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Overlapping Generations Models

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# Indeterminacy and Asset Price Volatility in Stochastic Overlapping Generations Models\*

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## Abstract

This paper addresses the effects of indeterminacy on the volatility of asset prices in a stochastic overlapping generations model with 3-period lived agents. With complete markets, the only indeterminacy is due to the selection of initial conditions for the economy. As with deterministic economies, the equilibrium set converges to the steady state in the long run. With incomplete markets, not only do the initial conditions introduce indeterminacy, but additionally each period a continuum of state price vectors can be selected as equilibrium continuation values. This additional indeterminacy in each period generates long-run price behavior that depends on both the fundamentals of the economy and the endogenous price expectations of agents. Using our innovative computational methodology, we characterize the entire set of sequential equilibria for an incomplete markets economy. Our numerical simulations suggest that asset price volatility has substantial welfare effects, persists in the long run, and is primarily driven by the endogenous price expectations of agents and not by endowment risk.

Key Words: OLG, Indeterminacy, Markov, Computation, Simulation.

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# 1 Introduction

Using a stochastic overlapping generation (OLG) model with 3-period lived agents, this paper characterizes the degree of indeterminacy in settings with both complete and incomplete financial markets. In the case of incomplete markets, indeterminacy is introduced in each period as the vector of possible equilibrium state prices is a manifold of dimension equal to the market deficiency (states of uncertainty minus number of assets). The characterization of indeterminacy in this setting is carried out by computing the entire set of sequential equilibria, and then running simulations of the economy by using a fixed selection rule to select sequences of equilibrium variables from the set.

It is well known that a deterministic OLG economy may have a continuum of equilibria [though not the first contribution, see Kehoe and Levine (1990) for the cleanest treatise on the issues involved]. To avoid such indeterminacy, strong sufficient conditions are required for the uniqueness of the equilibrium in these economies. In an OLG model with a single commodity and homogeneous 2-period lived agents, Gale (1973) shows that the sufficient condition for uniqueness is gross substitution in consumption. This result has been extended to a multi-commodity economy where homogeneous 2-period lived agents have either log-linear preference [Balasko and Shell (1981)] or inter-temporally separable preferences [Geanakoplos and Polemarchakis (1984); Kehoe and Levine (1984)]. Kehoe, Levine, Mas-Colell, and Woodford (1991) further extend these results to a multi-commodity, heterogeneous agent, non-monetary, pure-exchange economy and show that gross substitutability of excess demand ensures the determinacy of perfect-foresight equilibria.

Unlike the deterministic OLG model, less is known about indeterminacy in the stochastic OLG model. General conditions for either the existence or nonexistence of indeterminacy in this setting are unavailable. Several papers provide examples showing the existence of a continuum of stationary Markov equilibria in a stochastic OLG economy (Farmer and Woodford, 1984; Spear, Srivastava, and Woodford, 1990).<sup>1</sup> Our paper, however, suggests that it is short-sighted to only look at stationary Markov equilibria. By only looking at stationary Markov equilibria, the volatility of aggregate variables (such as prices) will be solely determined by the distribution of the exogenous shock, and not by individual endogenous variables, which could be much more volatile. Our paper characterizes the entire set of sequential equilibria by making an appropriate selection from the equilibrium transition correspondence, and in doing so, we observe that asset prices are an order of magnitude more volatile

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<sup>1</sup>As is standard in the macroeconomic literature, a stationary Markov equilibrium is often referred to simply as a recursive equilibrium.

compared to the set of stationary Markov equilibria.

Indeterminacy is not a phenomenon that is unique to the OLG model. It is often used as a mechanism to explain the propagation effects of business cycle. In monetary models, indeterminacy has been used to explain the transmission of money through the economy. In growth theory, there are models that use indeterminacy to explain the difference in growth paths across economies with identical fundamentals [c.f. Benhabib and Farmer (1998) for a comprehensive survey]. The existence of indeterminate equilibria poses challenge for economists conducting comparative statics analysis, meaning that care is taken to choose models with a determinate equilibrium. However, the sufficient condition for uniqueness in the class of OLG models considered in this paper is the property of gross substitution in consumption, which is a property that has been empirically refuted [Mankiw, Rotemberg and Summers (1985)].

Within the class of OLG models with 3-period lived agents, previous papers have only characterized equilibria in a local neighborhood around the steady state and have done so by linearizing the equilibrium conditions in that neighborhood (Kehoe and Levine, 1990). Of course, the properties of the linearized system are only valid in that open neighborhood of the steady state. Since we do not in practice know the size of this open neighborhood, then we cannot credibly claim that the properties of the linearized system hold globally. Gomis-Porqueras and Haro (2003) present some techniques to characterize all manifolds of a given dynamic OLG model, but their method is only applicable to deterministic models.

In this paper, we characterize the global properties of the system by characterizing the entire set of sequential equilibria using the novel methodological contribution from Feng et al. (2012). Our approach provides a measure of the size of indeterminacy by first computing the numerically obtained equilibrium set (in which the equilibrium transition is a correspondence), and then running simulations that select equilibrium variables from the correspondence according to a pre-specified selection rule. The baseline economy considered in our analysis is a generalization of the deterministic economy from Kehoe and Levine (1990), which was specifically chosen since the equilibrium set of the deterministic economy was shown by the authors to be indeterminate. We introduce a stochastic endowment process into the Kehoe and Levine (1990) economy, meaning that the endowments at each stage of the life cycle follow a Markov process.

A standard hypothesis, notably in the works of Spear, Srivastava, and Woodford (1990) and Wang (1993), is that in a deterministic OLG setting, the set of equilibria is indeterminate and yet all equilibria in the set converge asymptotically to the steady state. This fact has been numerically verified by Feng (2012). Numerical

simulations in that paper reveal that all equilibrium paths, which are selections from the equilibrium transition correspondences, do in fact converge to the steady state. The equilibrium paths are completely characterized by the initial period conditions. This paper proves that these same properties hold for a stochastic OLG economy with complete markets.

With incomplete markets, as with complete markets, the selection of initial conditions introduces indeterminacy into the model. Additionally, indeterminacy is introduced in all subsequent periods since the vector of equilibrium state prices is not uniquely defined under incomplete markets. The period-by-period indeterminacy means that the equilibrium transition is a correspondence, not a function.

While the indeterminacy caused by the initial conditions has no effect on the long run behavior of the economy, the equilibrium transition correspondence introduces persistent indeterminacy with significant implications for welfare and asset prices. These implications are characterized through a series of simulations of the economy. In each simulation, we maintain consistency in our period-by-period selection of equilibrium variables from the transition correspondence by specifying a selection rule. For example, one selection rule specifies that the asset price difference across successive periods is minimized. This particular selection rule is shown to generate a consumption equivalent welfare gain of 1.3% over an alternative selection rule.

Using our methodology, we can quantify the volatility of asset prices above and beyond the volatility from the Markov endowment process. To proceed in this direction, we introduce sunspots into our theoretical framework. In the baseline model, a Markov process determines which of the states is realized in each period, and each possible state corresponds to a distinct endowment level for each agent. This is intrinsic uncertainty, namely uncertainty about the fundamentals of the economy. In the model with sunspots, we maintain the same Markov process, but now each state contains identical fundamentals. This is extrinsic uncertainty, namely uncertainty that does not affect the fundamentals. Over several consistent selection rules, we show that the sunspot model has nearly as much asset price volatility as the baseline model with intrinsic uncertainty. From this comparative exercise, we conclude that only 10% (depending upon the selection rule) of asset price volatility is caused by the Markov endowment process, with the remaining volatility owing only to agents' expectations of prices and the indeterminacy generated by incomplete markets.

