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Paper No. 1248
Date: January, 2010

Institute for Research in the
Behavioral, Economic, and
Management Sciences

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"Wrong-product" delivery - the delivery of a product different from that desired - is a significant, but as yet unexplored problem in supply-chain management research. There are basically two reasons for wrong-product delivery: either the wrong product is mistakenly ordered or the right product is ordered but the wrong product is picked/shipped. This paper defines and analyzes the "wrong-product delivery" problem using a 2-product newsvendor model. Two non-substitutable products may be ordered at the beginning of each time period. However, whenever product i is ordered, then with known probability α_i, product j is delivered; $i, j = 1, 2 (i \neq j)$. First, we analyze the "no-recourse scenario", where management correctly stores whatever was received, but takes no other action. We establish the form of the optimal policy and conduct sensitivity analysis. Although our modeling framework is simple, our results are unexpected and non-intuitive. For example, it is well known that in the single-product newsvendor model, increasing the uncertainty of demand or supply will yield an increase in the corresponding target basestocks and safety stocks. However, increasing the risk of a wrong-product error yields a *decrease* in the corresponding basestocks and safety stocks. Further, although target basestocks in the single-product newsvendor model are invariant to increases in on-hand inventory, we show that the target basestock for either product is *non-decreasing* as its inventory increases. We also demonstrate that the cost impact of wrong-product uncertainty is comparable, if not larger than, the cost impact of demand uncertainty. Next, we analyze the "recourse scenario" where management is able to correct errors but only by incurring a fixed cost of $\$K$. We show that it is optimal to take recourse when the wrong-product uncertainty is sufficiently *small*, but not take recourse when the wrong-product uncertainty is *high*. In strategic terms, our analysis provides insight into the cost impact of wrong-product errors, and, hence, the importance of reducing them.

Key words: Supply chain management, Inventory management, Shipment errors, Ordering errors, Yield management, Unreliable supply

History: December 2009

1. Introduction

"Wrong-product" delivery - the delivery of a product different from that desired - is a significant, but as yet unexplored problem in supply-chain management research. There are basically two reasons for wrong-product delivery: either the wrong product is mistakenly ordered or the right product is ordered but the wrong product is picked/shipped. There are many underlying causes. In the healthcare-product setting that motivated this research, the "most likely suspect" is the lack of uniform standards (e.g., bar codes) for product identification throughout the supply chain. As an illustration, the Department of Defense Data Synchronization Study (Roberts 2009) noted that a single product, manufactured by 3M^{TM}(8630: DuraPrep^{TM}Surgical Solution), was cataloged at eight different distributors using eight different product numbers, which could potentially lead to ordering errors. In another example, the same product number (10313) represented a needle at one distributor, a cartridge replacement at another, an accessory traction-replacement strap at a third, and a chlorine test kit at a fourth. The same study estimates that wrong-product errors occur in between 2 and 30% of ordering/shipping transactions.

This paper develops and analyzes the wrong-product delivery problem using a 2-product newsvendor model. Two non-substitutable products experience independent, identically-distributed, periodic demand and identical per-unit acquisition, holding and stockout (i.e., backorder or lost-sale) costs. There is an opportunity to order either or both products at the beginning of each time period. There is no fixed ordering cost. Orders are received instantaneously. However, if product i is ordered, then with known probability α_i product j is delivered; $i, j = 1, 2$ $(i \neq j)$. Hence, four outcomes are possible. To illustrate: Assuming that 5 (10) units of product 1 (2) are ordered, then the possible outcomes are: (1) 5 units of product 1 and 10 units of product 2 are received (with probability $(1-\alpha_1)(1-\alpha_2)$); (2) 15 units of product 1 but no units of product 2 are received (with probability α_2 (1-α_1)); (3) 15 units of product 2 but no units of product 1 are received (with probability α_1 (1-α_2)); and (4) 10 units of product 1 and 5 units of product 2 are received (with probability $\alpha_1\alpha_2$). We assume that upon receipt the products are correctly identified. We first analyze the "no-recourse scenario", where management correctly stores and records whatever was received, but takes no other action. We then study the "recourse scenario" in which management is able to correct any errors (i.e., (2)-(4) above) but only by incurring a fixed cost of \$K. We begin with an analysis of the single-period, no-recourse scenario. We then examine two extensions: (1) the multi-period, no-recourse scenario; and (2) the single-period, recourse scenario. In each of these scenarios we establish the form of the optimal policy and conduct sensitivity analysis.

Although our modeling framework is simple, our results are unexpected and non-intuitive if based on the single-product newsvendor model. To illustrate: It is well known that in the single-product newsvendor model *increasing* the uncertainty of demand or supply will yield an *increase* in the corresponding target basestocks and safety stocks. However, increasing the risk of a wrong-product error yields a *decrease* in the corresponding basestocks and safety stocks. Further, although target basestocks in the single-product newsvendor model are invariant to increases in on-hand inventory, we show that the target basestocks for either product is *non-decreasing* as its inventory increases. Finally, perhaps the most significant insight is that the cost impact of wrong-product uncertainty is comparable, if not larger than, the cost impact of demand uncertainty. For the "recourse scenario", we show that it is optimal to take recourse when the wrong-product uncertainty is sufficiently *small*, but not take recourse when the wrong-product uncertainty is *high*.

The contributions of this work are as follows: To the best of our knowledge, ours is the first to model the wrong-product problem. We establish the form of the optimal policy for the single-period model and several extensions. We also establish and illustrate the non-intuitive characteristics of the optimal policies. In strategic terms, our analysis provides insight into the cost-impact of wrong-product errors, and, hence, the importance of reducing them.

In the next section, we review the related literature. In Section 3 we examine the no-recourse, single-period scenario. Section 4 discusses two extensions of the base model: multi-period model and "recourse scenario". Section 5 contains concluding remarks.

2. Literature Review

Our model links two newsvendor models, the linkage being on the supply side; i.e., with known probability α_i, an order placed for z_i units of product $i, i = 1, 2$, will result in the delivery of z_i units of product $j, j \neq i$. Hence, our model might be viewed as a type of supply substitution.

Demand substitution has been examined by Netessine and Rudi (2003), Mahajan and van Ryzin (2001) and others. In a demand-substitution scenario, demand for an out-of-stock product is supplied, deterministically by the supplier or deterministically or stochastically by the customer. Although our model might be viewed as involving supply substitution, this substitution doesnt occur as a consequence of an out-of-stock situation. Hence, as might be expected, both our analysis and results are quite different from demand-substitution models. In particular these models do not prescribe optimal inventory levels.

Our model can also be viewed as involving *unreliable supply*, as modeled by Dada et al. (2007), Anupindi and Akella (1993) and others. In these single-product papers, unreliable supply means that each given supplier has a known probability of delivering whatever was ordered (in the quantity ordered) or not, possibly at different purchase prices. The corresponding analysis focuses on how much should be ordered from each supplier. In the single-supplier scenario, Dada et al. (2007), demonstrate that both the optimal target basestock and safety stock increase as the risk of supply increases. As noted above, this is quite different from the wrong-product scenario, in which, the optimal target basestock and safety stock for each product decreases as the risk of supply increases.

Our model is also related to the literature on transaction errors. Iglehart and Morey (1972) develop a model for establishing a buffer (in addition to any existing safety stock) against transaction errors which can lead to discrepancies between the inventory record and actual inventory. Kok and Shang (2007) study an inventory-replenishment problem together with an inventory-audit policy to correct transaction errors. Atali et al. (2009) show how to design an optimal inventory-control policy in the presence of inventory discrepancies caused by shrinkage, misplacement, and transaction errors. See Lee and Ozer (2007) for an overview of these and related works. DeHoratius et al. (2008) and DeHoratius and Raman (2008) develop and test models for retail inventory management when records are inaccurate. Our model is different from all of these since wrong-product errors don't result in any inventory discrepancies and because wrong-product errors affect the inventory and the inventory-associated costs of two products.

Our model is most closely associated with the yield-management literature, in which, the number of units received may be less than, equal to, or more than the quantity ordered. See Yano and Lee (1995) for a review. Indeed, our results have interesting similarities and differences with those Henig and Gerchak (1990) in the single-product, stochastically-proportional yield scenario. Henig and Gerchak examine a single-product newsvendor model with stochastically-proportional yield (i.e., the amount delivered is a random multiple of the amount ordered). In order to see the connection, note that in the totally symmetric wrong-product model ($\alpha_1 = \alpha_2; x_1 = x_2$), if z of each product is ordered then the "yield" will be stochastically proportional for each product: 0 or $2z$, with probability $\alpha(1-\alpha)$; and z with probability $(1-2\alpha+2\alpha^2)$. Hence, Henig and Gerchak's Theorems 1-2 hold. See Section 3 for details.

3. Model Framework and Analysis

We first analyze a single-period, no-recourse scenario. We assume that stochastic demand for the two products is independent with known CDF $F(\cdot)$. Each product can be ordered at the beginning of the period and delivery is instantaneous[1]. We assume identical purchase costs \$ c per unit, inventory-holding costs \$ h per unit leftover, and shortage costs \$ p per unit of unsatisfied demand. Let α_i be the probability of making an error in ordering/shipping product i; i.e., α_i indicates the probability that product $j, j \neq i$ is received when an order for product i was planned.

The "no-recourse" scenario means that any error will be discovered upon receipt, but not corrected; i.e., the products are stored in the correct place and inventory records accurately adjusted to reflect what happened. In section 4, we analyze the "recourse" scenario, where errors are discovered on receipt and can be fixed at a cost.

Let $x_i, i = 1,2$ be the initial inventory, before the orders are placed, for product i. The goal is to determine the order-up-to levels y_i, and, hence, the order quantities $z_i = y_i - x_i$ for each product

[1] Fixed leadtimes can be incorporated provided backordering is permitted

$i=1,2$ in order to minimize the sum of the expected leftover and shortage costs. Thus the problem can be stated as follows:

$$C(x_1,x_2,\alpha_1,\alpha_2) = \min_{y_1\geq x_1, y_2\geq x_2} G(y_1,y_2,x_1,x_2) \tag{1}$$

Where

$$\begin{aligned} G(y_1,y_2,x_1,x_2) = &\{c\sum_{i=1}^{2}(y_i-x_i)+(1-\alpha_1)(1-\alpha_2)[L(y_1)+L(y_2)] \\ &+(1-\alpha_1)\alpha_2[L(y_1+y_2-x_2)+L(x_2)]+\alpha_1(1-\alpha_2)[L(x_1)+L(y_1+y_2-x_1)] \\ &+\alpha_1\alpha_2[L(x_1+y_2-x_2)+L(x_2+y_1-x_1)]\} \end{aligned} \tag{2}$$

Here, $L(\cdot)$ represents the one-period newsvendor expected cost for a product, i.e., $L(y)=h\int_0^y(y-t)f(t)dt+p\int_y^\infty(t-y)f(t)dt$. The first term in (2) represents the ordering costs incurred; the second term represents the expected costs if no errors are made (with probability $(1-\alpha_1)(1-\alpha_2)$); the third term represents the expected costs if an error occurs on product 2 order but not with the product 1 order (with probability $(1-\alpha_1)\alpha_2$); the fourth term represents the expected costs if an error occurs on product 1 order but not with the product 2 order (with probability $(1-\alpha_2)\alpha_1$); and the last term represents the expected costs if an error is made in shipping both products (with probability $\alpha_1\alpha_2$). Note that if the probability of an error is zero, i.e., $\alpha_1=\alpha_2=0$, then the problem reduces to solving two independent newsvendor problems. Mathematically, this can be stated as follows:

$$C(x_1,x_2,0,0) = \sum_{i=1}^{2}\min_{y_i\geq x_i}\{c(y_i-x_i)+L(y_i)\} \tag{3}$$

The next theorem establishes the convexity of the cost function $G(y_1,y_2,x_1,x_2)$

THEOREM 1. *The cost function $G(y_1,y_2,x_1,x_2)$ is jointly convex in y_1,y_2. Hence, a state-dependent basestock policy is optimal.*

(All proofs are provided in the Appendix)

The optimal basestock levels $y_i^*, i = 1,2$ can be found by solving the following first-order conditions:

$$(1-\alpha_1)(1-\alpha_2)F(y_1)+(1-\alpha_1)\alpha_2 F(y_1+y_2-x_2)+\alpha_1(1-\alpha_2)F(y_2+y_1-x_1)+\alpha_1\alpha_2 F(x_2+y_1-x_1)=\frac{p-c}{p+h} \tag{4}$$

$$(1-\alpha_1)(1-\alpha_2)F(y_2)+(1-\alpha_1)\alpha_2 F(y_1+y_2-x_2)+\alpha_1(1-\alpha_2)F(y_2+y_1-x_1)+\alpha_1\alpha_2 F(x_1+y_2-x_2)=\frac{p-c}{p+h} \tag{5}$$

Note that the righthand side of (4)-(5) is the "newsvendor fractile" and the lefthand sides are wrong-product generalizations of $F(x_i+z_i)$. Let $y_i^*(x_1,x_2,\alpha_1,\alpha_2)$ denote the optimal basestock levels when the initial inventories are (x_1,x_2) and the probabilities of wrong-product errors are (α_1,α_2). In the traditional Newsvendor model, it is well known that increased uncertainty in demand results in an increase in the target basestock levels, and, hence, the safety stock, provided the target service levels are sufficiently high. Similarly, in single-product models with supply uncertainty (e.g., yield loss), it is well known that an increase in supply uncertainty results in an increase in basestock levels (see Dada et al. (2007), Henig and Gerchak (1990), Yano and Lee (1995)). We next measure supply uncertainty in our model and study its impact on basestock levels y_i^*.

Let I_i be an indicator random variable that takes the value 1 if there is no error on the order for product i (with probability $1-\alpha_i$) and the value zero otherwise. Also, let R_i denote the actual amount of product i received. Then

$$R_i = Q_i I_i + Q_j(1-I_j) \tag{6}$$

The uncertainty in the product i received due to product i order is $\sigma_i = \alpha_i(1-\alpha_i)$ per unit ordered, while the uncertainty in the product i received due to product j order is $\sigma_j = \alpha_j(1-\alpha_j)$ per unit ordered. Thus σ_i is a measure of supply uncertainty in our model. Also, if $\sigma_1=\sigma_2=\sigma$, then σ measures the total uncertainty in the receipt of a product, per unit of product ordered. This can be written down mathematically as follows:

$$var(R_i) = Q_i\alpha_i(1-\alpha_i) + Q_j\alpha_j(1-\alpha_j) = \sum_{i=1}^{2} Q_i\sigma_i \tag{7}$$

If $\sigma_1 = \sigma_2 = \sigma$, then

$$var(R_i) = (Q_1 + Q_2)\sigma \tag{8}$$

We begin by presenting results for the totally-symmetric ($x_1 = x_2 = x$; $\alpha_1 = \alpha_2 = \alpha$), no-recourse scenario.

