, p^k.$$

The first-order condition for the incumbent's period 2 output is

$$2q_{11} + q_{21} = a - c_H.$$

If the incumbent neither buys nor sells capital in the second period but does buy capital in the third period,

$$K_{12} = (1-\delta)^2 K_m \ge .\overline{K} + a_K q_{12}, \quad \Rightarrow \lambda_{12} = 0.$$

From

$$(1+r)\frac{\partial V_1}{\partial K_{12}} = -\mu_{12} + \frac{\lambda_{12} + (1-\delta)\,\mu_{13}}{1+r} = 0$$

$$\mu_{12} = \frac{0 + (1-\delta)\, p^k}{1+r}$$

At this point, knowing the incumbent's rental cost of capital services, and knowing that the entrant buys capital, we can solve for equilibrium output and all other values that characterize period 2 equilibrium. Go forward in the same way.

The entrant's first-order conditions are as in the first solution: the entrant buys capital in every period, always has rental cost of capital $(r+\delta)\, p^k$, has in each period first-order condition for output

$$q_{1t} + 2q_{2t} = a - c_H.$$

This solution ends up replicating that of the case in which resale is impossible.