The remainder of the paper is organized as follows. Section 2 introduces the model and shows how we extend the Kehoe and Levine (1990) setting by introducing a Markov endowment process. Section 3 provides the theory that supports the computational procedure and describes how the simulations are conducted. Sections 4 presents the results of the simulations, including the analysis of the effects of inde-

terminacy on the long run behavior of the economy. Section 5 shows how the method can be applied to compute the set of stationary Markov equilibria, and analyzes how much of the long run behavior of the economy is omitted when imposing the stationary Markov equilibrium concept. Section 6 concludes, and the Appendix contains further details on the numerical algorithm.

## 2 Model

The model contains discrete time periods  $t \geq 0$ , and in every period, a new cohort of agents enters the economy. Each cohort consists of a representative agent that remains in the economy for 3 periods. The overlapping generations model contains stochastic endowment realizations. Time and uncertainty are represented by a countably infinite tree  $\Sigma$ . Each node of the tree,  $\sigma \in \Sigma$ , is a finite history of shocks  $\sigma = s^t = (s_0, s_1, \dots, s_t)$ , where the initial shock  $s_0$  is randomly determined with known probabilities  $\pi(s_0)$ . The agents are identified by the history of shocks at birth,  $\sigma = s^t$ , and the age of the agent,  $a \in \{0, 1, 2\}$ . For convenience, an agent of age  $a = 2$  in period  $s_0$  is given the birth period  $s_{-2}$ , while an agent of age  $a = 1$  in period  $s_0$  is given the birth period  $s_{-1}$ .

At each history  $s^t$ , a single consumption good is traded. The process of shocks  $\{s_t\}$  is assumed to be a Markov chain with finite support  $\mathbf{S} = \{1, \dots, S\}$ . From the Markov chain, define  $\pi(s^{t+k}|s^t)$  as the conditional probability that history  $s^{t+k}$  is realized given that the current history is  $s^t$ . Further, allow the notation  $s^{t+1} \geq s^t$  to indicate that  $s^{t+1} = (s^t, s)_{s \in \mathbf{S}}$  contains all  $S$  successor nodes of the history  $s^t$ . Likewise  $s^{t+2} \geq s^t$  indicates that  $s^{t+2} = (s^t, s, s')_{s, s' \in \mathbf{S}}$  contains all  $S^2$  histories that can be observed 2 periods after the history  $s^t$ .

### 2.1 Financial assets

For each history  $s^t$ , there exist  $N$  short-lived assets with fixed payouts in terms of the consumption good in all successor histories  $(s^t, s)_{s \in \mathbf{S}}$ . The number of assets  $N \leq S$ , where  $N = S$  refers to a complete markets asset structure and  $N < S$  refers to an incomplete markets asset structure. The  $N$  assets are indexed by a superscript  $j = 1, \dots, N$ . The equilibrium price of asset  $j$  in history  $s^t$  is denoted  $q^j(s^t)$ . The row vector  $q(s^t)$  contains the asset prices of all  $N$  assets. The asset payouts are a Markov chain such that the payout of asset  $j$  traded in history  $s^t$  is given by  $r^j = (r^j(s))_{s \in \mathbf{S}}$  for each of the  $S$  possible realizations  $(s^t, s)_{s \in \mathbf{S}}$  in period  $t + 1$ . Additionally, define  $r(s) = (r^1(s), \dots, r^N(s))$  as the vector of portfolio payouts for the realization  $s$ . The

asset payouts can be collected into the  $S \times N$  payout matrix

$$R = (r^1, \dots, r^N) = (r(s))_{s \in \mathbf{S}}.$$

The payout matrix is assumed to be a nonnegative and full rank matrix.

Let  $\theta_a^j(s^t)$  denote the amount of asset  $j$  purchased by an agent of age  $a$  in history  $s^t$ . Market clearing for assets requires that the net asset holdings of all agents alive at history  $s^t$  must be zero:  $\sum_{a=0}^1 \theta_a^j(s^t) = 0 \forall j = 1, \dots, N$ . It should be noted that nobody is willing to purchase assets from agents of age  $a = 2$  as they will no longer be in the economy in the following period to fulfill their commitment. The column vector  $\theta_a(s^t)$  contains the entire portfolio of all assets positions of the agent of current age  $a$  in history  $s^t$ .

## 2.2 Household behavior

An agent born at node  $s^t$  has endowment and makes consumption decisions at all contingent histories  $s^t$ ,  $(s^t, s)_{s \in \mathbf{S}}$ , and  $(s^t, s, s')_{s, s' \in \mathbf{S}}$ . An agent's individual endowments follow a Markov process that only depends on the shock and age, i.e., for all  $a \in \{0, 1, 2\}$  and all histories  $s^t$ ,  $e_a(s^{t+a}) = \mathbf{e}_a(s_{t+a})$ .

For an agent born in history  $s^t$ , define the lifetime contingent consumption vector as  $c(s^t) = (c_0(s^t), (c_1(s^t, s))_{s \in \mathbf{S}}, (c_2(s^t, s, s'))_{s, s' \in \mathbf{S}}) \in \mathbb{R}_+^{1+S+S^2}$ . The agent preferences are assumed to be identical and are represented by the time-separable utility function  $U : \mathbb{R}_+^{1+S+S^2} \rightarrow \mathbb{R}$

$$U(c(s^t)) = \sum_{a=0}^2 \beta^a \sum_{s^{t+a} \geq s^t} \pi(s^{t+a} | s^t) u(c_a(s^{t+a})) \quad (1)$$

The one-period utility  $u$  satisfies the following conditions:

**Assumption 2.1** *The one-period utility function  $u : \mathbb{R}_+ \rightarrow \mathbb{R} \cup \{-\infty\}$  is increasing, strictly concave, and continuous. These functions are also continuously differentiable at every interior point  $c > 0$ .*

For simplicity, define  $\theta(s^t) = (\theta_0(s^t), (\theta_1(s^t, s))_{s \in \mathbf{S}}) \in \mathbb{R}^{N(1+S)}$  as the entire vector of lifetime contingent portfolios for an agent born in history  $s^t$ . Similarly, define the asset prices  $q(s^t) = (q(s^t), (q(s^t, s))_{s \in \mathbf{S}}) \in \mathbb{R}_+^{N(1+S)}$  as the vector of contingent asset prices that an agent born in history  $s^t$  will observe in its lifetime. Given asset prices  $q(s^t)$ , an agent born at node  $s^t$  solves

$$\max_{c(s^t), \theta(s^t)} U(c) \tag{2}$$

s.t.

$$c_0(s^t) + q(s^t) \cdot \theta_0(s^t) \leq e_0(s_t)$$

$$c_1(s^{t+1}) + q(s^{t+1}) \cdot \theta_1(s^{t+1}) \leq e_1(s_{t+1}) + r(s_{t+1}) \theta_0(s^t) \quad \forall s^{t+1} \geq s^t$$

$$c_2(s^{t+2}) \leq e_2(s_{t+2}) + r(s_{t+2}) \theta_1(s^{t+1}) \quad \forall s^{t+2} \geq s^t$$

The model contains both a representative agent born in period  $t = -2$  (identified by birth history  $s_{-2}$ ) and a representative agent born in period  $t = -1$  (identified by birth history  $s_{-1}$ ). Both agents only solve an optimization problem in period  $t = 0$ , where the asset positions  $\theta_1(s_{-1})$  and  $\theta_0(s_{-1})$  are fixed parameters of the model. Market clearing requires that  $\theta_1(s_{-1}) + \theta_0(s_{-1}) = 0$ .