THEOREM 2. *In the totally-symmetric scenario, where $x_1 = x_2 = x$; $\alpha_1 = \alpha_2 = \alpha$:*

(i) *Optimal basestock levels y_i^* are equal, i.e., $y_1^*(x,x,\alpha,\alpha) = y_2^*(x,x,\alpha,\alpha)$ for all x and $0 \le \alpha \le 1$.*

(ii) *Optimal basestock levels y_i^* are decreasing in the uncertainty measure σ.*

(iii) *Optimal basestock levels y_i^* are less than the corresponding no-error basestock levels; i.e., $y_i^*(x,x,\alpha,\alpha) < y_i^*(x,x,0,0)$ for all x and $0 < \alpha < 1$.*

(iv) *Optimal basestock levels y_i^* are non-decreasing in the individual initial inventory level x.*

(v) *Optimal order quantities z_i^* are monotone decreasing in the individual initial inventory level x.*

(vi) *Optimal cost $C(x,x,\alpha,\alpha)$ is increasing in the uncertainty measure σ and is decreasing in x for $x \le y^*(x,x,\alpha,\alpha)$.*

Result *(i)* is intuitive, even obvious, given equal cost drivers, equal on-hand inventories and equal α's. However, it is noteworthy that this optimal basestock *decreases* in the uncertainty measure σ (result *(ii)*). This is the *opposite* of the effect of increasing demand or supply uncertainty in the single-product newsvendor model. For intuition, consider the optimal basestock in the no-error scenario, $y^*(x,x,0,0)$. The optimal basestock equalizes the marginal expected holding cost, $hF(y^*)$, and the marginal expected penalty cost, $p[1 - F(y^*)]$, for each product, and, hence, minimizes total expected cost for both products. Introducing wrong-product errors *increases* the marginal expected holding cost from $hF(y)$ to $h[(1-\alpha)^2F(y) + 2\alpha(1-\alpha)F(2y-x) + (\alpha)^2F(y)]$ for all values of y, and, hence, decreases the marginal expected penalty cost. Consequently, the corresponding optimal basestocks decrease from their no-error values (result *(iii)*) and decrease with increasing σ (result *(ii)*). This is illustrated in Figure 1, which plots $y^*(0,0,\alpha,\alpha)$ versus α (and σ) given uniformly-distributed demand on the interval $[0,10]$ with $c = 0$, $h = \$1$/unit and $p = \$9$/unit.

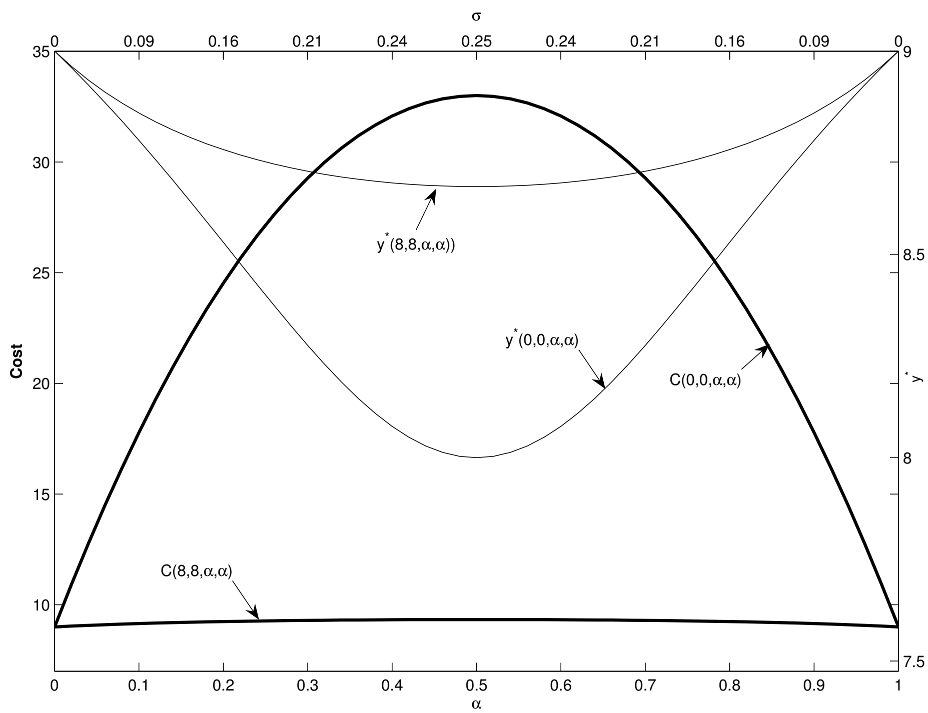

Figure 1 **Graph of optimal cost (C) and basestocks (y^*) as a function of α and σ**

Hence, $y^*(x, x, \alpha = 0, \alpha = 0) = 9$ for all x. Result *(iv)* is noteworthy since, as just noted, in the no-error scenario (i.e., $\alpha = 0$) the optimal basestock is invariant in the on-hand inventory, x. The intuition into why it is non-decreasing is that the smaller the quantity ordered, the smaller the cost consequences if an error should occur. Hence, the target basestock is non-decreasing as on-hand inventory increases. This is also illustrated in Figure 1, which plots $y^*(8, 8, \alpha, \alpha)$ versus α (and σ) for the same demand and cost parameters. Note that $y^*(8, 8, \alpha, \alpha) > y^*(0, 0, \alpha, \alpha)$ for all $0 < \alpha < 1$. Figure 1 also plots $C(0, 0, \alpha, \alpha)$ and $C(8, 8, \alpha, \alpha)$. This shows that the cost impact of wrong-product error is high when inventory levels are *low*, because order-sizes are large in this case. Note that $C(x, x, \alpha, \alpha)$ is increasing in σ and decreasing in x (result *(vi)*). However, note (result *(v)*), that the optimal order quantity, z^*, decreases in x.

The fact that $C(x, x, \alpha, \alpha)$ is increasing in the uncertainty measure σ is consistent with the intuition based on the single-product, no-error model; i.e., that increased uncertainty - in this case, wrong-product supply uncertainty - increases the expected cost of the optimal policy. Indeed, it

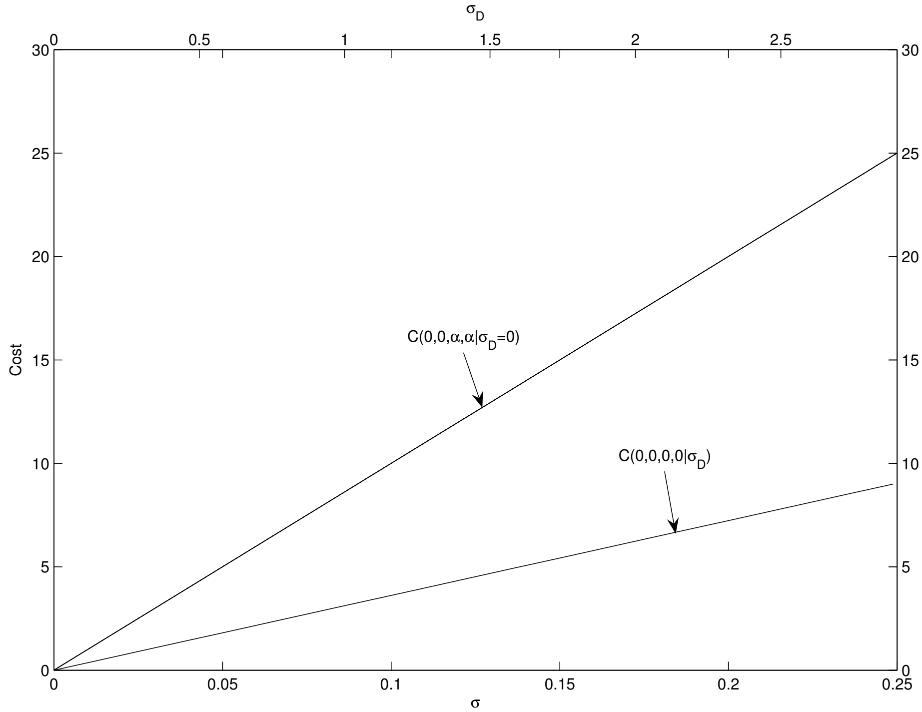

Figure 2 Graph of optimal cost as a function of σ (with $\sigma_D = 0$) and σ_D (with $\sigma = 0$)

is straightforward to show that the marginal increase in optimal expected cost with supply uncertainty (when demand uncertainty is absent) is greater than its marginal increase with demand uncertainty (when supply uncertainty is absent), except for extremely low values of demand. And in either case, given uniformly-distributed demand and $x = 0$, it can be shown that the expected cost of the optimal policy is linear in either uncertainty measure: σ_D in the case of demand uncertainty and σ in the case of wrong-product supply uncertainty (Refer Appendix for a formal analysis). See Figure 2, which compares the expected cost of the optimal policy when there is supply uncertainty but no demand uncertainty, denoted $C(0, 0, \alpha, \alpha|\sigma_D = 0)$ versus σ (lower scale), with the expected cost of the optimal policy when there is no supply uncertainty but (uniform-distribution) demand uncertainty, denoted $C(0, 0, 0, 0|\sigma_D)$ versus σ_D (upper scale). Like Figure 1, unless stated otherwise, all figures are based on uniformly-distributed demand on the interval $[0, 10]$ with $c = 0$, $h = \$1$/unit and $p = \$9$/unit. It is also straightforward to show that any given percentage increase in either demand uncertainty, σ_D, (when there is no supply uncertainty) or wrong-product supply uncertainty, σ,(when there is no demand uncertainty) yields the same percentage increase in the expected cost of the optimal policy.

Although these per-unit and percentage increases are interesting, it should also be noted that σ is limited to the range $[0, 0.25]$ while σ_D is theoretically unlimited. However, management typically has *much* more control over wrong-product supply uncertainty - for example, by improving its own business processes - than management has over demand uncertainty, which is typically beyond its control.

As we noted in the literature review, there are similarities between the single-product, single-period yield model of Henig and Gerchak (1990) and our symmetric, single-period, wrong-product model. We restate their Theorems 1-2 below, using the language of our model.

H&G Theorem 1: In the totally-symmetric wrong-product scenario if $z^* = 0$ for $\alpha = 0$, then $z^* = 0$ for $\alpha > 0$.

H&G Theorem 2: In the totally-symmetric wrong-product scenario if for a given x, $z^*(x) > 0$ when $\alpha = 0$, then $z^*(x) > 0$ for $\alpha > 0$.

Next, we present results for the asymmetric-α, symmetric-x, no-recourse scenario case ($x_1 = x_2 = x$, but $\alpha_1 \neq \alpha_2$).

THEOREM 3. *In the asymmetric-α, symmetric-x case, where $\alpha_1 \neq \alpha_2$, but $x_1 = x_2 = x$:*

(i) Optimal basestock levels y_i^* are equal, i.e., $y_1^*(x, x, \alpha_1, \alpha_2) = y_2^*(x, x, \alpha_1, \alpha_2)$ for all x and $0 < \alpha_1, \alpha_2 < 1$.

(ii) Optimal basestock levels y_i^* are monotone decreasing in α_i if $\alpha_j(fixed) < 0.5$.

(iii) Optimal basestock levels y_i^* are monotone increasing in α_i if $\alpha_j(fixed) > 0.5$.

(iv) Optimal basestock levels y_i^* are less than no-error basestock levels, i.e., $y_i^*(x, x, \alpha_1, \alpha_2) < y_i^*(x, x, 0, 0)$ for all x and $0 < \alpha_i < 1$.

(v) Optimal basestock levels y_i^* are non-decreasing in the individual initial inventory level x.

(vi) Optimal order quantities z_i^* are monotone decreasing in the individual initial inventory level x.

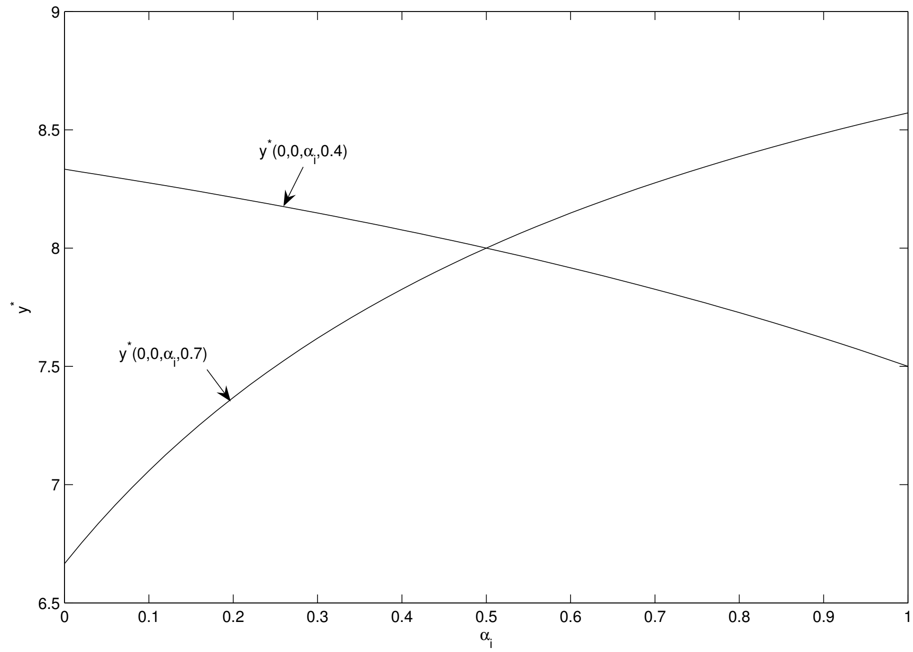

Figure 3 **Graph of basestock (y^*) as a function of α_i (with α_j fixed)**

The intuition for *(iv)* - *(vi)* above is the same as for the corresponding results in Theorem 2. The intuition behind result *(i)* follows from the observation that, regardless of which error occurs, the vector of after-delivery, on-hand inventory has only four possible values: (y_1, y_2), (y_2, y_1), $(x, y_2 + y_1 - x)$, and $(y_1 + y_2 - x, x)$. Since the initial inventories are equal $(x_1 = x_2 = x)$, $y_1 = y_2 = y$ is the optimal solution to the first-order conditions, (4) and (5). Results *(ii)* and *(iii)* follow from the fact that the cost function is super-modular in y_i and α_i if $\alpha_i < 0.5$, and sub-modular otherwise. Example results are provided in Figure 3

Finally, we present results for the asymmetric-x, symmetric-α case (i.e., $\alpha_1 = \alpha_2 = \alpha$, but $x_1 \neq x_2$)

Theorem 4. *For the asymmetric-x, symmetric-α case, where $\alpha_1 = \alpha_2 = \alpha$ and $x_1 \neq x_2$:*

(i) *Total optimal basestock y_T^* is decreasing in the uncertainty measure σ.*

(ii) *Total optimal basestock y_T^* is less than total no-error basestock level, i.e., $y_T^*(x_1, x_2, \alpha, \alpha) < y_T^*(x_1, x_2, 0, 0)$ for all x_i and $0 < \alpha < 1$.*

(iii) *Optimal basestock levels y_i^* are monotone increasing in individual initial inventory x_i, while keeping x_j fixed.*