## 2.3 Equilibrium

**Definition 1** *A sequential competitive equilibrium (SCE) is given by a collection of prices and choices of individuals  $\{q(s^t), \theta(s^t), c(s^t)\}$  such that:*

(i) *For each  $s^t$ , taking as given the prices  $q(s^t)$ , the representative agent born in  $s^t$  solves (2).*

(ii) *Market clearing for each  $s^t$  :*

$$\sum_{a=0}^1 \theta_a(s^t) = 0. \tag{3}$$

The existence of a SCE can be verified by standard methods [e.g., Balasko and Shell (1980); Schmachtenberg (1988)]. Moreover, the cited authors prove that every sequence of equilibrium asset prices  $\{q(s^t)\}$  is uniformly bounded.

In a deterministic version of the OLG model, it is known [e.g., Kehoe and Levine (1990)] that a continuum of SCE may exist. The previous literature has verified the existence of such indeterminacy for the deterministic setting by linearizing the model around the steady state. They show that when a continuum of SCE exist, they are characterized completely by the initial asset positions  $\theta_1(s_{-1})$  and  $\theta_0(s_{-1})$ . Despite the fact that the equilibrium set is indeterminate, every equilibrium in that set converges asymptotically to the steady state. Thus, the indeterminacy does not have any long run implications.

Our computation methodology allows us to compute all SCE in a stochastic setting, not only those contained within the open neighborhood for which the linearization method is valid. This is important, because the method is able to show that

the long run dynamics of SCE are vastly different in a stochastic setting compared to the deterministic setting.

### 3 Computation Methodology

In this section, we introduce the transition correspondence that we use to characterize the set of SCE. We refer to the correspondence as the "Markov equilibrium correspondence." We then provide theoretical results to justify the computational method and simulation techniques that we will employ to numerically approximate the set of SCE.

#### 3.1 Computation

Recall that market clearing in any history  $s^t$  is given by  $\theta_0(s^t) + \theta_1(s^t) = 0$ . Using this fact, we can rewrite the budget constraints faced by agents of all 3 ages alive in history  $s^t$  :

$$c_0(s^t) - q(s^t) \cdot \theta_1(s^t) \leq e_0(s_t) \quad (4)$$

$$c_1(s^t) + q(s^t) \cdot \theta_1(s^t) \leq e_1(s_t) - r(s_t) \theta_1(s^{t-1}) \quad (5)$$

$$c_2(s^t) \leq e_2(s_t) + r(s_t) \theta_1(s^{t-1}) \quad (6)$$

It is clear that the asset position of the household who was born at  $s^{t-1}$ ,  $\theta_1(s^{t-1})$ , and the current shock  $s_t$  summarize the state of the economy.

We define shadow values of investment in each asset as follows:

$$m^j(\theta_1(s^{t-1}), s_t) := q^j(s^t) u_c(c_1(s^t)). \quad (7)$$

Let's assume that the  $N$ -dimensional vector  $m(\theta_1(s^{t-1}), s_t) = \{m^j(\theta_1(s^{t-1}), s_t)\}_{j=1}^N$  is fixed, the previous asset positions  $\theta_1(s^{t-1})$  are known, and the current shock  $s_t$  is known. The current period variables  $\{q^j(s^t), \theta_1^j(s^t)\}_{j=1}^N$ , which number  $2N$ , are determined by the following  $2N$  equilibrium equations:

$$m^j(\theta_1(s^{t-1}), s_t) = q^j(s^t) u_c[e_1(s_t) - q(s^t) \cdot \theta_1(s^t) - r(s_t) \theta_1(s^{t-1})] \quad (8)$$

$$m^j(\theta_1(s^{t-1}), s_t) = \beta \sum_{s_{t+1} \in \mathbf{S}} \pi(s_{t+1}|s_t) u_c[e_2(s_{t+1}) + r(s_{t+1}) \theta_1(s^t)] r^j(s_{t+1}) \quad (9)$$

where equation (8) is the definition of  $m$  and holds  $\forall j = 1, \dots, N$ , and (9) are the first order conditions (with respect to asset choices) of the age  $a = 1$  agent and hold  $\forall j = 1, \dots, N$ .

The state variables  $m(\theta_1(s^{t-1}), s_t)$  must also be consistent with the first order conditions (with respect to asset choices) of the age  $a = 0$  agent. This yields the following  $S^2(N - 1) + SN$  equilibrium equations:

$$\begin{aligned} \frac{m^j(\theta_1(s^{t-1}), s_t)}{q^j(s^t)} &= \frac{m^1(\theta_1(s^{t-1}), s_t)}{q^1(s^t)} \quad \forall (s_{t-1}, s_t) \in \mathbf{S}^2 \\ q^j(s^{t-1})u_c[e_0(s_{t-1}) + q(s^{t-1}) \cdot \theta_1(s^{t-1})] &= \beta \sum_{s_t \in \mathbf{S}} \pi(s_t | s_{t-1}) \frac{m^1(\theta_1(s^{t-1}), s_t)}{q^1(s^t)} r^j(s_t) \cdot \end{aligned} \quad (10)$$

The second equation of (10) holds  $\forall j = 1, \dots, N$  and for all possible shocks  $s_{t-1} \in \mathbf{S}$ .

**Definition 2** *The Markov equilibrium correspondence  $\mathbf{V}^* : \mathbf{R}^N \times \mathbf{S} \rightarrow \mathbf{R}^N$  is defined such that for any history  $s^t$ :*

$$\mathbf{V}^*(\theta_1(s^{t-1}), s_t) = \left\{ m(\theta_1(s^{t-1}), s_t) : \begin{array}{l} (10) \text{ are satisfied and} \\ \exists \{q^j(s^t), \theta_1^j(s^t)\}_{j=1}^N \text{ satisfying (8) and (9)} \end{array} \right\}.$$

**Theorem 3** *The Markov equilibrium correspondence  $\mathbf{V}^*$  characterizes the entire set of SCE. Specifically, any SCE corresponds to a sequence of selections from the Markov equilibrium correspondence, where the remaining variables  $\{q^j(s^t), \theta_1^j(s^t)\}_{j=1}^N$  are determinate solutions of (8) and (9).*

**Proof.** Fix a history  $s^t$ . Define  $\xi = \{q^j(s^t), \theta_1^j(s^t)\}_{j=1}^N$  as the variables and  $\psi = (\{\theta_1^j(s^{t-1}), m^j(\theta_1(s^{t-1}), s_t)\}_{j=1}^N, \{e_1(s), e_2(s)\}_{s \in \mathbf{S}})$  as the parameters. Variables are elements of the open set  $\Xi$  and parameters are elements of the open set  $\Psi$ . We will define the mapping  $\Phi : \Xi \times \Psi \rightarrow \mathbb{R}^{2N}$  such that  $\Phi(\xi, \psi) = 0$  iff (8) and (9) are satisfied. In particular, the mapping is defined by  $(\xi, \psi) \mapsto$

$$\begin{aligned} &q^1(s^t)m^j(\theta_1(s^{t-1}), s_t) - q^j(s^t)m^1(\theta_1(s^{t-1}), s_t) \quad \forall j = 2, \dots, N \\ &m^1(\theta_1(s^{t-1}), s_t) - q^1(s^t)u_c[e_1(s_t) - q(s^t) \cdot \theta_1(s^t) - r(s_t)\theta_1(s^{t-1})] \\ &m^j(\theta_1(s^{t-1}), s_t) - \beta \sum_{s_{t+1} \in \mathbf{S}} \pi(s_{t+1} | s_t) u_c[e_2(s_{t+1}) + r(s_{t+1})\theta_1(s^t)] r^j(s_{t+1}) \quad \forall j = 1, \dots, N \end{aligned}$$

Define the projection  $\rho : \Xi \times \Psi \rightarrow \Psi$  as the parameters  $\psi$  such that for some  $\xi \in \Xi : \Phi(\xi, \psi) = 0$ . This proof will demonstrate that  $\psi$  is a regular value of  $\rho$ , meaning that  $\rho^{-1}(\psi)$  contains only regular points of  $\rho$ . Using Sard's Theorem [see Villanacci et al. (2002)], then over a generic subset of parameters, the set of variables solving (8) and (9) is finite and for each solution, there exists a  $C^1$  bijection from an open set of parameters to an open set of those solution variables. Further, the bijections across each of the solutions are disjoint, meaning that the solutions are locally unique. The

properties of finite and locally unique are precisely what is meant by a determinate solution.

The mapping  $\rho$  is trivially proper [see Villanacci et al. (2002) for further details]. The proof requires the verification that the derivative matrix  $D_{\xi,\psi}\Phi(\xi, \psi)$  has full row rank. The derivative matrix  $D_{\xi,\psi}\Phi(\xi, \psi)$  is of the form

$$\begin{bmatrix} M_1 & M_2 \\ 0 & M_3 \end{bmatrix}$$

where the  $N \times N$  submatrix  $M_1 = \begin{bmatrix} \{m^j(\theta_1(s^{t-1}), s_t)\}_{j=2}^N & I_{N-1} \\ -u_c[\cdot] + q^1(s^t)u_{cc}[\cdot] \sum_j \frac{m^j}{m^1} \theta_1^j(s^t) & 0 \end{bmatrix}$  has full rank provided that  $-u_c(\cdot) + q^1(s^t)u_{cc}(\cdot) \sum_j \frac{m^j}{m^1} \theta_1^j(s^t) \neq 0$ . Taking the derivative with respect to  $e_1(s_t)$  guarantees that full rank holds over a generic subset of parameters (specifically, the endowment  $e_1(s_t)$ ). The  $N \times N$  submatrix  $M_3 = -\beta R^T \begin{bmatrix} .. & 0 & 0 \\ 0 & \pi(s_{t+1}|s_t)u_c[\cdot] & 0 \\ 0 & 0 & .. \end{bmatrix} R$  has full rank given that  $R$  has full rank (full column rank) and  $u_c[\cdot] > 0$  by Assumption 2.1. Thus, the derivative matrix  $D_{\xi,\psi}\Phi(\xi, \psi)$  has full row rank, completing the argument. ■

The correspondence  $\mathbf{V}^*$  is recursively defined as the fixed point of an operator  $\mathbf{B} : \mathbf{V} \rightarrow \mathbf{B}(\mathbf{V})$  that links state variables to future equilibrium states. This operator incorporates all equilibrium conditions: (8), (9), and (10). The Euler equation is given by:

$$q^j \cdot u_c(c_0) = \beta \frac{m_+^j(\theta_1, s)}{q_+^j}, \quad (11)$$

where the prices  $q_+^j$  satisfy (8) and (9).

The following result is proved in Feng et al. (2012).

**Theorem 4 (convergence)** *Let  $\mathbf{V}_0$  be a compact-valued correspondence such that  $\mathbf{V}_0 \supset \mathbf{V}^*$ . Let  $\mathbf{V}_n = \mathbf{B}(\mathbf{V}_{n-1})$ ,  $n \geq 1$ . Then,  $\mathbf{V}_n \rightarrow \mathbf{V}^*$  as  $n \rightarrow \infty$ . Moreover,  $\mathbf{V}^*$  is the largest fixed point of the operator  $\mathbf{B}$ , i.e., if  $\mathbf{V} = \mathbf{B}(\mathbf{V})$ , then  $\mathbf{V} \subset \mathbf{V}^*$ .*

Let's assume that the initial conditions  $(\theta_1(s_{-1}), s_0)$  of the economy are fixed. The initial conditions have dimension  $NS$ . By definition of  $\mathbf{V}^*$ , there exists  $m(\theta_1(s_{-1}), s_0) \in \mathbf{V}^*(\theta_1(s_{-1}), s_0)$ . Thus the degree of indeterminacy introduced by the initial conditions has dimension  $SN$  and is completely summarized by the vector  $m(\theta_1(s_{-1}), s_0)$ . The  $2N$  unknowns  $\{q^j(s_0), \theta_1^j(s_0)\}_{j=1}^N$  are the determinate solutions to the following

system of  $2N$  equations:

$$m^j(\theta_1(s_{-1}), s_0) = q^j(s_0)u_c[e_1(s_0) - q(s_0) \cdot \theta_1(s_0) - r(s_0)\theta_1(s_{-1})] \quad (12)$$

$$m^j(\theta_1(s_{-1}), s_0) = \beta \sum_{s_1 \in \mathbf{S}} \pi(s_1|s_0)u_c[e_2(s_1) + r(s_1)\theta_1(s_0)]r^j(s_1) \quad (13)$$

where equation (12) is the definition of  $m$ , and equation (13) is the first order condition for agent born in period  $t = -1$ .

In any future period  $s^t$ , the known variables are the previous period asset positions  $\{\theta_1^j(s^{t-1})\}_{j=1}^N$ . What are unknown are the  $SN$  asset prices  $\{q^j(s^{t-1}, s)\}_{s \in \mathbf{S}}^{j=1, \dots, N}$ , the  $SN$  asset positions  $\{\theta^j(s^{t-1}, s)\}_{s \in \mathbf{S}}^{j=1, \dots, N}$ , and the  $S^2N$  shadow values  $\{m^j(\theta_1(s^{t-2}, s_-), s)\}_{s_-, s \in \mathbf{S}}^{j=1, \dots, N}$ . By construction, the shadow values depend upon the realizations in consecutive periods. The total number of unknowns is  $S^2N + 2SN$ .