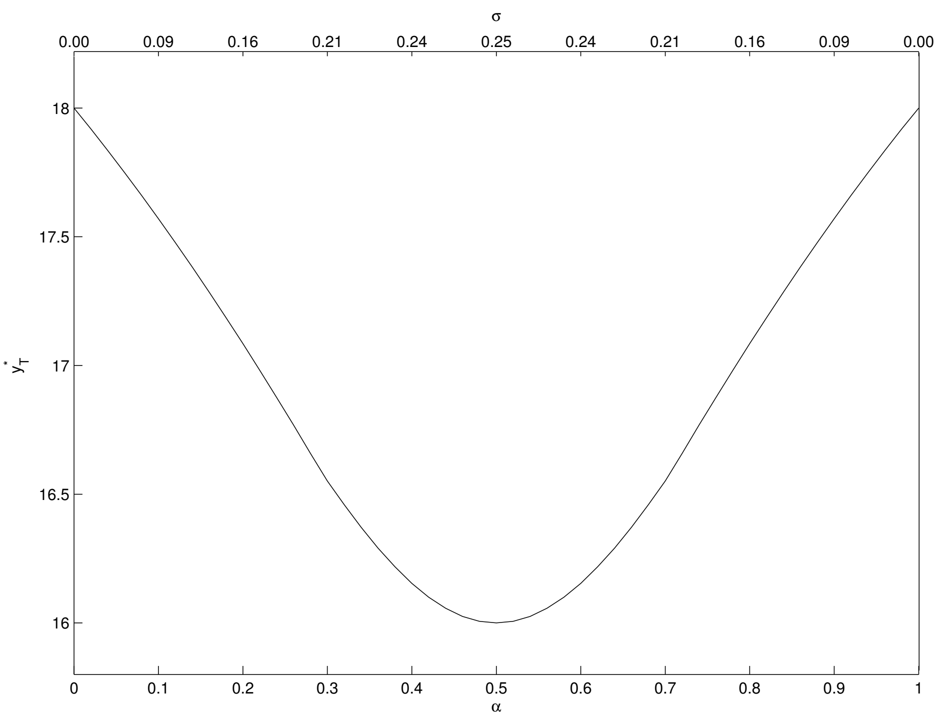

Figure 4 **Graph of total basestock (y_T^*) as a function of α (and σ) for $x_1 = 5$ and $x_2 = 3$**

(iv) Total optimal order quantity z_T^* is monotone decreasing in individual initial inventory x_i, while keeping x_j fixed.

(v) Optimal order quantities z_i^* and z_j^* have the following property: $z_i^*(\alpha) = z_j^*(1-\alpha)$, $i \neq j$.

Intuition on results *(i)*-*(iv)*, which involve total basestock, is the same as for the corresponding results in Theorems 2 and 3 above. Result *(v)* can be explained as follows: In words this result says that the order quantity for product i with a wrong product probability α is the same as the order quantity of product j when wrong product error probability is $1-\alpha$. This is because the marginal costs are symmetric with respect to the order quantities z_1 and z_2. This symmetry happens because if a cross-over error happens (i.e., both products shipped incorrectly), then it is equivalent to the case where the order quantities of the two products were switched and the wrong-product error was $1-\alpha$ instead of α.

3.1. Uniform Demand Distribution

In this sub-section, we provide closed-form expressions for the optimal order quantities and total expected cost, if demand for the two products has a uniform distribution over the range $[0, M]$ and the critical fractile is expressed as F_c. In this case the first-order conditions, (4) and (5), reduce to:

$$z_1^* + (\alpha_1 + \alpha_2 - 2\alpha_1\alpha_2)z_2^* = MF_c - (1-\alpha_1)x_1 - \alpha_1 x_2 \tag{9}$$

$$(\alpha_1 + \alpha_2 - 2\alpha_1\alpha_2)z_1^* + z_2^* = MF_c - \alpha_2 x_1 - (1-\alpha_1)x_2 \tag{10}$$

which can be solved to get the following optimal base-stock levels:

$$y_1^*(x_1, x_2, \alpha_1, \alpha_2) = x_1 + z_1^* = \frac{MF_c - x}{1 + \alpha_1 + \alpha_2 - 2\alpha_1\alpha_2} + x + \frac{\alpha_1\epsilon}{1 - (\alpha_1 + \alpha_2 - 2\alpha_1\alpha_2)^2} - \frac{(\epsilon - \alpha_2\epsilon)(\alpha_1 + \alpha_2 - 2\alpha_1\alpha_2)}{1 - (\alpha_1 + \alpha_2 - 2\alpha_1\alpha_2)^2} \tag{11}$$

$$y_2^*(x_1, x_2, \alpha_1, \alpha_2) = x_2 + z_2^* = \frac{MF_c - x}{1 + \alpha_1 + \alpha_2 - 2\alpha_1\alpha_2} + x + \frac{(\epsilon - \alpha_2\epsilon)}{1 - (\alpha_1 + \alpha_2 - 2\alpha_1\alpha_2)^2} - \epsilon - \frac{(\alpha_1\epsilon)(\alpha_1 + \alpha_2 - 2\alpha_1\alpha_2)}{1 - (\alpha_1 + \alpha_2 - 2\alpha_1\alpha_2)^2} \tag{12}$$

where $x = x_1$ and $\epsilon = x_1 - x_2 = x - x_2$.

For the totally symmetric case ($x_1 = x_2 = x$; $\alpha_1 = \alpha_2 = \alpha$) with uniform demand; the optimal expected cost C is given by,

$$C(x, x, \alpha, \alpha) = (pM - 2px)(1 + 2\alpha - 2\alpha^2) - 2p(MF_c - x) + (h+p)(MF_c^2 + 2\alpha(1-\alpha)\frac{x^2}{M}) \tag{13}$$

It is known that the no-error expected cost ($\alpha = 0$) is given by, $C(x, x, 0, 0) = M\{p(F_c - 1)^2 + hF_c^2\}$. Hence, a *cost penalty*(C_p) can be defined as the ratio of the optimal expected cost with wrong-product errors to the corresponding optimal expected cost when there are no errors.

$$C_p = \frac{C(x, x, \alpha, \alpha)}{C(x, x, 0, 0)} = \{\frac{(pM - 2px)(1 + 2\alpha - 2\alpha^2) - 2p(MF_c - x) + (h+p)(MF_c^2 + 2\alpha(1-\alpha)\frac{x^2}{M})}{M\{p(F_c - 1)^2 + hF_c^2\}}\}$$

C_p can be regarded as a measure of the additional cost that is incurred (penalty) because of wrong-product errors.

(Note that the above closed-form expressions are valid for $0 < x < M$, $0 < x + z_1^* < M$, $0 < x + z_2^* < M$ and $0 < x + z_1^* + z_2^* < M$. For a detailed discussion, see appendix.)

4. Extensions of the Base Model

In this section: 1) we extend the single-period model to a multi-period setting; and 2) we extend the base model to a scenario where recourse to *fix* wrong-product errors can be taken when errors occur.

4.1. Multi-period model with wrong-product errors

We first consider a T-period version of the base model. Inventory now has to be managed over $t = 1, ..., T$ periods. The sequence of events in each period t is as follows.

i. The beginning inventory levels for each period t are x_1^t and x_2^t for the two products

ii. Order quantities Q_1^t and Q_2^t are placed after observing the state (x_1^t, x_2^t).

iii. Wrong-product errors could result, with probability α_1 and α_2, as described above. Shipments are received (instantaneously) for the two products. Let $R_i^t = Q_i^t I_i^t + Q_j^t(1 - I_j^t)$ be the amount received for product $i, i = 1, 2$.

iv. Random demand for each product for period t is realized and satisfied using the available inventory $x_i^t + R_i^t$.

v. Inventory-holding and shortage costs are assessed for period t at the end of the period. Any leftover inventory is carried over to the next period. Any unmet demand in period t is backordered to the next period.[2]

The optimality equation for the T-period model can be written as:

$$C^t(x_1^t, x_2^t) = \min_{y_1^t \geq x_1^t, y_2^t \geq x_2^t} G(y_1^t, y_2^t, x_1^t, x_2^t) \tag{14}$$

where $C^t(x_1^t, x_2^t)$ is the total expected discounted cost of the optimal policy, β is the discount factor, and,

$$G(y_1^t, y_2^t, x_1^t, x_2^t)$$

$$= \{c \sum_{i=1}^{2} (y_i^t - x_i^t) + (1 - \alpha_1)(1 - \alpha_2)[L(y_1^t) + L(y_2^t)$$

[2] The analysis in the lost-sale case is similar.

$$+\beta\int_0^\infty\int_0^\infty C^{t+1}(y_1^t-\zeta_1^t, y_2^t-\zeta_2^t)f(\zeta_1^t)f(\zeta_2^t)d\zeta_1 d\zeta_2]$$
$$+(1-\alpha_1)\alpha_2[L(y_1^t+y_2^t-x_2^t)+L(x_2^t)$$
$$+\beta\int_0^\infty\int_0^\infty C^{t+1}(y_1^t+y_2^t-x_2^t-\zeta_1^t, x_2^t-\zeta_2^t)f(\zeta_1^t)f(\zeta_2^t)d\zeta_1 d\zeta_2]$$
$$+\alpha_1(1-\alpha_2)[L(x_1^t)+L(y_1^t+y_2^t-x_1^t)$$
$$+\beta\int_0^\infty\int_0^\infty C^{t+1}(x_1^t-\zeta_1^t, y_1^t+y_2^t-x_1^t-\zeta_2^t)f(\zeta_1^t)f(\zeta_2^t)d\zeta_1 d\zeta_2]$$
$$+\alpha_1\alpha_2[L(x_1^t+y_2^t-x_2^t)+L(x_2^t+y_1^t-x_1^t)$$
$$+\beta\int_0^\infty\int_0^\infty C^{t+1}(x_1^t+y_2^t-x_2^t-\zeta_1^t, x_2^t+y_1^t-x_1^t-\zeta_2^t)f(\zeta_1^t)f(\zeta_2^t)d\zeta_1 d\zeta_2]\}$$

The next theorem establishes the convexity of the cost function.

THEOREM 5. *The cost function $C^t(x_1^t, x_2^t)$ is convex in x_1^t, x_2^t. Hence, a state-dependent basestock policy is optimal.*

4.2. The Recourse Scenario

In this extension we permit management to instantaneously correct any wrong-product errors by incurring a fixed cost of \$K. More specifically, at the beginning of each period, after receiving whatever products were shipped, we continue to assume that management correctly identifies whatever was shipped. Now, however, depending on what was received, management chooses either to have any errors instantaneously corrected (i.e., to swap any wrong-product receipts for whatever management intended to order), by incurring a fixed cost of \$K; or, to "live" with those errors, as in the no-recourse scenario. We begin by stating the 1-period problem.

$$C^R(x_1, x_2, \alpha_1, \alpha_2) = \min_{y_1\ge x_1, y_2\ge x_2} G^R(y_1, y_2, x_1, x_2) \tag{15}$$

Where

$$\begin{aligned} G^R(y_1, y_2, x_1, x_2) = \{&c\sum_{i=1}^{2}(y_i-x_i)+(1-\alpha_1)(1-\alpha_2)[L(y_1)+L(y_2)] \\ &+(1-\alpha_1)\alpha_2\min[L(y_1+y_2-x_2)+L(x_2), L(y_1)+L(y_2)+K] \\ &+\alpha_1(1-\alpha_2)\min[L(x_1)+L(y_1+y_2-x_1), L(y_1)+L(y_2)+K]] \end{aligned} \tag{16}$$

$$+\alpha_1\alpha_2 \min[L(x_1+y_2-x_2)+L(x_2+y_1-x_1), L(y_1)+L(y_2)+K]]\}$$

We next provide some structural results for the symmetric-x case (where $x_1 = x_2 = x$). If $x_1 = x_2 = x$, then (15)-(16) can be written as

$$C^R(x,x,\alpha_1,\alpha_2) = \min_{y_1 \geq x_1, y_2 \geq x_2} G^R(y_1,y_2,x,x) \tag{17}$$

Where

$$\begin{aligned} G^R(y_1,y_2,x,x) = \{& c\sum_{i=1}^{2}(y_i - x) + [(1-\alpha_1)(1-\alpha_2)+\alpha_1\alpha_2][L(y_1)+L(y_2)] \\ & +[(1-\alpha_1)\alpha_2+\alpha_1(1-\alpha_2)]\min[L(y_1+y_2-x)+L(x), L(y_1)+L(y_2)+K] \end{aligned} \tag{18}$$

Note that this can be stated as the minimization of two problems as follows:

$$C^R(x,x,\alpha_1,\alpha_2) = \min\{C(x,x,\alpha_1,\alpha_2), C(x,x,0,0)+[(1-\alpha_1)\alpha_2+\alpha_1(1-\alpha_2)]K\} \tag{19}$$

The first term in (19) is the cost of the optimal policy in the no-recourse scenario above, while the second term is the traditional newsvendor problem under the no-errors scenario plus the expected cost of fixing errors. Each of these problems can be solved independently, and, hence, the optimal solution to the recourse scenario can be found by comparing the optimal values of these independent problems. Hence, if $C(x,x,\alpha_1,\alpha_2) < C(x,x,0,0)+[(1-\alpha_1)\alpha_2+\alpha_1(1-\alpha_2)]K$, then it is never optimal to fix errors because the cost of fixing errors is too high compared to the benefit obtained from fixing them. However if $C(x,x,\alpha_1,\alpha_2)) \geq C(x,x,0,0)+[(1-\alpha_1)\alpha_2+\alpha_1(1-\alpha_2)]K$, then it is always optimal to take recourse actions if errors occur. Thus, one can determine *ex-ante* whether it is economical to take recourse action if wrong-product errors occur.

It is intuitive that the optimality of recourse is a function of the fixed cost K. That is, as the fixed cost K increases, the expected cost of fixing errors, i.e., $[(1-\alpha_1)\alpha_2+\alpha_1(1-\alpha_2)]K$ also increases. Let K_T represent the value of K when the cost of no-recourse scenario and the recourse scenario are equal; i.e.,

$$C(x,x,\alpha_1,\alpha_2) = C(x,x,0,0)+[(1-\alpha_1)\alpha_2+\alpha_1(1-\alpha_2)]K_T \tag{20}$$

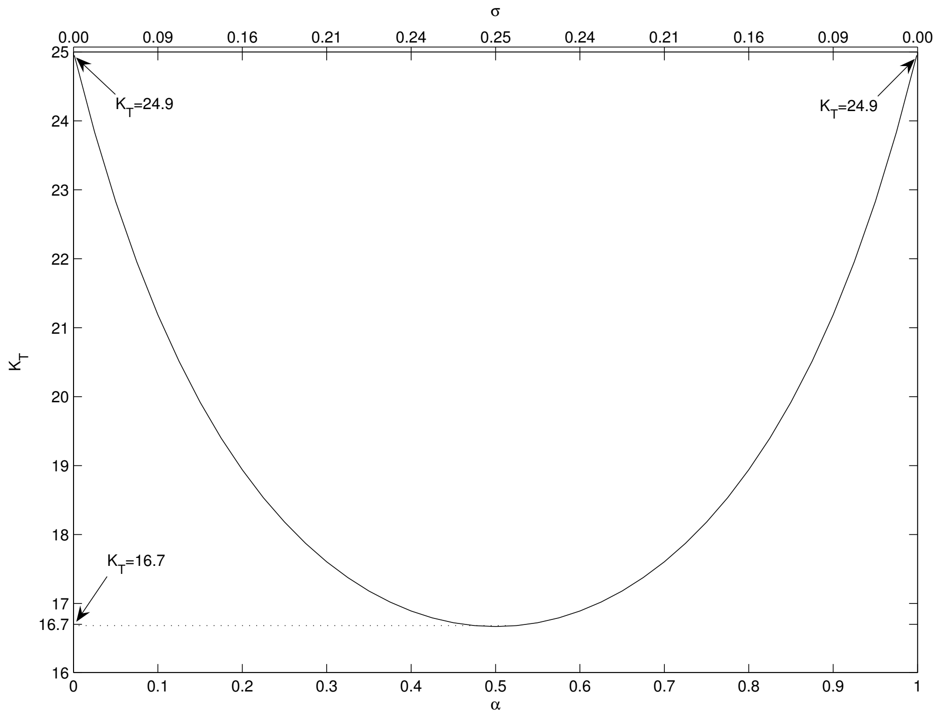

Figure 5 **Graph of Threshold fixed cost (K_T) as a function of α (and σ), for $x = 0$ and critical fractile=50%**

Here, K_T represents the threshold value for the cost of fixing errors above which it is no longer optimal to take recourse, i.e., if $K > K_T$, then it is optimal not to take recourse.

Theorem 6 presents results related to the order-up-to levels and the threshold value, K_T.

THEOREM 6. *For the totally-symmetric case where $x_1 = x_2 = x$ and $\alpha_1 = \alpha_2 = \alpha$;*

(i) A basestock policy is optimal

(ii) The optimal basestock levels are either the basestock levels in the no-recourse scenario or the basestock levels for the newsvendor problem with no errors.

(iii) K_T is decreasing in the uncertainty measure σ.

(iv) K_T is decreasing in the initial inventory x.

Since the recourse problem is the minimum of the no-recourse problem and the newsvendor problem, the optimal basestock corresponds to the basestock level of the no-recourse problem or the newsvendor problem. Theorem 6 also states that K_T is decreasing in the uncertainty measure σ. This is because the right-hand side of the breakeven equation, (20), is quadratic in α; and,

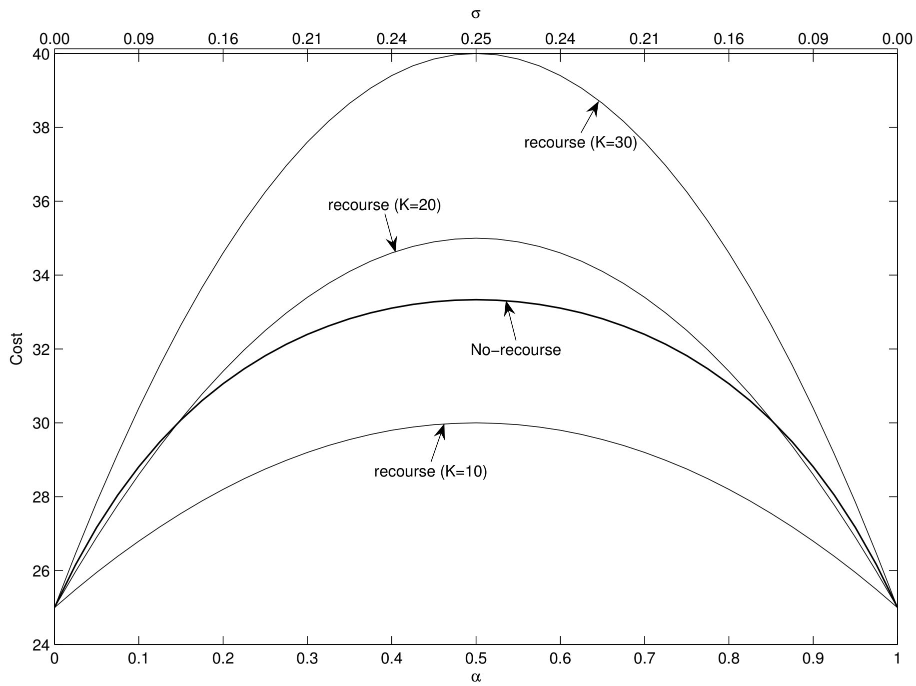

Figure 6 Graph of optimal cost as a function of α (and σ), for $x = 0$ and critical fractile=50%

although the left-hand side, $C(x,x,\alpha,\alpha)$, also increases in α, it increases more slowly. Similarly, as the initial inventory x increases, the cost impact of errors decreases, since order quantities decrease. As a result, recourse is less likely to be taken, and K_T decreases. Figure 5 shows the impact of α and σ on K_T (result (iii)), where K_T decreases as the uncertainty measure $\sigma = \alpha(1-\alpha)$ increases.

Figure 6 shows the impact of α on both the optimal expected recourse and no-recourse costs. Note that, for $K = 30 > max\{K_T\} = 24.9$ (from Figure 5), it is never optimal to take recourse and for $K = 10 < min\{K_T\} = 16.7$ (from Figure 5), it is always optimal to take recourse. If the chosen K is such that it is greater than K_T for some α's but less for others (e.g. $K = 20$), then it is optimal to take recourse for very small and high values of α; but, otherwise take no recourse. This shows that it is optimal to take recourse actions when the wrong-product uncertainty is sufficiently *small*, and taking no recourse actions is optimal when the wrong-product uncertainty is *high*.

Figure 7 shows the effect of initial inventory level x on the expected cost of the optimal policy. Note that when x is small it is optimal to take recourse, hence, the cost curve is flat (because newsvendor costs are independent of initial inventory). As x increases it becomes less costly to

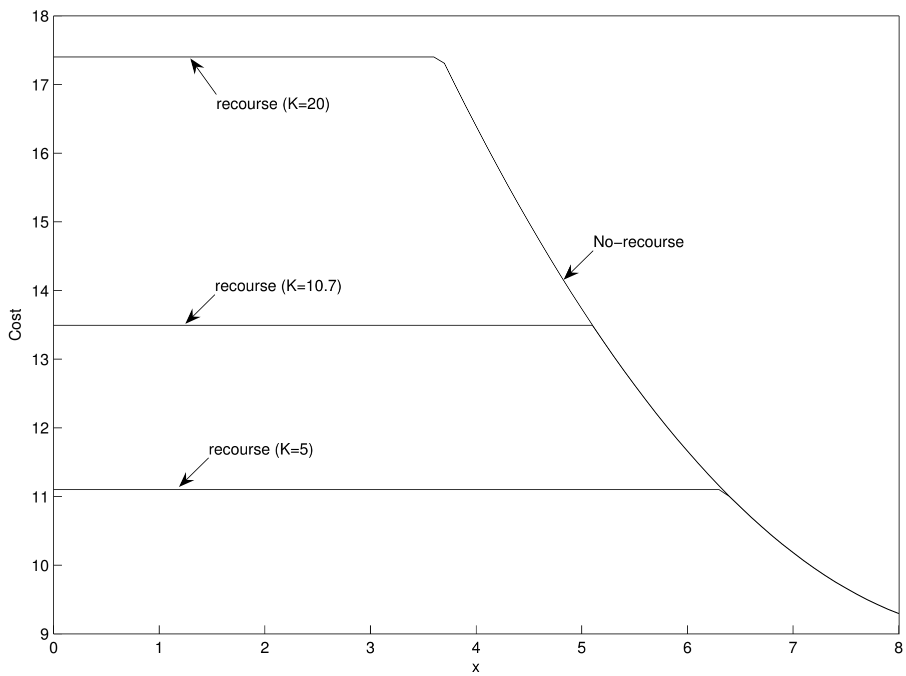

Figure 7 **Graph of optimal cost as a function of x, for $\alpha = 0.3$ and critical fractile=90%**

choose no-recourse, and, hence, the cost curve follows the same trajectory as that of a no-recourse scenario. The graph also shows that as K increases, the range of x over which recourse is optimal decreases, thus resulting in a smaller value of K_T.

The next theorem establishes the optimality of the basestock policy for the general case; i.e., $x_1 \neq x_2$ and $\alpha_1 \neq \alpha_2$.

Theorem 7. *A state-dependent basestock policy is optimal in the one-period recourse scenario*

Unfortunately, it is very difficult to describe the structure of the optimal solution in more detail for the general case ($x_1 \neq x_2$ and $\alpha_1 \neq \alpha_2$). Although there are only three types of errors (cases (2)-(4) in section 1), there are seven possible "triggers" for recourse. As described in section 1, if z_1^* and z_2^* are the optimal order quantities when initial inventories are x_1 and x_2 respectively, then the four possible cases are (1) $(x_1 + z_1^*, x_2 + z_2^*)$, (2) $(x_1 + z_1^* + z_2^*, x_2)$, (3) $(x_1, x_2 + z_1^* + z_2^*)$ and (4) $(x_1 + z_2^*, x_2 + z_1^*)$ (cross-over errors). Figure 8 shows the structure of the optimal policy over the entire state space (x_1, x_2) given uniformly distributed customer demand $U[0, 10]$ for parameters

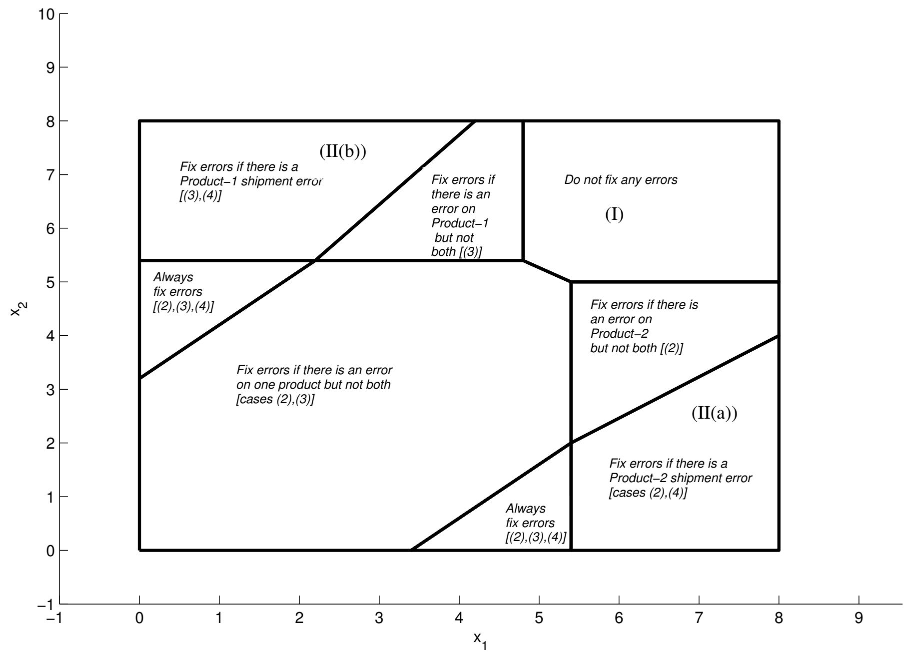

Figure 8 Inventory Range Graph for $\alpha_1 = \alpha_2 = 0.3$ **and** $K = 10$

$\alpha_1 = \alpha_2 = 0.3$ and $K = 10$. Figure 8 can be interpreted as follows. For large values of x_1 and x_2, the corresponding order-quantities are small; and thus the cost consequences of wrong-product errors are also small. This is region I where it is optimal to not take recourse. Similar intuition applies in the regions II(a) and II(b) where x_1 (x_2) is large but x_2 (x_1) is small; thus denoting that it is optimal to correct errors with product 2 (product 1), i.e., cases (2) and (4) (cases (3) and (4)). Note that in these regions, it is optimal to fix cross-over errors, (4), in spite of x_1 (x_2) being very high. This is because x_2 (x_1) is so low that the cost impact of an error is high. Now, as we keep x_1 (x_2) high but increase x_2 (x_1), the cost impact of an x_2 (x_1) error decreases, and it is no longer optimal to fix cross-over errors. Thus, it is only optimal to correct errors in product 2 (product 1) only. Now, when x_1 and x_2 are not very high and their values are close to each other; the results are in-line with the symmetric-x case; where it is optimal to fix uni-directional errors (cases (2) and (3)), but it is no longer optimal to fix cross-over errors since cross-over errors cancel out each other. Finally, as expected, all errors are fixed under two cases: (1) x_1 is very low and x_2 is not too high (2) x_2 is very low and x_1 is not too high.

The complicated nature of interaction between the inventory of both products makes it difficult to establish a structure for the optimal policy for the general case with asymmetric inventories or to establish the form of the optimal policy for multiple time periods.

5. Conclusions

'Wrong-product" delivery - the delivery of a product different from that desired - is a significant, but as yet unexplored problem in supply-chain management research. This paper has defined and analyzed the 2-product "wrong-product delivery" problem using a newsvendor-modeling framework. Two non-substitutable products may be ordered at the beginning of each time period. However, whenever product i is ordered, then with known probability α_i, product j is delivered; $i, j = 1, 2 (i \neq j)$. We first analyzed the "no-recourse scenario" where management correctly stores whatever was received, but takes no other action. Next, we analyzed the "recourse scenario" where management is able to correct any errors but only by incurring a fixed cost of $\$K$.

Although our model is simple, our results are unexpected and non-intuitive if based on the single-product newsvendor model. The most non-intuitive results are: First, that target basestocks (and safety stocks) *decrease* with increasing wrong-product uncertainty. Hence, effective inventory service levels will be lower than would be expected from the corresponding newsvendor target fractile. Second, that the target basestock for either product is non-decreasing as its inventory increases. Third, that wrong-product supply uncertainty can be *much* more costly than demand uncertainty. Finally, if recourse is allowed, we show that it is optimal to take recourse when the wrong-product uncertainty is sufficiently *small*, but not take recourse when the wrong-product uncertainty is *high*.

These results suggest that management should be very attentive to the existence of wrong-product supply uncertainty. And, once identified, management should carefully assess its service-level and cost consequences. In particular, we believe that insights and estimates based on the single-product newsvendor model may be grossly overestimate service levels and underestimate expected costs. Finally, although reducing or eliminating wrong-product errors may be very costly

- in the healthcare-product supply chain, for example, this will probably require the adoption of uniform standards for product identification - doing so may very well be cost effective.

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Appendix

A. Proof of Theorem 1

The joint convexity of the cost function in y_1, y_2 is the result of its Hessian matrix being positive.