The equilibrium equations include the  $S^2(N-1) + SN$  equations in (10). With  $m^j(\theta_1(s^{t-1}), s_t)$  as unknown variables, the second set of equations are the  $SN$  equations in (9), which must be satisfied  $\forall j = 1, \dots, N$  and for all possible shocks  $s_t \in \mathbf{S}$ . The third set of equations are variations from (8) and must be satisfied for all consecutive shocks  $(s_{t-1}, s_t) \in \mathbf{S}$  ( $S^2$  equations):

$$m^1(\theta_1(s^{t-1}), s_t) = q^1(s^t)u_c[e_1(s_t) - q(s^t) \cdot \theta_1(s^t) - r(s_t)\theta_1(s^{t-1})]. \quad (14)$$

The equations (14) are simply the definition of  $m$  and use the relation in the first equation of (10). The total number of equations is  $S^2N + 2SN$ , same as the total number of unknowns.

In settings with either complete or incomplete markets, the indeterminacy introduced by the initial conditions (of degree  $SN$ ) is present. From Theorem 3, any additional indeterminacy, which may arise period-by-period, is entirely due to the Markov equilibrium correspondence. If this correspondence is a function, then no additional indeterminacy arises. The degree of additional indeterminacy equals the dimension of indeterminacy in the image of the correspondence.

### 3.1.1 Complete markets

With complete markets, given Assumption 2.1, the equilibrium consumption is such that the shadow value  $m(\theta_1(s^{t-1}), s_t)$  is constant across all time periods, regardless of uncertainty realization. This implies that the correspondence  $\mathbf{V}^*(\theta_1(s^{t-1}), s_t)$  maps into a singleton (a 0-dimensional set). No additional indeterminacy is introduced into the economy beyond the indeterminacy of the initial conditions  $(\theta_1(s_{-1}), s_0)$ . Thus, the total degree of indeterminacy equals  $SN$ . The economy is observationally

equivalent to the deterministic economy and has the same long run properties. The Markov endowment process is neutralized with complete markets and the only fluctuations in the economy result from the initial conditions, which we know will vanish in the long run from the study of deterministic economies by Feng (2012).

The effects of indeterminacy on the long run economy in deterministic OLG model, as documented in Feng (2012), can be summarized as follows:

1. The indeterminacy is indexed by the initial conditions and all equilibrium paths asymptotically converge to the same steady state.
2. The choice of initial conditions results in different transition paths and different welfare implications.

### 3.1.2 Incomplete markets

The shadow values are proportional to the equilibrium state prices (equivalently known as stochastic discount factors in the finance literature). In a standard 2-period setting with incomplete markets, the state prices are not uniquely determined, but belong to a manifold of dimension equal to the market deficiency ( $S - N$ ). The exact same outcome occurs in this stochastic OLG model with incomplete markets. The additional indeterminacy that is added each and every period is proportional to the market deficiency  $S - N$ . This is exactly represented in the dimension of the image of the Markov equilibrium correspondence.

There is no theoretical basis to make comparisons of the degree of indeterminacy, or its welfare consequences, across economies. Such comparisons require the computational approximation of the Markov equilibrium correspondence. This is achieved using the methodological contribution in Feng et al. (2012). The next subsection discusses how this approximation takes place, but more importantly the types of simulations of the economy that are carried out. The type of simulations are important, because we are fundamentally tasked with choosing continuation variables from a continuum of possible values.

## 3.2 Simulation

We apply the numerical algorithm detailed in Feng (2012) to approximate the equilibrium set  $\mathbf{V}^*$ . It is well established that the presence of rounding and truncation errors makes it infeasible to numerically compute the exact equilibrium in finite time. Following Kubler and Schmedders (2005), we construct a Markov  $\epsilon$ -equilibrium as a collections of policy function and transition function such that the maximum error

in agents' equilibrium conditions are bounded by some bound  $\epsilon > 0$ . The Markov  $\epsilon$ -equilibrium of Kubler and Schmedders (2009) is an approximation of a stationary Markov equilibrium, but a Markov  $\epsilon$ -equilibrium can in principle be constructed as an approximation to any SCE. The concept of a Markov  $\epsilon$ -equilibrium carries meaning as Kubler and Polemarchakis (2004) verify that in the limit as  $\epsilon \rightarrow 0$ , a Markov  $\epsilon$ -equilibrium approaches a SCE.

In the final section of results, we define a second Markov  $\epsilon$ -equilibrium to approximate a stationary Markov equilibrium. Our numerical simulations suggest that by focusing on stationary Markov equilibria, a subset of SCE, we are missing nearly all of the important long run dynamics of the economy. Thus, besides the issue that stationary Markov equilibria have been shown to not exist in some OLG settings [e.g., Kubler and Polemarchakis (2004)], we focus on the computation of SCE to fully understand the long run dynamics, particularly related to asset price volatility, of the economy.

**Definition 5** *A Markov  $\epsilon$ -equilibrium is defined in terms of a finite state space  $\Theta$ , a Markov correspondence  $\mathbf{V} : \Theta^N \times \mathbf{S} \rightarrow \mathbb{R}^N$ , and  $\epsilon > 0$ , such that:*

1. *for any  $(\theta, s, m) \in \text{graph}(\mathbf{V})$  under the action of operator  $\mathbf{B}$ , we can generate a sequence  $\{q(s^t), \theta(s^t), c(s^t)\}$  satisfying (8) and (9) with the Euler equation residuals from (11) bounded above by  $\epsilon$ .*
2. *for any  $(\theta, s, m^*) \in \text{graph}(\mathbf{V}^*)$  and  $\epsilon > 0$ , there exists  $(\theta, s, m) \in \text{graph}(\mathbf{V})$  such that  $\text{dist}\{m, m^*\} < \epsilon$  and we can generate a SCE  $\{q(s^t), \theta(s^t), c(s^t)\}$  based on  $(\theta, s, m)$  by applying operator  $\mathbf{B}$ .*

In all numerical examples we considered, we are able to characterize the Markov correspondence  $V$  at any given  $\epsilon > 0$ .

The numerical simulations are run under 4 different selection rules. The selection rules specify certain properties that the continuation variables must satisfy. The properties are held constant across all time periods of a particular simulation.

1. Maximize difference in asset prices

We choose  $m(\theta_1(s^{t-1}), s_t)$  from the image of the Markov correspondence  $\mathbf{V}(\theta_1(s^{t-1}), s_t)$  such that (8) and (9) are satisfied and  $q(s^t)$  is such that the difference  $|q(s^t) - q(s^{t-1})|$  is maximized. Formally, we pick  $m(\theta_1(s^{t-1}), s_t)$  such that

$$q(s^{t-1}, s_t) \in \arg \max |q(s^{t-1}, s_t) - q(s^{t-1})| \quad (15)$$

*s.t.*

$$m(\theta_1(s^{t-1}), s_t) \in \mathbf{V}(\theta_1(s^{t-1}), s_t)$$

(8) and (9) are satisfied

In our numerical simulations, we consider an economy with a single asset, so the maximization problem (15) is easy to solve. Further, for the economy with  $N = 1$  and  $S = 2$  (as considered in our simulations), the maximum dimension of the image of  $V$  equals the market deficiency ( $S - N = 1$ ) times the number of possible states ( $S = 2$ ) for a total dimension of 2. Thus, by choosing an asset price value  $q(s^{t-1}, s_t)$  for each state  $s_t$  that solves (15), we have a determinate solution period-by-period.

2. Minimize difference in asset prices

We choose  $m(\theta_1(s^{t-1}), s_t)$  from the equilibrium set  $\mathbf{V}(\theta_1(s^{t-1}), s_t)$  such that all contemporaneous equilibrium conditions are satisfied and the corresponding  $q(s^t)$  is such that the difference  $|q(s^t) - q(s^{t-1})|$  is minimized.

3. Maximize difference in asset holdings

We choose  $m(\theta_1(s^{t-1}), s_t)$  from the equilibrium set  $\mathbf{V}(\theta_1(s^{t-1}), s_t)$  such that all contemporaneous equilibrium conditions are satisfied and the corresponding  $\theta_1(s^t)$  is such that the difference  $|\theta_1(s^t) - \theta_1(s^{t-1})|$  is maximized.

4. Minimize difference in asset holdings

We choose  $m(\theta_1(s^{t-1}), s_t)$  from the equilibrium set  $\mathbf{V}(\theta_1(s^{t-1}), s_t)$  such that all contemporaneous equilibrium conditions are satisfied and the corresponding  $\theta_1(s^t)$  is such that the difference  $|\theta_1(s^t) - \theta_1(s^{t-1})|$  is minimized.

## 4 Indeterminacy and Asset Price Volatility

In this section, we simulate an economy that is derived from Kehoe and Levine (1990). From the simulation we obtain important implications of indeterminacy on the long run behavior of the economy.

### 4.1 Numerical specifications

There is an exogenous shock that affects the endowments of the household. We assume that the shock takes two values: good and bad. Given the shock realization, the endowments of the age  $a = 2$  agent change, while the other endowments remain unchanged. Specifically, we assume  $\{e^a(s_t)\}_{a=0}^2 = \{3, 12, 1 \pm \epsilon\}$ , where  $\epsilon = 0.05$ . The transition matrix that governs the Markov chain is given by

$$\pi = \begin{bmatrix} 0.95 & 0.05 \\ 0.05 & 0.95 \end{bmatrix}.$$

The utility function is given by:  $u(c) = \frac{c^{1-\sigma}}{1-\sigma} - 1$ , where  $\sigma = 4$ .

There is a single bond available for trade. At each  $s^t$ , the bond is in zero net supply. Its price is denoted by  $q(s^t) \in \mathbf{R}_+$ , and agent  $a$ 's bond-holding is  $\theta_a(s^t) \in \mathbf{R}$  for  $a \in \{0, 1\}$ .

At the root node,  $s_0$ , a representative age  $a = 2$  agent is present (born in state  $s_{-2}$ ) and a representative age  $a = 1$  agent is present (born in state  $s_{-1}$ ). The initial conditions of the economy are the bond holdings that these two agents bring into that initial state  $s_0$ :  $\theta_1(s_{-1})$  and  $\theta_0(s_{-1})$ , where market clearing requires  $\theta_1(s_{-1}) + \theta_0(s_{-1}) = 0$ .

We solve for the entire set of SCE and present the Markov correspondence in Figure 1. Figure 1 clearly demonstrates that the image of the Markov correspondence is a 2-dimensional manifold for each of the 2 shocks. This verifies the theoretical fact that the dimension of the image of  $V$  equals the market deficiency ( $S - N = 1$ ) times the number of possible states ( $S = 2$ ) for a total dimension of 2.

## 4.2 Simulation results

We choose the initial conditions so that  $\theta_1(s_{-1}) = 3.0$  and the initial shock in state  $s_0$  is the bad shock, meaning  $e_2(s_0) = 1 - \epsilon$ . Simulations last for 5,000 periods and the first 1,000 periods are ignored when computing simulated moments and simulated conditional moments.

### 4.2.1 Effects of the selection rules

We run simulations under each of the 4 selection rules introduced at the end of the previous section. Comparing simulation 1 and simulation 2, the average welfare (equal weights provided to all agents) is 3.96% higher in simulation 1 compared to simulation 2, with a consumption equivalent welfare gain of 1.3%. Recall simulation 1 seeks to maximize the difference in asset prices, while simulation 2 seeks to minimize this difference. The maximum difference in asset prices allows for greater risk sharing on the part of agents.

Clearly, in a stochastic setting, the simulated unconditional variances will be strictly positive, given that the prices and bond holdings adjust when the shock changes. Our interest is in determining the long run behavior of asset prices, conditioning on the realization of uncertainty. The simulated conditional variances capture

this effect. As evident from Table 2, these variances are strictly positive and roughly as large as the simulated unconditional variances, suggesting that the variance is driven by the indeterminacy of the equilibrium set rather than by the Markov endowment process.

Table 1: Simulation results for 4 selection rules: unconditional moments

	$std(\theta)$	$mean(\theta)$	$std(q)$	$mean(q)$	$mean(U)$	$max(ee)$
Sim. 1	0.4166	3.6953	0.2351	0.8421	-0.0073	$4.5 * 1e^{-5}$
Sim. 2	0.2184	3.6902	0.1061	0.8232	-0.0071	$4.5 * 1e^{-5}$
Sim. 3	0.4847	3.8492	0.1984	0.7618	-0.0071	$4.5 * 1e^{-5}$
Sim. 4	0.2000	3.6776	0.1075	0.8296	-0.0071	$4.5 * 1e^{-5}$

	$std(q 0)$	$mean(q 0)$	$std(q 1)$	$mean(q 1)$
Sim. 1	0.2121	0.9034	0.2411	0.7876
Sim. 2	0.0890	0.8812	0.0927	0.7717
Sim. 3	0.2092	0.8176	0.1739	0.7123
Sim. 4	0.0799	0.8880	0.1022	0.7777

Table 2: Simulation results for 4 selection rules: state contingent moments

#### 4.2.2 Robustness check on initial conditions

We analyze the effects of the initial conditions on the behavior of the economy. For each of the following experiments, we remain consistent by applying the same selection rule (chosen from one of the 4 possibilities previously introduced) for both the benchmark economy and for economies with different initial conditions. Recall that the benchmark economy specifies  $\{\theta_1(s_{-1}), s_0, m_0\} = \{3.0, 1 - \epsilon, 5.50\}$ . In what follows, we only mention the differences from the benchmark. The first experiment specifies  $m_0 = 5.10$ , the second chooses  $s_0$  such that  $e^2 = 1 + \epsilon$ , while the third one picks  $\theta_1(s_{-1}) = 4.3128$ . After we drop the first 1,000 periods, the simulated moments are identical to those for the benchmark economy.

#### 4.2.3 Excess volatility in asset pricing

Our conjecture in this paper is that indeterminacy generates excess volatility in the asset prices. In the economy we are computing, the endowment of the age  $a = 2$  agents fluctuates 5% around its long-term average, while under simulation 1, the standard deviation for asset prices is roughly 30% of the long-term average. The

first direction we take to decompose asset price volatility into the volatility owing to changes in fundamentals (endowments) and volatility owing to indeterminacy (self-fulfilling beliefs of agents) is to consider two economies with a determinate equilibrium. Both determinate economies are nearby "less-risky" versions of our benchmark economy.

For the first economy, we reduce the parameter for agent risk-aversion from  $\sigma = 4$  to  $\sigma = 3.2$ , while keeping all other parameters unchanged. For the second economy, we reduce the endowment inequality across generations by specifying an endowment process of  $e = \{3, 8, 2 \pm 5\%\}$ . For both economies, the equilibrium set is determinate. We compute these models and simulate the economies. The simulated moments are given in Table 3. For both determinate economies, the standard deviation for asset prices is less than 10% of the long-term average.