First order derivatives are given by,

$$\begin{aligned}\frac{\partial G}{\partial y_1} = \; & \{c + (1-\alpha_1)(1-\alpha_2)[L^{'}(y_1)] + (1-\alpha_1)(\alpha_2)[L^{'}(y_1+y_2-x_2)] \\ & + (\alpha_1)(1-\alpha_2)[L^{'}(y_1+y_2-x_1)] + (\alpha_1)(\alpha_2)[L^{'}(x_2+y_1-x_1)]\}\end{aligned}$$

$$\frac{\partial G}{\partial y_2} = \{c + (1-\alpha_1)(1-\alpha_2)[L^{'}(y_2)] + (1-\alpha_1)(\alpha_2)[L^{'}(y_1+y_2-x_2)] + (\alpha_1)(1-\alpha_2)[L^{'}(y_1+y_2-x_1)] + (\alpha_1)(\alpha_2)[L^{'}(x_1+y_2-x_2)]\}$$

Second order derivatives are given by,

$$\frac{\partial^2 G}{\partial y_1^2} = \{(1-\alpha_1)(1-\alpha_2)[L^{''}(y_1)] + (1-\alpha_1)(\alpha_2)[L^{''}(y_1+y_2-x_2)] + (\alpha_1)(1-\alpha_2)[L^{''}(y_1+y_2-x_1)] + (\alpha_1)(\alpha_2)[L^{''\text{`}}(x_2+y_1-x_1)]\}$$

$$\frac{\partial^2 G}{\partial y_2^2} = \{(1-\alpha_1)(1-\alpha_2)[L^{''}(y_2)] + (1-\alpha_1)(\alpha_2)[L^{''}(y_1+y_2-x_2)] + (\alpha_1)(1-\alpha_2)[L^{''}(y_1+y_2-x_1)] + (\alpha_1)(\alpha_2)[L^{''}(x_1+y_2-x_2)]\}$$

$$\frac{\partial^2 G}{\partial y_1 \partial y_2} = \{(1-\alpha_1)(\alpha_2)[L^{''}(y_1+y_2-x_2)] + (\alpha_1)(1-\alpha_2)[L^{''}(y_1+y_2-x_1)]\}$$

$$\frac{\partial^2 G}{\partial y_2 \partial y_1} = \{(1-\alpha_1)(\alpha_2)[L^{''}(y_1+y_2-x_2)] + (\alpha_1)(1-\alpha_2)[L^{''}(y_1+y_2-x_1)]\}$$

The Hessian matrix is given by,

$$H(y_1,y_2,x_1,x_2) = \begin{bmatrix} \frac{\partial^2 G}{\partial y_1^2} & \frac{\partial^2 G}{\partial y_1 \partial y_2} \\ \frac{\partial^2 G}{\partial y_2 \partial y_1} & \frac{\partial^2 G}{\partial y_2^2} \end{bmatrix}$$

Now, from the above equations,

$$\{\frac{\partial^2 G}{\partial y_1^2}.\frac{\partial^2 G}{\partial y_2^2}\} \geq \{\frac{\partial^2 G}{\partial y_1 \partial y_2}.\frac{\partial^2 G}{\partial y_2 \partial y_1}\} = \{\frac{\partial^2 G}{\partial y_1 \partial y_2}\}^2$$

$$\Rightarrow \ \det \left| H(y_1,y_2,x_1,x_2) \right| \geq 0 \quad \Box$$

B. Proof of Theorem 2

(i) First-order conditions are given by,

$$(1-\alpha)^2[F(x+z_2^*)] + (1-\alpha)(\alpha)[F(x+z_1^*+z_2^*)] + (\alpha)(1-\alpha)[F(x+z_1^*+z_2^*)] + (\alpha)^2[F(x+z_2^*)] = \frac{p-c}{p+h}$$

$$(1-\alpha)^2[F(x+z_1^*)] + (1-\alpha)(\alpha)[F(x+z_1^*+z_2^*)] + (\alpha)(1-\alpha)[F(x+z_1^*+z_2^*)] + (\alpha)^2[F(x+z_1^*)] = \frac{p-c}{p+h}$$

$$\Rightarrow F(x+z_2^*)[(1-\alpha)^2+(\alpha)^2] = F(x+z_1^*)[(1-\alpha)^2+(\alpha)^2]$$

$$\Rightarrow F(x+z_2^*) = F(x+z_1^*)$$

$$\Rightarrow z_1^* = z_2^*$$

$$\Rightarrow y_1^* = y_2^* \quad \Box$$

*(ii)*First-order condition is given by,

$$f_1 = c + (1-\alpha)^2[L^{'}(x+z_1^*)] + (\alpha)(1-\alpha)[L^{'}(x+z_T^*)] + (\alpha)(1-\alpha)[L^{'}(x+z_T^*)] + (\alpha)^2[L^{'}(x+z_1^*)] = 0$$

$$Let \quad w = 1 - \{(\alpha)(1-\alpha) + (\alpha)(1-\alpha)\}$$

$$\Rightarrow f_1 = c + w[L'(x+z_1^*)] + (1-w)[L'(x+z_T^*)] = 0$$

$$\Rightarrow \frac{df_1}{dw} = \frac{w}{2}L''(x+\frac{z_T^*}{2})\frac{dz_T^*}{dw} + L'(x+\frac{z_T^*}{2}) + (1-w)L''(x+z_T^*)\frac{dz_T^*}{dw} - L'(x+z_T^*) = 0$$

$$\Rightarrow \frac{dz_T^*}{dw} = \frac{L'(x+z_T^*) - L'(x+\frac{z_T^*}{2})}{\frac{w}{2}L''(x+\frac{z_T^*}{2}) + +(1-w)L''(x+z_T^*)}$$

Since $L(\cdot)$ is convex, $L^{''}(\cdot)$ is non negative and $L^{'}(x+z_T^*) > L^{'}(x+\frac{z_T^*}{2})$. Thus, $\frac{dz_T^*}{dw} > 0$ and $\frac{dy_T^*}{dw} = \frac{d(2x+z_T^*)}{dw} = \frac{dz_T^*}{dw} > 0$. Since our uncertainty measure $\sigma = \frac{1-w}{2}$; as w decreases, σ increases. Thus, y_T^* decreases as σ increases. From *(i)*, $y_1^* = y_2^* = \frac{y_T^*}{2}$. Thus, y_i^*'s decrease as σ increases. $\square$

(iii) It follows from *(i)* and *(ii)* above that individual basestock levels in the wrong product scenario are always lesser than no-error case. $\square$

(iv) As defined previously,

$$f_1 = c + w[L'(x+z_1^*)] + (1-w)[L'(x+z_T^*)] = 0$$

Differentiating w.r.t x gives,

$$(wL^{''}(x+\frac{z_T^*}{2}) + (1-w)L^{''}(x+z_T^*)) + \frac{dz_T^*}{dx}(\frac{w}{2}L^{''}(x+\frac{z_T^*}{2}) + (1-w)L^{''}(x+z_T^*)) = 0$$

$$\Rightarrow \frac{dy_T^*}{dx} = \frac{d(2x+z_T^*)}{dx} = 2 + \frac{dz_T^*}{dx} = 2 - \frac{(wL^{''}(x+\frac{z_T^*}{2}) + (1-w)L^{''}(x+z_T^*))}{(\frac{w}{2}L^{''}(x+\frac{z_T^*}{2}) + (1-w)L^{''}(x+z_T^*))}$$

It is to be shown that $\frac{dy_T^*}{dx} \geq 0$. *Proof by contradiction* is used to prove the result. Let us assume that $\frac{dy_T^*}{dx} < 0$.

$$\Rightarrow (wL^{''}(x+\frac{z_T^*}{2}) + (1-w)L^{''}(x+z_T^*)) > 2(\frac{w}{2}L^{''}(x+\frac{z_T^*}{2}) + (1-w)L^{''}(x+z_T^*))$$

$$\Rightarrow (1-w)L^{''}(x+z_T^*) < 0$$

The above result is never possible since $0 \leq w \leq 0.25$ and $L^{''}(\cdot) \geq 0$. Thus our assumption that $\frac{dy_T^*}{dx} < 0$ is incorrect. $\square$

(v) From *(iv)*, it is known that,

$$\frac{dz_T^*}{dx} = -\{\frac{(wL^{''}(x+\frac{z_T^*}{2}) + (1-w)L^{''}(x+z_T^*))}{(\frac{w}{2}L^{''}(x+\frac{z_T^*}{2}) + (1-w)L^{''}(x+z_T^*))}\}$$

Since w, $(1-w)$ and $L^{''}(\cdot)$ are always positive, $\frac{dz_T^*}{dx} < 0$. Also, from *(i)*, $z_1^* = z_2^* = \frac{z_T^*}{2}$ and thus $\frac{dz_i^*}{dx} < 0$; i=1,2.

$\square$

(vi) Optimal Cost C is given by,

$$C(x,x,\alpha,\alpha) = 2c[\frac{y_T^*}{2} - x] + 2[L(\frac{y_T^*}{2})][(1-\alpha)^2+\alpha^2] + 2[L(x)+L(y_T^*-x)][\alpha(1-\alpha)]$$

where $x_1 = x_2 (= x)$, from *(i)*, $y_1^* = y_2^* = \frac{y_T^*}{2}$. Differentiating $w.r.t$ α gives,

$$\begin{aligned}\frac{dC}{d\alpha} &= c[\frac{dy_T^*}{d\alpha}] + 2\{\ [L(\frac{y_T^*}{2})][2\alpha - 2(1-\alpha)] + \frac{[(1-\alpha)^2+\alpha^2]}{2}[L^{'}(\frac{y_T^*}{2})][\frac{dy_T^*}{d\alpha}]\ \}\\ &+2\{\ [L(x)+L(y_T^*-x)][1-2\alpha] + [\alpha(1-\alpha)][L^{'}(y_T^*-x)][\frac{dy_T^*}{d\alpha}]\ \}\end{aligned}$$

$$\Rightarrow \frac{dz_T^*}{d\alpha} = (\frac{dz_T^*}{dw})(\frac{dw}{d\alpha})$$

$$\Rightarrow \frac{dz_T^*}{d\alpha} = \frac{L'(x+z_T^*) - L'(x+\frac{z_T^*}{2})}{\frac{w}{2}L''(x+\frac{z_T^*}{2}) + +(1-w)L''(x+z_T^*)}(\frac{dw}{d\alpha})$$

Since $\frac{dw}{d\alpha} = 2(2\alpha - 1) = 0$ at $\alpha = 0.5$, $\frac{dz_T^*}{d\alpha} = 0$ and thus $\frac{dy_T^*}{d\alpha} = 0$. Thus, at $\alpha = 0.5$, i.e., $\sigma = 0.25$, $\frac{dC}{d\alpha} = 0$.

$$\begin{aligned}\frac{d^2C}{d\alpha^2} &= [\frac{d^2y_T^*}{d\alpha^2}]\{\ c + [(1-\alpha)^2+\alpha^2][L^{'}(\frac{y_T^*}{2})] + 2[\alpha(1-\alpha)][L^{'}(y_T^*-x)]\ \}\\ &+[\frac{dy_T^*}{d\alpha}]\{\ [4\alpha-2][L^{'}(\frac{y_T^*}{2})] + 2[1-2\alpha][L^{'}(y_T^*-x)]\ \}\\ &+4\{\ 2L(\frac{y_T^*}{2}) - L(x) - L(y_T^*-x)\ \}\end{aligned}$$

This expression can be reduced to,

$$\frac{d^2C}{d\alpha^2} = 4\{\ 2L(\frac{y_T^*}{2}) - L(x) - L(y_T^*-x)\ \} \Rightarrow \frac{d^2C}{d\alpha^2} \le 0$$

$(c + [(1-\alpha)^2+\alpha^2][L^{'}(\frac{y_T^*}{2})] + 2[\alpha(1-\alpha)][L^{'}(y_T^*-x)] = 0$, as per the first order condition.)

Thus, $C(x,x,\alpha,\alpha)$ reaches its maximum value at $\alpha = 0.5$. At $\alpha = 0.5$, the uncertainty measure $\sigma = \alpha(1-\alpha) = 0.25$ is also maximized.

Optimal cost can also be written as,

$$C(x,x,\alpha,\alpha) = c(z_T^*) + (1-A)(2L(x+\frac{z_T^*}{2})) + (A)(L(x)+L(x+z_T^*))$$

where $A = 1 - w = 2\alpha(1-\alpha)$. Differentiating w.r.t x gives,

$$\frac{dC(x,x,\alpha,\alpha)}{dx} = c\frac{dz_T^*}{dx} + (1-A)(2L^{'}(x+\frac{z_T^*}{2})(1+\frac{1}{2}\frac{dz_T^*}{dx})) + (A)(L^{'}(x) + L^{'}(x+z_T^*)(1+\frac{dz_T^*}{dx}))$$

$$\Rightarrow \frac{dC(x,x,\alpha,\alpha)}{dx} = \frac{dz_T^*}{dx}(c + (1-A)L^{'}(x+\frac{z_T^*}{2}) + (A)L^{'}(x+z_T^*))$$

$$+2(1-A)L^{'}(x+\frac{z_T^*}{2})+(A)L^{'}(x)+(A)L^{'}(x+z_T^*)$$

Using the first-order condition, i.e., $c+(1-A)L^{'}(x+\frac{z_T^*}{2})+(A)L^{'}(x+z_T^*)=0$,

$$\Rightarrow \frac{dC(x,x,\alpha,\alpha)}{dx}=-c+(A)[L^{'}(x)-L^{'}(x+z_T^*)]+L^{'}(x+\frac{z_T^*}{2})$$

From the above equation and first-order condition, $\frac{dC(x,x,\alpha,\alpha)}{dx}<0$. □

C. Analysis of Figure 2

(i) Proof of linearity of $C(0,0,\alpha,\alpha|\sigma_D=0)$ *in* σ

Let $P=[(1-\alpha)^2+\alpha^2]$ and $c=0$, expected cost of the optimal policy is given by,

$$C(0,0,\alpha,\alpha|\sigma_D=0)=2P(L(z_1^*))+(1-P)(L(0)+L(2z_1^*))$$

$$C(0,0,\alpha,\alpha|\sigma_D=0)=(1-P)(p(\mu-0)+h(2\mu-\mu))$$

$$C(0,0,\alpha,\alpha|\sigma_D=0)=2\sigma(h+p)\mu$$

$$\frac{dC(0,0,\alpha,\alpha|\sigma_D=0)}{d\sigma}=2(h+p)\mu$$

Thus, $C(0,0,\alpha,\alpha|\sigma_D=0)$ is constant in σ when $\sigma_D=0$.

(ii) Proof of linearity of $C(0,0,0,0|\sigma_D)$ *in* σ_D

In case of no errors, the optimal order quantity of both products is given as, $z^*(0,0,0,0)=F_c(b-a)+a$ (where F_c is the critical-fractile), and the expected cost of the optimal policy is,

$$C(0,0,0,0|\sigma_D)=\sqrt{1}2\sigma_D[hF_c^2+p(1-F_c)^2]$$

$$\frac{dC(0,0,0,0|\sigma_D)}{d\sigma_D}=\sqrt{1}2[hF_c^2+p(1-F_c)^2]$$

Thus, $C(0,0,0,0|\sigma_D)$ is constant in σ_D when $\sigma=0$.

(iii) Proof of $\{\frac{dC(0,0,\alpha,\alpha|\sigma_D=0)}{d\sigma}>\frac{dC(0,0,0,0|\sigma_D)}{d\sigma_D}\}$, *if* $\mu>\frac{\sqrt{1}2F_c(1-F_c)}{2}$

Using *Proof by contradiction*, let $2(h+p)\mu\leq\sqrt{1}2[hF_c^2+p(1-F_c)^2]$.

$$\Rightarrow 2(h+p)\mu\leq\sqrt{1}2\frac{hp}{h+p}$$

$$\Rightarrow 2\mu\leq\sqrt{1}2(\frac{p}{h+p})(\frac{h}{h+p})$$

$$\Rightarrow 2\mu\leq\sqrt{1}2F_c(1-F_c)$$

The above result contradicts our assumption that, $\mu>\frac{\sqrt{1}2F_c(1-F_c)}{2}$. □

D. Proof of Theorem 3

(i) First-order conditions are given by,

$$(1-\alpha_1)(1-\alpha_2)[F(x+z_2^*)]+(1-\alpha_1)(\alpha_2)[F(x+z_1^*+z_2^*)]+(\alpha_1)(1-\alpha_2)[F(x+z_1^*+z_2^*)]+(\alpha_1)(\alpha_2)[F(x+z_2^*)] = \frac{p-c}{p+h}$$

$$(1-\alpha_1)(1-\alpha_2)[F(x+z_1^*)]+(1-\alpha_1)(\alpha_2)[F(x+z_1^*+z_2^*)]+(\alpha_1)(1-\alpha_2)[F(x+z_1^*+z_2^*)]+(\alpha_1)(\alpha_2)[F(x+z_1^*)] = \frac{p-c}{p+h}$$

$$\Rightarrow F(x+z_2^*)[(1-\alpha_1)(1-\alpha_2)+(\alpha_1)(\alpha_2)] = F(x+z_1^*)[(1-\alpha_1)(1-\alpha_2)+(\alpha_1)(\alpha_2)]$$

$$\Rightarrow F(x+z_2^*) = F(x+z_1^*)$$

$$\Rightarrow z_1^* = z_2^*$$

$$\Rightarrow y_1^* = y_2^* \quad \square$$

(ii) Let the first-order conditions be represented as,

$$f_1 = c+(1-\alpha_1)(1-\alpha_2)L^{'}(x+z_1^*)+\alpha_1(1-\alpha_2)L^{'}(x+z_T^*)+\alpha_2(1-\alpha_1)L^{'}(x+z_T^*)+\alpha_1\alpha_2L^{'}(x+z_1^*) = 0$$

Let α_1 be fixed. Then, differentiating w.r.t α_2 gives,

$$[\frac{(1-\alpha_1)(1-\alpha_2)}{2}L^{"}(x+\frac{z_T^*}{2})+(1-\alpha_1)(\alpha_2)L^{"}(x+z_T^*)+(\alpha_1)(1-\alpha_2)L^{"}(x+z_T^*)+\frac{(\alpha_1\alpha_2)}{2}L^{"}(x+\frac{z_T^*}{2})](\frac{dz_T^*}{d\alpha_2}) = (1-\alpha_1)(L^{'}(x+\frac{z_T^*}{2})-L^{'}(x+z_T^*))+(\alpha_1)(L^{'}(x+z_T^*)-L^{'}(x+\frac{z_T^*}{2}))$$

$$\Rightarrow \frac{dy_T^*}{d\alpha_2} = \{\frac{(1-2\alpha_1)(L^{'}(x+\frac{z_T^*}{2})-L^{'}(x+z_T^*))}{\frac{(1-\alpha_1)(1-\alpha_2)}{2}L^{"}(x+\frac{z_T^*}{2})+(1-\alpha_1)(\alpha_2)L^{"}(x+z_T^*)+(\alpha_1)(1-\alpha_2)L^{"}(x+z_T^*)+\frac{(\alpha_1\alpha_2)}{2}L^{"}(x+\frac{z_T^*}{2})}\}$$

It is known that $L^{'}(x+\frac{z_T^*}{2}) < L^{'}(x+z_T^*)$ and from the above equation, if $\alpha_1(fixed) < 0.5$, $\frac{dy_T^*}{d\alpha_2} < 0$, i.e., total basestock (and individual basestocks) is monotonically decreasing in α_2. $\square$

(iii) From *(ii)*, if $\alpha_1(fixed) > 0.5$, $\frac{dy_T^*}{d\alpha_2} > 0$, i.e., total basestock (and individual basestocks) is monotonically increasing in α_2. $\square$

(iv) The proof is divided into two parts,

(a) $\alpha_1(fixed) < 0.5$

It is known from *(ii)* that total basestock decreases in α_2. Thus, it will suffice to prove that $y_T^*(x,x,\alpha_1,0) <$

$y_T^*(x,x,0,0)$.

The first order conditions at $\alpha_1(fixed) < 0.5$ and $\alpha_2 = 0$ can be written as,

$$c + (1-\alpha_1)L^{'}(x + \frac{z_T^*}{2}) + \alpha_1 L^{'}(x + z_T^*) = 0$$

$$\Rightarrow L^{'}(x + \frac{z_T^*}{2}) = \alpha_1 (L^{'}(x + \frac{z_T^*}{2}) - L^{'}(x + z_T^*)) - c$$

For the no-error case, i.e., at $\alpha_1 = \alpha_2 = 0$, the first-order condition is given as,

$$\Rightarrow L^{'}(x + \frac{z_T^*(x,x,0,0)}{2}) = -c$$

Comparing the above two expressions, it is clear that $z_T^*(x,x,\alpha_1,0) < z_T^*(x,x,0,0)$ (since $L'(y_1) > L'(y_2)$ if $y_1 > y_2$). Also, the initial inventories being the same, $y_T^*(x,x,\alpha_1,0) < y_T^*(x,x,0,0)$.

(b) $\alpha_1(fixed) > 0.5$

It is known from *(iii)* that total basestock increases in α_2. Thus, it will suffice to prove that $y_T^*(x,x,\alpha_1,1) < y_T^*(x,x,0,0)$.

The first order conditions at $\alpha_1(fixed) > 0.5$ and $\alpha_2 = 1$ can be written as,

$$c + (1-\alpha_1)L^{'}(x + z_T^*) + \alpha_1 L^{'}(x + \frac{z_T^*}{2}) = 0$$

$$\Rightarrow L^{'}(x + \frac{z_T^*}{2}) = (1 - \frac{1}{\alpha_1})L^{'}(x + z_T^*) - c$$

Comparing the above expression with the no-error case in *(a)*, it is clear that $z_T^*(x,x,\alpha_1,1) < z_T^*(x,x,0,0)$. Also, the initial inventories being the same, $y_T^*(x,x,\alpha_1,1) < y_T^*(x,x,0,0)$. $\square$

(v) Rewriting the first-order condition,

$$f_1 = c + (1-\alpha_1)(1-\alpha_2)L^{'}(x + \frac{z_T^*}{2}) + \alpha_1(1-\alpha_2)L^{'}(x + z_T^*) + \alpha_2(1-\alpha_1)L^{'}(x + z_T^*) + \alpha_1\alpha_2 L^{'}(x + \frac{z_T^*}{2}) = 0$$

Differentiating w.r.t x,

$$\begin{aligned}\Rightarrow \frac{dz_T^*}{dx}\{&\frac{(1-\alpha_1)(1-\alpha_2)}{2}L^{"}(x + \frac{z_T^*}{2}) + \frac{(\alpha_1\alpha_2)}{2}L^{"}(x + \frac{z_T^*}{2}) + \alpha_1(1-\alpha_2)L^{"}(x + z_T^*) + \alpha_2(1-\alpha_1)L^{"}(x + z_T^*)\} \\ &= -\{(1-\alpha_1)(1-\alpha_2)L^{"}(x + \frac{z_T^*}{2}) + (\alpha_1\alpha_2)L^{"}(x + \frac{z_T^*}{2}) + \alpha_1(1-\alpha_2)L^{"}(x + z_T^*) + \alpha_2(1-\alpha_1)L^{"}(x + z_T^*)\}\end{aligned}$$

$$\Rightarrow \frac{dy_T^*}{dx} = \frac{d(z_T^* + 2x)}{dx} = 2 - \frac{A}{B}$$

where $A = (1-\alpha_1)(1-\alpha_2)L"(x+\frac{z_T^*}{2}) + (\alpha_1\alpha_2)L"(x+\frac{z_T^*}{2}) + \alpha_1(1-\alpha_2)L"(x+z_T^*) + \alpha_2(1-\alpha_1)L"(x+z_T^*)$

and $B = \frac{(1-\alpha_1)(1-\alpha_2)}{2}L"(x+\frac{z_T^*}{2}) + \frac{(\alpha_1\alpha_2)}{2}L"(x+\frac{z_T^*}{2}) + \alpha_1(1-\alpha_2)L"(x+z_T^*) + \alpha_2(1-\alpha_1)L"(x+z_T^*)$.

Using *Proof by contradiction*, let $\frac{dy_T^*}{dx} < 0$, i.e., $A > 2B$. Then,

$$\{(1-\alpha_1)(1-\alpha_2)L"(x+\frac{z_T^*}{2}) + (\alpha_1\alpha_2)L"(x+\frac{z_T^*}{2}) + \alpha_1(1-\alpha_2)L"(x+z_T^*) + \alpha_2(1-\alpha_1)L"(x+z_T^*)\}$$
$$> 2\{\frac{(1-\alpha_1)(1-\alpha_2)}{2}L"(x+\frac{z_T^*}{2}) + \frac{(\alpha_1\alpha_2)}{2}L"(x+\frac{z_T^*}{2}) + \alpha_1(1-\alpha_2)L"(x+z_T^*) + \alpha_2(1-\alpha_1)L"(x+z_T^*)\}$$

$$\Rightarrow \alpha_1(1-\alpha_2)L"(x+z_T^*) + \alpha_2(1-\alpha_1)L"(x+z_T^*) < 0$$

$$\Rightarrow L"(x+z_T^*)(\alpha_1+\alpha_2-2\alpha_1\alpha_2) < 0$$

Now, $L"(\cdot) > 0$, $\alpha_1\alpha_2 < \alpha_1$ and $\alpha_1\alpha_2 < \alpha_2$; thus the above expression is invalid. Hence, our assumption that $\frac{dy_T^*}{dx} < 0$ is invalid. □

(vi) From *(v)*, $\frac{dz_T^*}{dx} = -\{\frac{A}{B}\}$. Also, $A > 0$ and $B > 0$ (since $L"(\cdot) > 0$ and $0 < \alpha_1, \alpha_2 < 1$); thus, $\frac{dz_T^*}{dx} < 0$. Since $x_1 = x_2$, $z_1^* = z_2^* = \frac{z_T^*}{2}$, $\frac{dz_1^*}{dx} = \frac{dz_2^*}{dx} < 0$. □

E. Proof of Theorem 4

(i) Let the first-order conditions be represented as,

$$f_1 = c + (1-\alpha)^2(L'(x_1+z_1^*)) + \alpha(1-\alpha)(L'(x_1+z_T^*)) + \alpha(1-\alpha)(L'(x_2+z_T^*)) + (\alpha^2)(L'(x_2+z_1^*)) = 0$$

$$f_2 = c + (1-\alpha)^2(L'(x_2+z_T^*-z_1^*)) + \alpha(1-\alpha)(L'(x_1+z_T^*)) + \alpha(1-\alpha)(L'(x_2+z_T^*)) + (\alpha^2)(L'(x_1+z_T^*-z_1^*)) = 0$$

Differentiating the above equations w.r.t α gives,

$$B\frac{dz_T^*}{d\alpha} = A + C\frac{dz_1^*}{d\alpha}$$

$$B'\frac{dz_T^*}{d\alpha} = A' + C'\frac{dz_1^*}{d\alpha}$$

where A, B, C, A', B' and C' are defined as,

$$A = 2(1-\alpha)(L'(x_1+z_1^*)) - (1-2\alpha)(L'(x_1+z_T^*)) - (1-2\alpha)(L'(x_2+z_T^*)) - (2\alpha)(L'(x_2+z_1^*))$$

$$B = \alpha(1-\alpha)(L"(x_1+z_T^*)) + \alpha(1-\alpha)(L"(x_2+z_T^*))$$

$$C = -[(1-\alpha)^2(L"(x_1+z_1^*)) + (\alpha^2)(L"(x_2+z_1^*))]$$

$$A^{'} = 2(1-\alpha)(L^{'}(x_2+z_T^*-z_1^*))-(1-2\alpha)(L^{'}(x_1+z_T^*))-(1-2\alpha)(L^{'}(x_2+z_T^*))-(2\alpha)(L^{'}(x_1+z_T^*-z_1^*))$$

$$B^{'} = (1-\alpha)^2(L^{"}(x_2+z_T^*-z_1^*))+\alpha(1-\alpha)(L^{"}(x_1+z_T^*))+\alpha(1-\alpha)(L^{"}(x_2+z_T^*))+(\alpha^2)(L^{"}(x_1+z_T^*-z_1^*))$$

$$C^{'} = (1-\alpha)^2(L^{"}(x_2+z_T^*-z_1^*))+(\alpha^2)(L^{"}(x_1+z_T^*-z_1^*))$$

Eliminating $\frac{dz_1^*}{d\alpha}$ from the above equations gives,

$$\frac{dz_T^*}{d\alpha} = \frac{AC^{'}-A^{'}C}{BC^{'}-B^{'}C}$$

Now, $AC^{'}-A^{'}C$, at $\alpha=0.5$ can be given as,

$$\begin{aligned} AC^{'}-A^{'}C = & \{[L^{'}(x_1+z_1^*)-L^{'}(x_2+z_1^*)][L^{"}(x_1+z_T^*-z_1^*)+L^{"}(x_2+z_T^*-z_1^*)] \\ & [L^{'}(x_2+z_T^*-z_1^*)-L^{'}(x_1+z_T^*-z_1^*)][L^{"}(x_1+z_1^*)+L^{"}(x_2+z_1^*)]\} \end{aligned}$$

Using the first-order conditions, it can be shown that $z_1^*=z_2^*=\frac{z_T^*}{2}$ at $\alpha=0.5$.

$$\begin{aligned} \Rightarrow (AC^{'}-A^{'}C)|(\alpha=0.5) = & \{[L^{'}(x_1+\frac{z_T^*}{2})-L^{'}(x_2+\frac{z_T^*}{2})][L^{"}(x_1+\frac{z_T^*}{2})+L^{"}(x_2+\frac{z_T^*}{2})] \\ & [L^{'}(x_2+\frac{z_T^*}{2})-L^{'}(x_1+\frac{z_T^*}{2})][L^{"}(x_1+\frac{z_T^*}{2})+L^{"}(x_2+\frac{z_T^*}{2})]\} \end{aligned}$$

$$\Rightarrow (AC^{'}-A^{'}C)|(\alpha=0.5)=0$$

Thus, $\frac{dz_T^*}{d\alpha}=\frac{dy_T^*}{d\alpha}=0$ at $\alpha=0.5$. It is also straightforward to show that $\frac{d^2z_T^*}{d\alpha^2}>0$ at $\alpha=0.5$. Thus the total basestock decreases as the risk $\sigma=\alpha(1-\alpha)$ increases. □