Model	$std(\theta)$	$mean(\theta)$	$std(q)$	$mean(q)$
$\sigma = 3.2, e = \{3, 12, 1 \pm 5\%\}$	0.0431	5.0300	0.0132	0.3306
$\sigma = 4.0, e = \{3, 8, 2 \pm 5\%\}$	0.0441	2.7297	0.0306	0.3455

Table 3: Simulation results for determinate economies

#### 4.2.4 Sunspot equilibrium

The second direction we take to isolate the contributing effects on asset price volatility is to introduce sunspots into our benchmark economy. We maintain the same Markov chain, but the shocks are now states of extrinsic uncertainty, meaning that the endowments remain unchanged, rather than states of intrinsic uncertainty. Simply, regardless of the state realization, the endowment process is given by  $\{\mathbf{e}^a(s_t)\}_{a=0}^2 = \{3, 12, 1\}$ . There still remain  $S = 2$  states of uncertainty, and agents need not have the same price expectations for both states. If the price expectations differ, then any asset price volatility is owing only to the indeterminacy, since the fundamentals (endowments) remain unchanged.

Not surprisingly, we can apply the same theoretical insights previously discussed to verify that the image of the Markov correspondence is indeterminate. Using our computation methodology, we can characterize the same 4 simulations that were previously considered.

The results are presented in Tables 4 and 5. We compare the simulated asset price variances in Table 4 with the corresponding values in Table 1 (for the case with endowment risk). In simulations 3 and 4, the asset price variances without endowment risk are more than 90% of the respective values with endowment risk.

Table 4: Simulation of sunspot equilibrium

	$std(\theta)$	$mean(\theta)$	$std(q)$	$mean(q)$	$mean(U)$
Sim. 1	0.3081	3.7740	0.1297	0.7828	-0.0070
Sim. 2	0.2042	3.6749	0.1079	0.8307	-0.0071
Sim. 3	0.5181	3.8791	0.1943	0.7489	-0.0071
Sim. 4	0.1740	3.6664	0.0980	0.8340	-0.0071

	$std(q 0)$	$mean(q 0)$	$std(q 1)$	$mean(q 1)$
Sim. 1	0.1386	0.7803	0.1212	0.7851
Sim. 2	0.1083	0.8345	0.1074	0.8273
Sim. 3	0.2002	0.7407	0.1887	0.7562
Sim. 4	0.0898	0.8324	0.1047	0.8354

Table 5: Simulation results for sunspot equilibrium, state contingent moments

This strongly suggests the first order determinant of asset price volatility is the indeterminacy of equilibria.

The reason that we have focused your attention on simulations 3 and 4 is because these simulations are run by making selections using asset holdings as the criterion. Thus, the selection rule does not directly affect the measure of interest: asset price variances. With simulations 1 and 2, this is not the case.

## 5 Stationary Markov equilibria

Finally, we seek to apply our computation methodology to approximate the set of stationary Markov equilibria. This equilibrium concept has been the focus of previous studies, including Farmer and Woodford (1984) and Spear, Srivastava, and Woodford (1990). We first introduce the stationary Markov definition and discuss how the computation methodology can be applied as a special case of our previously discussed approach. We then provide simulation results that strongly suggest that important long run dynamics of the economy are ignored by only computing the set of stationary Markov equilibria. In our simulations, the asset price volatility is an order of magnitude smaller when limiting our analysis to the set of stationary Markov equilibria.

**Definition 6** *A stationary Markov equilibrium is described by continuous functions  $\mathbf{f}^\theta : \mathbb{R}^N \times \mathbf{S} \rightarrow \mathbb{R}^N$  and  $\mathbf{f}^q : \mathbb{R}^N \times \mathbf{S} \rightarrow \mathbb{R}^N$  such that for every  $(\theta_1(s_{-1}), s_0)$ , there exists a SCE with  $\theta_1(s^t) = \mathbf{f}^\theta(\theta_1(s^{t-1}), s_t)$  and  $q(s^t) = \mathbf{f}^q(\theta_1(s^{t-1}), s_t)$ .*

The initial conditions  $(\theta_1(s_{-1}), s_0)$  continue to introduce indeterminacy of degree  $SN$  as in all other settings considered in this paper. We discuss the computational methodology and theory in the context of incomplete markets. With complete markets, such an approach is not required since the only indeterminacy in the system is caused by the initial conditions.

The computational algorithm assumes that a stationary Markov equilibrium exists. We recognize that the non-existence of stationary Markov equilibria is a question of theoretical importance, but as our objective is only to compare the asset price volatility in a specific numerically simulated economy, we are content with applying the algorithm and adopting the numerical solution as an  $\epsilon$ -approximation to a stationary Markov equilibrium.

The iterative method begins by specifying a grid of intervals for all asset positions  $\Theta = \Theta^1 \times \dots \times \Theta^J$ , where  $\Theta^j \subset \mathbb{R}$  is an interval. Using Theorem 3, the policy functions  $\mathbf{f}^\theta$  and  $\mathbf{f}^q$  are determinate functions of  $(m(\theta_1(s^{t-1}), s_t), \theta_1(s^{t-1}), s_t)$ . That is, once the vector  $m(\theta_1(s^{t-1}), s_t)$  is known, then  $\theta_1(s^t)$  and  $q(s^t)$  are finite and locally unique solutions to the standard equilibrium equations (8) and (9).

Define  $\mathbf{f}^m : \Theta \times \mathbf{S} \rightarrow \mathbb{R}^N$  as the stationary transition function that maps  $(\theta_1(s^{t-1}), s_t) \mapsto m(\theta_1(s^{t-1}), s_t)$ . The stationary Markov equilibrium concept requires that not only is  $\mathbf{f}^m$  time-invariant, but that it is a selection from the equilibrium Markov correspondence  $V^*$ . Our method constructs approximates this transition function using an iterative method.

Define the determinate policy function  $\mathbf{f}^{\theta^m} : \mathbb{R}^N \times \mathbf{S} \rightarrow \Theta$  as the asset positions determined as a function of shadow values  $m(\theta_1(s^{t-1}), s_t)$ . If we can show that  $\mathbf{f}^{\theta^m} \circ \mathbf{f}^m : \Theta \times \mathbf{S} \rightarrow \Theta$  is surjective, then we know that  $\mathbf{f}^m$  is a stationary transition function. Under the assumption that a stationary Markov equilibrium exists, there exists a nonempty ergodic set  $\Theta^* \neq \emptyset$  that corresponds to the equilibrium Markov correspondence  $V^*$ . The objective of the algorithm is to approximate this set of asset positions that serves as the range for the stationary transition function  $\mathbf{f}^m$ .

For any grid of intervals  $\Theta_n$ , define the updated grid of intervals  $\Theta_{n+1}$  as the image of  $\mathbf{f}^{\theta^m}$  over all possible  $\theta_1^- \in \Theta_n$ , all possible states  $s \in \mathbf{S}$ , and all possible  $m \in V(\theta_1^-, s)$ . Here, as in the previous section,  $V$  is the Markov correspondence obtained via numerical approximation. Thus, the lower boundary of the interval  $\Theta_n^j$  is defined as  $\min_{(\theta_1^-, s) \in \Theta_{n-1} \times \mathbf{S}, m \in V(\theta_1^-, s)} \theta_1^j(s)$ , where  $\theta_1(s) = \mathbf{f}^{\theta^m}(m, s)$  is the determinate implicit function of the system (8) and (9). In a similar manner, the boundaries (both upper and lower) can be specified for each of the intervals.