(ii) From Theorem (i), it follows that as α (or σ) is increased from 0, the total basestock will drop. □

(iii) Let x_2 be fixed and x_1 is varying. The first-order conditions are given by,

$$f_1 = c+(1-\alpha)^2L^{'}(y_1^*)+\alpha(1-\alpha)L^{'}(y_T^*-x_2)+\alpha(1-\alpha)L^{'}(y_T^*-x_1)+(\alpha^2)L^{'}(y_1^*+x_2-x_1)=0$$

$$f_2 = c+(1-\alpha)^2L^{'}(y_2^*)+\alpha(1-\alpha)L^{'}(y_T^*-x_2)+\alpha(1-\alpha)L^{'}(y_T^*-x_1)+(\alpha^2)L^{'}(y_2^*+x_1-x_2)=0$$

Differentiating w.r.t. x_1 gives,

$$\begin{aligned} \frac{df_1}{dx_1} = & (1-\alpha)^2L^{"}(y_1^*)\frac{dy_1^*}{dx_1}+\alpha(1-\alpha)L^{"}(y_T^*-x_2)\frac{dy_T^*}{dx_1} \\ & +\alpha(1-\alpha)L^{"}(y_T^*-x_1)(\frac{dy_T^*}{dx_1}-1)+(\alpha^2)L^{"}(y_1^*+x_2-x_1)(\frac{dy_1^*}{dx_1}-1)=0 \end{aligned}$$

$$\frac{df_2}{dx_1} = (1-\alpha)^2L^{"}(y_T^*-y_1^*)(\frac{dy_T^*}{dx_1}-\frac{dy_1^*}{dx_1})+\alpha(1-\alpha)L^{"}(y_T^*-x_2)\frac{dy_T^*}{dx_1}$$

$$+\alpha(1-\alpha)L^{"}(y_T^*-x_1)(\frac{dy_T^*}{dx_1}-1)+(\alpha^2)L^{"}(y_T^*-y_1^*+x_1-x_2)(\frac{dy_T^*}{dx_1}-\frac{dy_1^*}{dx_1}+1)=0$$

The above equations can be represented as,

$$A_1\frac{dy_1^*}{dx_1}+B_1\frac{dy_T^*}{dx_1}=C_1$$

$$A_2\frac{dy_1^*}{dx_1}+B_2\frac{dy_T^*}{dx_1}=C_2$$

where,

$$A_1=(1-\alpha)^2L^{"}(y_1^*)+(\alpha^2)L^{"}(y_1^*+x_2-x_1)$$

$$B_1=\alpha(1-\alpha)L^{"}(y_T^*-x_2)+\alpha(1-\alpha)L^{"}(y_T^*-x_1)$$

$$C_1=\alpha(1-\alpha)L^{"}(y_T^*-x_1)+(\alpha^2)L^{"}(y_1^*+x_2-x_1)$$

$$A_2=-\{(1-\alpha)^2L^{"}(y_T^*-y_1^*)+(\alpha^2)L^{"}(y_T^*-y_1^*+x_1-x_2)\}$$

$$B_2=\{(1-\alpha)^2L^{"}(y_T^*-y_1^*)+(\alpha^2)L^{"}(y_T^*-y_1^*+x_1-x_2)+\alpha(1-\alpha)L^{"}(y_T^*-x_2)+\alpha(1-\alpha)L^{"}(y_T^*-x_1)\}$$

$$C_2=\alpha(1-\alpha)L^{"}(y_T^*-x_1)-(\alpha^2)L^{"}(y_T^*-y_1^*+x_1-x_2)$$

Eliminating $\frac{dy_T^*}{dx_1}$ from the above equations,

$$\frac{dy_1^*}{dx_1}=\frac{B_1C_2-B_2C_1}{B_1A_2-B_2A_1}$$

$$\begin{aligned}B_1C_2-B_2C_1 = {} & \alpha^2(1-\alpha)^2L^{"}(y_T^*-x_2)L^{"}(y_T^*-x_1)-\alpha^3(1-\alpha)L^{"}(y_T^*-x_2)L^{"}(y_T^*-y_1^*+x_1-x_2)\\
&+\alpha^2(1-\alpha)^2L^{"}(y_T^*-x_1)L^{"}(y_T^*-x_1)-\alpha^3(1-\alpha)L^{"}(y_T^*-x_1)L^{"}(y_T^*-y_1^*+x_1-x_2)\\
&-\alpha(1-\alpha)^3L^{"}(y_T^*-y_1^*)L^{"}(y_T^*-x_1)-\alpha^2(1-\alpha)^2L^{"}(y_T^*-x_2)L^{"}(y_T^*-x_1)\\
&-\alpha^2(1-\alpha)^2L^{"}(y_T^*-x_1)L^{"}(y_T^*-x_1)-\alpha^3(1-\alpha)L^{"}(y_T^*-x_1)L^{"}(y_T^*-y_1^*+x_1-x_2)\\
&-\alpha^2(1-\alpha)^2L^{"}(y_T^*-y_1^*)L^{"}(y_1^*+x_2-x_1)-\alpha^3(1-\alpha)L^{"}(y_T^*-x_2)L^{"}(y_1^*+x_2-x_1)\\
&-\alpha^3(1-\alpha)L^{"}(y_T^*-x_1)L^{"}(y_1^*+x_2-x_1)-\alpha^4L^{"}(y_T^*-y_1^*+x_1-x_2)L^{"}(y_1^*+x_2-x_1)\end{aligned}$$

It is easy to observe that $B_1C_2-B_2C_1<0$. Also, $B_1A_2-B_2A_1<0$. Thus, $\frac{dy_1^*}{dx_1}>0$. Similarly, using the symmetry of the first-order conditions w.r.t x_1 and x_2, it is clear that $\frac{dy_2^*}{dx_2}>0$ when x_1 is fixed. $\square$

(iv) From (i), the expressions for f_1 and f_2 are known. Defining the following terms,

$$A_1 = (1-\alpha)^2(L^{"}(x_1+z_1^*)) + \alpha^2(L^{"}(x_2+z_1^*))$$

$$B_1 = \alpha(1-\alpha)(L^{"}(x_1+z_T^*)) + \alpha(1-\alpha)(L^{"}(x_2+z_T^*))$$

$$C_1 = -[(1-\alpha)^2(L^{"}(x_1+z_1^*)) + \alpha(1-\alpha)(L^{"}(x_1+z_T^*))]$$

$$A_2 = -[(1-\alpha)^2(L^{"}(x_2+z_T^*-z_1^*)) + \alpha^2(L^{"}(x_1+z_T^*-z_1^*))]$$

$$B_2 = [(1-\alpha)^2(L^{"}(x_2+z_T^*-z_1^*)) + \alpha(1-\alpha)(L^{"}(x_1+z_T^*)) + \alpha(1-\alpha)(L^{"}(x_2+z_T^*)) + \alpha^2(L^{"}(x_1+z_T^*-z_1^*))]$$

$$C_2 = -[\alpha(1-\alpha)(L^{"}(x_1+z_T^*)) + \alpha^2(L^{"}(x_1+z_T^*-z_1^*))]$$

In this case, differentiating the first-order conditions w.r.t x_1 (with x_2 fixed) gives,

$$A_1\frac{dz_1^*}{dx_1} + B_1\frac{dz_T^*}{dx_1} = C_1$$

$$A_2\frac{dz_1^*}{dx_1} + B_2\frac{dz_T^*}{dx_1} = C_2$$

Eliminating $\frac{dz_1^*}{dx_1}$ from the above two equations gives,

$$\frac{dz_T^*}{dx_1} = \frac{A_1C_2 - A_2C_1}{A_1B_2 - A_2B_1}$$

It is easy to observe that $(A_1B_2 - A_2B_1) > 0$ while $(A_1C_2 - A_2C_1) < 0$ since $L^{"}(.)$ is always positive. Thus, $\frac{dz_T^*}{dx_1} < 0$. Similarly, it can be shown that if x_1 is fixed and x_2 is changed, the total order-quantity would still decrease. □

(v) First-order conditions at $(x_1, x_2, \alpha, \alpha)$ are given by,

$$f_1 = c + (1-\alpha)^2(L^{'}(x_1+z_1^*)) + \alpha(1-\alpha)(L^{'}(x_1+z_T^*)) + \alpha(1-\alpha)(L^{'}(x_2+z_T^*)) + (\alpha^2)(L^{'}(x_2+z_1^*)) = 0$$

$$f_2 = c + (1-\alpha)^2(L^{'}(x_2+z_2^*)) + \alpha(1-\alpha)(L^{'}(x_1+z_T^*)) + \alpha(1-\alpha)(L^{'}(x_2+z_T^*)) + (\alpha^2)(L^{'}(x_1+z_2^*)) = 0$$

where z_1^*, z_2^* are the optimal order quantities and $z_T^* = z_1^* + z_2^*$.

Now, first-order conditions at $(x_1, x_2, 1-\alpha, 1-\alpha)$ are given by,

$$f_1^{'} = c + (1-\alpha)^2(L^{'}(x_2+z_{11}^*)) + \alpha(1-\alpha)(L^{'}(x_1+z_{TT}^*)) + \alpha(1-\alpha)(L^{'}(x_2+z_{TT}^*)) + (\alpha^2)(L^{'}(x_1+z_{11}^*)) = 0$$

$$f_2^{'} = c + (1-\alpha)^2(L^{'}(x_1+z_{22}^*)) + \alpha(1-\alpha)(L^{'}(x_1+z_{TT}^*)) + \alpha(1-\alpha)(L^{'}(x_2+z_{TT}^*)) + (\alpha^2)(L^{'}(x_2+z_{22}^*)) = 0$$

where z_{11}^*, z_{22}^* are the optimal order quantities and $z_{TT}^* = z_{11}^* + z_{22}^*$. It is easy to observe that z_{11}^* and z_{22}^* when substituted by z_2^* and z_1^* respectively, is optimal. □

F. General results for Uniform Demand Distribution

This section discusses all possible cases for the totally-symmetric case, when demand is uniformly distributed over the range U[a,b]. The two first-order conditions, (4) and (5), reduce to:

$$PF(x+z_1^*)+(1-P)F(x+2z_1^*)=F_c$$

where $P=[(1-\alpha)^2+(\alpha)^2]$, $(1-P)=[2\alpha(1-\alpha)]$, $c=0$ and $z_1^*=z_2^*$.

Also, expected cost is given by,

$$C(x,x,\alpha,\alpha)=2P[L(x+z_1^*)]+(1-P)[L(x)+L(x+2z_1^*)]$$

There are eight realizations of the above equations, depending on the optimal value of z_1^* and the initial inventory x. These realizations stem from the nature of the loss function, defined as,

$L(y)=h(y-\mu)+\frac{(h+p)}{2(b-a)}(b-y)^2$, if $a\leq y\leq b$

$L(y)=h(y-\mu)$, if $y>b$

$L(y)=p(\mu-y)$, if $y<a$, where $\mu=\frac{(a+b)}{2}$, p and h being the penalty and holding costs respectively.

However, three of these realizations are not feasible, thus, the other five cases are discussed below.

(i) $(x+z_1^*)>a$, $(x+2z_1^*)<b$ and $x>a$

Optimal order-quantity can be easily shown to be,

$$z_1^*=z_2^*=\frac{(F_c(b-a)-(x-a))}{(2-P)}$$

Similarly, expected cost is given by,

$$\begin{aligned}C(x,x,\alpha,\alpha) &= 2P[h(x+z_1^*-\mu)+\frac{(h+p)}{2(b-a)}(b-x-z_1^*)^2]\\ &\quad +(1-P)[h(2x+2z_1^*-2\mu)+\frac{(h+p)}{2(b-a)}((b-x)^2+(b-x-2z_1^*)^2)]\end{aligned}$$

In case of no errors($\alpha=0$), it is obvious that, $z_1^*(x,x,0,0)=F_c(b-a)-(x-a)$ and the corresponding expected cost is given by, $C(x,x,0,0)=(b-a)[hF_c^2+p(1-F_c)^2]$.

$$C_p=\frac{C(x,x,\alpha,\alpha)}{C(x,x,0,0)}$$

$$=\frac{2P[h(x+z_1^*-\mu)+\frac{(h+p)}{2(b-a)}(b-x-z_1^*)^2]+(1-P)[h(2x+2z_1^*-2\mu)+\frac{(h+p)}{2(b-a)}((b-x)^2+(b-x-2z_1^*)^2)]}{(b-a)[hF_c^2+p(1-F_c)^2]}$$

(ii) $(x+z_1^*)>a$, $(x+2z_1^*)<b$ and $x<a$

Optimal order-quantity is given by,

$$z_1^* = z_2^* = \frac{(F_c(b-a)-(x-a))}{(2-P)}$$

Expected cost is given by,

$$\begin{aligned} C(x,x,\alpha,\alpha) =\ & 2P[h(x+z_1^*-\mu)+\frac{(h+p)}{2(b-a)}(b-x-z_1^*)^2] \\ & +(1-P)[p(\mu-x)+h(x+2z_1^*-\mu)+\frac{(h+p)}{2(b-a)}((b-x-2z_1^*)^2)] \end{aligned}$$

$$C_p = \frac{2P[h(x+z_1^*-\mu)+\frac{(h+p)}{2(b-a)}(b-x-z_1^*)^2]+(1-P)[p(\mu-x)+h(x+2z_1^*-\mu)+\frac{(h+p)}{2(b-a)}((b-x-2z_1^*)^2)]}{(b-a)[hF_c^2+p(1-F_c)^2]}$$

(iii) $(x+z_1^*)>a$, $(x+2z_1^*)>b$ and $x>a$

Optimal order-quantity is given by,

$$z_1^* = z_2^* = \frac{(F_c+P-1)(b-a)}{P}-(x-a)$$

Expected cost is given by,

$$\begin{aligned} C(x,x,\alpha,\alpha) =\ & 2P[h(x+z_1^*-\mu)+\frac{(h+p)}{2(b-a)}(b-x-z_1^*)^2] \\ & +(1-P)[h(x-\mu)+\frac{(h+p)}{2(b-a)}(b-x)^2+h(x+2z_1^*-\mu)] \end{aligned}$$

$$C_p = \frac{2P[h(x+z_1^*-\mu)+\frac{(h+p)}{2(b-a)}(b-x-z_1^*)^2]+(1-P)[h(x-\mu)+\frac{(h+p)}{2(b-a)}(b-x)^2+h(x+2z_1^*-\mu)]}{(b-a)[hF_c^2+p(1-F_c)^2]}$$

(iv) $(x+z_1^*)>a$, $(x+2z_1^*)>b$ and $x<a$

Optimal order-quantity is given by,

$$z_1^* = z_2^* = \frac{(F_c+P-1)(b-a)}{P}-(x-a)$$

Expected cost is given by,

$$\begin{aligned} C(x,x,\alpha,\alpha) =\ & 2P[h(x+z_1^*-\mu)+\frac{(h+p)}{2(b-a)}(b-x-z_1^*)^2] \\ & +(1-P)[p(\mu-x)+h(x+2z_1^*-\mu)] \end{aligned}$$

$$C_p = \frac{2P[h(x+z_1^*-\mu)+\frac{(h+p)}{2(b-a)}(b-x-z_1^*)^2]+(1-P)[p(\mu-x)+h(x+2z_1^*-\mu)]}{(b-a)[hF_c^2+p(1-F_c)^2]}$$

(v) $(x+z_1^*) < a$, $(x+2z_1^*) < b$ and $x < a$

Optimal order-quantity is given by,

$$z_1^* = z_2^* = \frac{F_c(b-a)}{2(1-P)} - \frac{(x-a)}{2}$$

Expected cost is given by,

$$\begin{aligned} C(x,x,\alpha,\alpha) = & \; 2P[p(\mu - x - z_1^*)] \\ & +(1-P)[p(\mu-x) + h(x+2z_1^* - \mu) + \frac{(h+p)}{2(b-a)}(b-x-2z_1^*)^2] \end{aligned}$$

$$C_p = \frac{2P[p(\mu-x-z_1^*)] + (1-P)[p(\mu-x) + h(x+2z_1^*-\mu) + \frac{(h+p)}{2(b-a)}(b-x-2z_1^*)^2]}{(b-a)[hF_c^2 + p(1-F_c)^2]}$$

G. Proof of Theorem 5

Considering all possible cases for the one-period scenario,

If $z_1^* = z_1^* = 0$;

$$\begin{aligned} C(x_1,x_2) = L(x_1) + L(x_2) \\ \Rightarrow \frac{dC(x_1,x_2)}{dx_1} = L^{'}(x_1) \end{aligned}$$

If $z_1^* > 0$ and $z_2^* = 0$;

$$C(x_1,x_2) = cz_1^* + (1-\alpha)(L(x_1+z_1^*) + L(x_2)) + (\alpha)(L(x_2+z_1^*) + L(x_1))$$

$$\begin{aligned} \Rightarrow \frac{dC(x_1,x_2)}{dx_1} = & \; c\frac{dz_1^*}{dx_1} + (1-\alpha)[L^{'}(x_1+z_1^*)(\frac{dz_1^*}{dx_1}+1) + 0] \\ & +(\alpha)[L^{'}(x_2+z_1^*)(\frac{dz_1^*}{dx_1}) + L^{'}(x_1))] \end{aligned}$$

As $z_1^* \to 0$ from R.H.S.,

$$\begin{aligned} \Rightarrow \frac{dC(x_1,x_2)}{dx_1} &= (1-\alpha)[L^{'}(x_1)] + (\alpha)[L^{'}(x_1)] \\ \Rightarrow \frac{dC(x_1,x_2)}{dx_1} &= L^{'}(x_1) \end{aligned}$$

If $z_2^* > 0$ and $z_1^* = 0$;

$$C(x_1,x_2) = cz_2^* + (1-\alpha)(L(x_2+z_2^*) + L(x_1)) + (\alpha)(L(x_1+z_2^*) + L(x_2))$$

$$\Rightarrow \frac{dC(x_1,x_2)}{dx_1} = c\frac{dz_2^*}{dx_1} + (1-\alpha)[L^{'}(x_2+z_2^*)(\frac{dz_2^*}{dx_1}) + L^{'}(x_1))]$$

$$+(\alpha)[L^{'}(x_1+z_2^*)(1+\frac{dz_2^*}{dx_1})+0]$$

As $z_2^* \to 0$ from R.H.S.,

$$\Rightarrow \frac{dC(x_1,x_2)}{dx_1} = (1-\alpha)[L^{'}(x_1)]+(\alpha)[L^{'}(x_1)]$$
$$\Rightarrow \frac{dC(x_1,x_2)}{dx_1} = L^{'}(x_1)$$

If $z_2^* > 0$ and $z_1^* > 0$;

$$\begin{aligned} C(x_1,x_2) = \; & c(z_1^*+z_2^*)+(1-\alpha)^2[L(x_1+z_1^*)+L(x_2+z_2^*)] \\ & +(\alpha)^2[L(x_1+z_2^*)+L(x_2+z_1^*)] \\ & +\alpha(1-\alpha)[L(x_1+z_1^*+z_2^*)+L(x_2)] \\ & +\alpha(1-\alpha)[L(x_2+z_1^*+z_2^*)+L(x_1)] \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{dC(x_1,x_2)}{dx_1} = \; & c(\frac{dz_1^*}{dx_1}+\frac{dz_2^*}{dx_1})+(1-\alpha)^2[L^{'}(x_1+z_1^*)(1+\frac{dz_1^*}{dx_1})+L^{'}(x_2+z_2^*)\frac{dz_2^*}{dx_1}] \\ & +(\alpha)^2[L^{'}(x_1+z_2^*)(1+\frac{dz_2^*}{dx_1})+L^{'}(x_2+z_1^*)\frac{dz_1^*}{dx_1}] \\ & +\alpha(1-\alpha)[L^{'}(x_1+z_1^*+z_2^*)(1+\frac{dz_1^*}{dx_1}+\frac{dz_2^*}{dx_1})+0] \\ & +\alpha(1-\alpha)[L^{'}(x_2+z_1^*+z_2^*)(\frac{dz_1^*}{dx_1}+\frac{dz_2^*}{dx_1})+L^{'}(x_1)] \end{aligned}$$

As $z_1^* \to 0$ and $z_2^* \to 0$ from R.H.S.,

$$\begin{aligned} \Rightarrow \frac{dC(x_1,x_2)}{dx_1} = \; & (1-\alpha)^2[L^{'}(x_1)]+(\alpha)^2[L^{'}(x_1)] \\ & \alpha(1-\alpha)[L^{'}(x_1)]+\alpha(1-\alpha)[L^{'}(x_1)] \\ & = L^{'}(x_1) \end{aligned}$$

Thus the derivative of the cost function in x_1, as shown above, are equal. Similar results can be shown for $\frac{dC(x_1,x_2)}{dx_2}$. This shows that the optimal cost of the expected policy for one-period is jointly convex in x_1 and x_2. Since $L(\cdot)$ is convex, by induction, $C^t(x_1^t, x_2^t)$ is convex in x_1^t, x_2^t. □

H. Proof of Theorem 6

(i) Equation (19) shows that the expected cost of the optimal policy is given as $C^R(x,x,\alpha_1,\alpha_2) = \min\{C(x,x,\alpha_1,\alpha_2), C(x,x,0,0)+[(1-\alpha_1)\alpha_2+\alpha_1(1-\alpha_2)]K\}$. It is known that a basestock policy is optimal for the no-recourse scenario. For the recourse scenario, if, $C^K(x,x,\alpha,\alpha) = C(x,x,0,0)+[(1-\alpha_1)\alpha_2+\alpha_1(1-\alpha_2)]K$, then,

$$C^K(x,x,\alpha,\alpha) = c(z_{1k}^*+z_{2k}^*)+L(x+z_{1k}^*)+L(x+z_{2k}^*)+[2\alpha(1-\alpha)]K$$

where z^*_{1k} and z^*_{2k} are order quantities for product-1 and product-2 respectively; $\alpha_1 = \alpha_2$, $x_1 = x_2$ and K is the fixed cost of fixing errors.

The above cost expression is a newsvendor problem. Thus, a basestock policy is optimal. $\square$

(ii) Equation (19) shows that the optimal cost for the recourse scenario is the minimization of two problems. Thus the optimal basestock levels are either the basestock levels in the no-recourse scenario or the recourse scenario (newsvendor problem). $\square$

(iii) If $\alpha_1 = \alpha_2$, $x_1 = x_2$ and K is the fixed cost of fixing errors; the two costs are given by,

$$C(x,x,\alpha,\alpha) = cz^*_T + [(1-\alpha)^2 + (\alpha)^2][L(x+\frac{z^*_T}{2}) + L(x+\frac{z^*_T}{2})] + [2\alpha(1-\alpha)][L(x) + L(x+z^*_T)]$$

where $z^*_1 = z^*_2 = z^*_T/2$.

$$C^K(x,x,\alpha,\alpha) = cz^*_k + L(x+\frac{z^*_k}{2}) + L(x+\frac{z^*_k}{2}) + [2\alpha(1-\alpha)]K$$

where $z^*_{1k} = z^*_{2k} = z^*_k/2$.

Since this is a sum of two newsvendor problems, it is known that $c + L'(x+\frac{z^*_k}{2}) = 0$. If $K = K_T$, the threshold value of fixed cost after which it is no longer optimal to take recourse, K_T is given by equating both costs, i.e., $C(x,x,\alpha,\alpha) = C^{K_T}(x,x,\alpha,\alpha)$. Let $A = [2\alpha(1-\alpha)]$. Thus, $(1-A) = [(1-\alpha)^2 + (\alpha)^2]$. Then, $\frac{dA}{d\alpha} = 2(1-2\alpha)$, $\frac{d^2A}{d\alpha^2} = -4$ and $\frac{d(1-A)}{d\alpha} = 4\alpha - 2$.

$$\Rightarrow \frac{cz^*_T}{A} + (\frac{1}{A})[2L(x+\frac{z^*_T}{2})] - 2L(x+\frac{z^*_T}{2}) + L(x) + L(x+z^*_T) = \frac{cz^*_k}{A} + (\frac{2}{A})[L(x+\frac{z^*_k}{2})] + K_T$$

Differentiating w.r.t α,

$$\frac{c}{A}(\frac{dz^*_T}{d\alpha}) - \frac{cz^*_T}{A^2}(\frac{dA}{d\alpha}) + (\frac{1}{A})(2L'(x+\frac{z^*_T}{2}))(\frac{1}{2})(\frac{dz^*_T}{d\alpha}) - (\frac{2}{A^2})(L(x+\frac{z^*_T}{2}))(\frac{dA}{d\alpha}) - (2L'(x+\frac{z^*_T}{2}))(\frac{1}{2})(\frac{dz^*_T}{d\alpha}) + L'(x+z^*_T)\frac{dz^*_T}{d\alpha}$$

$$= \frac{c}{A}(\frac{dz^*_k}{d\alpha}) - \frac{cz^*_k}{A^2}(\frac{dA}{d\alpha}) + (\frac{-2}{A^2})(L(x+\frac{z^*_k}{2}))(\frac{dA}{d\alpha}) + (\frac{2}{A})(L'(x+\frac{z^*_k}{2}))(\frac{1}{2})(\frac{dz^*_k}{d\alpha}) + \frac{dK_T}{d\alpha}$$

Now, at $\alpha = 0.5$, (i) $\frac{dz^*_T}{d\alpha} = 0$, (ii) $\frac{dA}{d\alpha} = 0$ and $\frac{dz^*_k}{d\alpha} = 0$, for any α. Thus, $\frac{dK_T}{d\alpha} = 0$ at $\alpha = 0.5$.

$$\begin{aligned}\Rightarrow \frac{d^2K_T}{d\alpha^2} &= (\frac{1}{A})(\frac{d^2z^*}{d\alpha^2})[c + (1-A)L'(x+\frac{z^*_T}{2}) + AL'(x+z^*_T)] \\ &\quad + (\frac{1}{A^2})(\frac{d^2A}{d\alpha^2})[cz^*_k - cz^*_T + 2L(x+\frac{z^*_k}{2}) - 2L(x+\frac{z^*_T}{2})]\end{aligned}$$

$$\Rightarrow \frac{d^2K_T}{d\alpha^2} = (\frac{1}{A^2})(\frac{d^2A}{d\alpha^2})[c(z^*_k - z^*_T) + 2(L(x+\frac{z^*_k}{2}) - L(x+\frac{z^*_T}{2}))]$$

(since $c+(1-A)(L^{'}(x+\frac{z_T^*}{2}))+(A)(L^{'}(x+z_T^*))=0$)

Now, it is known that $z_k^* > z_T^*$. Also, $c+L^{'}(x+\frac{z_k^*}{2})=0$. Thus, $L(x+\frac{z_T^*}{2}) > L(x+\frac{z_k^*}{2})$. It has also been shown that $\frac{d^2A}{d\alpha^2}=-4<0$. Thus, the expression for $\frac{d^2K_T}{d\alpha^2}$ is positive at $\alpha=0.5$, for high service levels (i.e., low values of c). This implies that we have $\frac{dK_T}{d\alpha}=0$ and $\frac{d^2K_T}{d\alpha^2}>0$ at $\alpha=0.5$. Thus, the value of K_T decreases as the risk $\sigma=\alpha(1-\alpha)$ increases. □

(iv) We know that,

$$cz_T^*+[(1-\alpha)^2+\alpha^2][2L(x+\frac{z_T^*}{2})]+[2\alpha(1-\alpha)][L(x)+L(x+z_T^*)]$$

$$=cz_k^*+2L(x+\frac{z_k^*}{2})+[2\alpha(1-\alpha)]K_T$$

Differentiating w.r.t x,

$$[(1-\alpha)^2+\alpha^2](L^{'}(x+\frac{z_T^*}{2}))+[\alpha(1-\alpha)](L^{'}(x)+(L^{'}(x+z_T^*))$$
$$+(\frac{dz_T^*}{dx})[\frac{c}{2}+\frac{1}{2}((1-\alpha)^2+\alpha^2)(L^{'}(x+\frac{z_T^*}{2}))+(\alpha(1-\alpha))(L^{'}(x+z_T^*))]$$

$$=(\alpha(1-\alpha))\frac{dK_T}{dx}+L^{'}(x+\frac{z_k^*}{2})+\frac{dz_k^*}{dx}(\frac{c+L^{'}(x+\frac{z_k^*}{2})}{2})$$

$$\Rightarrow [(1-\alpha)^2+\alpha^2](L^{'}(x+\frac{z_T^*}{2}))+[2\alpha(1-\alpha)](L^{'}(x+z_T^*))-[\alpha(1-\alpha)](L^{'}(x+z_T^*))+[\alpha(1-\alpha)](L^{'}(x))$$

$$+(\frac{1}{2})(\frac{dz_T^*}{dx})[c+((1-\alpha)^2+\alpha^2)(L^{'}(x+\frac{z_T^*}{2}))+(2\alpha(1-\alpha))(L^{'}(x+z_T^*))]$$

$$=(\alpha(1-\alpha))\frac{dK_T}{dx}-c$$

Now, using the first-order condition gives,

$$(\alpha(1-\alpha))[L^{'}(x)-L^{'}(x+z_T^*)]-c=(\alpha(1-\alpha))\frac{dK_T}{dx}-c$$

$$\Rightarrow \frac{dK_T}{dx}=L^{'}(x)-L^{'}(x+z_T^*)$$

$$\Rightarrow \frac{dK_T}{dx}<0 \quad \square$$