The updated grid of intervals satisfies  $\Theta_{n+1} \supset \Theta^*$  for all  $n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} \Theta_n \supset \Theta^*$ . This means that the stationary set  $\Theta = \lim_{n \rightarrow \infty} \Theta_n \neq \emptyset$ . Without a stationary

Markov equilibrium, the approximated set  $\Theta = \lim_{n \rightarrow \infty} \Theta_n$  may not be well-defined (our numerical simulations for the economy reveal that the approximated set  $\Theta$  is nonempty).

Given this stationary set, there exists a stationary transition function  $\mathbf{f}^m$  such that the stationary Markov equilibrium functions  $\mathbf{f}^\theta$  and  $\mathbf{f}^q$  can be defined.

The methodological approach guarantees that the transition function  $\mathbf{f}^m$  is a stationary selection from  $V$ . It does not claim that such a function is unique. The numerical simulations suggest that a continuum of possible transition functions  $\mathbf{f}^m$  can be found as stationary selections from the Markov correspondence  $V$ . This is not surprising given Figure 1, which demonstrates a continuum of possible continuation values for all possible state variables.

Our simulations present 2 polar cases of stationary Markov equilibria. Simulation A characterizes the stationary Markov equilibrium in which the stationary transition function  $\mathbf{f}^m$  takes the highest possible continuation values  $m$  from the image of the Markov correspondence  $V$ , while simulation B characterizes the stationary Markov equilibrium when  $\mathbf{f}^m$  selects the lowest values for  $m$ .

Table 6 shows that no matter which polar case is considered, the asset price volatility is at least an order of magnitude smaller than the asset price volatility of the SCE. This strongly suggests that it will be misleading if we implement policy based only on the computation of stationary Markov equilibria.

	$std(\theta)$	$mean(\theta)$	$std(q)$	$mean(q)$
Simulation A	0.0789	2.9794	0.0330	1.3215
Simulation B	0.0188	5.5682	0.0030	0.2069

Table 6: Simulation of stationary Markov equilibria

## 6 Conclusion

In this paper, we study the effects of indeterminacy in a stochastic OLG model with incomplete financial markets. In order to derive the implications of the indeterminacy on the long run behavior of the economy, we numerically characterize the entire set of sequential competitive equilibria. Numerical simulations indicate that the selections from the equilibrium transition correspondence have important welfare effect. We require the selection rule to be consistent across all time periods and show that the welfare difference between selection rules can be as high as 1.3% in terms of consumption equivalent. Additionally, the simulations reveal that the primary factor

generating asset price volatility is the indeterminacy of the equilibrium set, where the Markov endowment process plays a secondary role. This suggests that in models with indeterminate equilibria, agents' expectations of prices are the factors that drive the allocation of resources, and it is only in understanding the effects of indeterminacy that we can propose and implement welfare-improving policies. Suggestions for welfare-improving policies in this class of models are left for future research.

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## 7 Appendix

### 7.1 Numerical Algorithm

The vector of possible values for bond-holding and shocks are given by  $\hat{\Theta} = \{\theta_0^{i_1}\}_{i_1=1}^{N_\theta}$ ,  $\hat{S} = \{s_0^{i_2}\}_{i_2=1}^{N_s}$ , and for each pair of the bond-holding and shock grids,  $(\theta_0^{i_1}, s_0^{i_2})$ , we also define a finite vector of possible values for  $\hat{V}_0^{\mu,\varepsilon}(\theta_0^{i_1}, s_0^{i_2}) = \{m_0^{i_1, i_2, j}\}_{j=1}^{N_v}$ .<sup>2</sup> Notice,  $\lim_{N_\theta \rightarrow \infty} \hat{\Theta} = \Theta$ ,  $\lim_{N_v \rightarrow \infty} \hat{V}_0^{\mu,\varepsilon}(\theta_0^{i_1}, s_0^{i_2}) = \tilde{V}_0^{\mu,\varepsilon}(\theta_0^{i_1}, s_0^{i_2})$ . Finally, we construct the discrete version of operator  $\mathbf{B}^{h,\mu,N}$  by eliminating points that cannot be continued (in the Euler equation, for a predetermined tolerance  $\varepsilon > 0$ ) as follows:

1. Given  $(\theta_0^{i_1}, s_0^{i_2})$ , pick a point  $m_0^{i_1, i_2, j}$  in the vector  $\hat{V}_0^{\mu,\varepsilon}(\theta_0^{i_1}, s_0^{i_2})$ . From  $m_0^{i_1, i_2, j}$  we can determine the values of  $(\theta_+^{i_1, i_2, j}, q_+^{i_1, i_2, j})$  by solving for

$$m_0^{i_1, i_2, j} - (e^1(s_0^{i_2}) + \theta_0^{i_1} - q_+^{i_1, i_2, j} \theta_+^{i_1, i_2, j}) = 0. \quad (16)$$

$$q_+^{i_1, i_2, j} \cdot u_c(m_0^{i_1, i_2, j}) - \beta \sum_{s_+} \pi(s_+ | s_0) u_c(e^2(s_+) + \theta_+^{i_1, i_2, j}) = 0 \quad (17)$$

Thus, if for all  $m_+ \in \hat{V}_0^{\mu,\varepsilon}(\theta_+^{i_1, i_2, j}, s_+) = \{m_+^l(\theta_+^{i_1, i_2, j}, s_+)\}_{l=1}^{N_v}$  we have

$$\min_{m_+ \in \{m_+^l\}_{l=1}^{N_v}} \left\| q_+^{i_1, i_2, j} \cdot u_c(e^0(s_0^{i_2}) - q_+^{i_1, i_2, j} \theta_+^{i_1, i_2, j}) - \beta \sum \pi(s_+ | s_0^{i_2}) u_c(m_+) \right\| > \varepsilon \quad (18)$$

then  $\hat{V}_1^{\mu,\varepsilon}(\theta_0^{i_1}, s_0^{i_2}) = \hat{V}_0^{\mu,\varepsilon}(\theta_0^{i_1}, s_0^{i_2}) - m_0^{i_1, i_2, j}$ .

2. Iterate over all possible values  $m_0^{i_1, i_2, j} \in \hat{V}_0^{\mu,\varepsilon}(\theta_0^{i_1}, s_0^{i_2})$ , and all possible  $(\theta_0^{i_1}, s_0^{i_2}) \in \hat{\Theta} \times \hat{S}$ .

3. Iterate until convergence is achieved  $\sup \left\| \hat{V}_n^{\mu,\varepsilon} - \hat{V}_{n-1}^{\mu,\varepsilon} \right\| = 0$ .

At the limit of the above algorithm, we have  $\lim_{n \rightarrow \infty} \hat{V}_n^{\mu,\varepsilon} = \hat{V}^{\mu,\varepsilon*}$ .

### 7.2 Figure

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<sup>2</sup>Notice the portfolio of the household has  $\mathbf{S}$  components in stochastic case. In the case of two shocks,  $\theta_0 = (\theta_{0,1}, \theta_{0,2})$ .

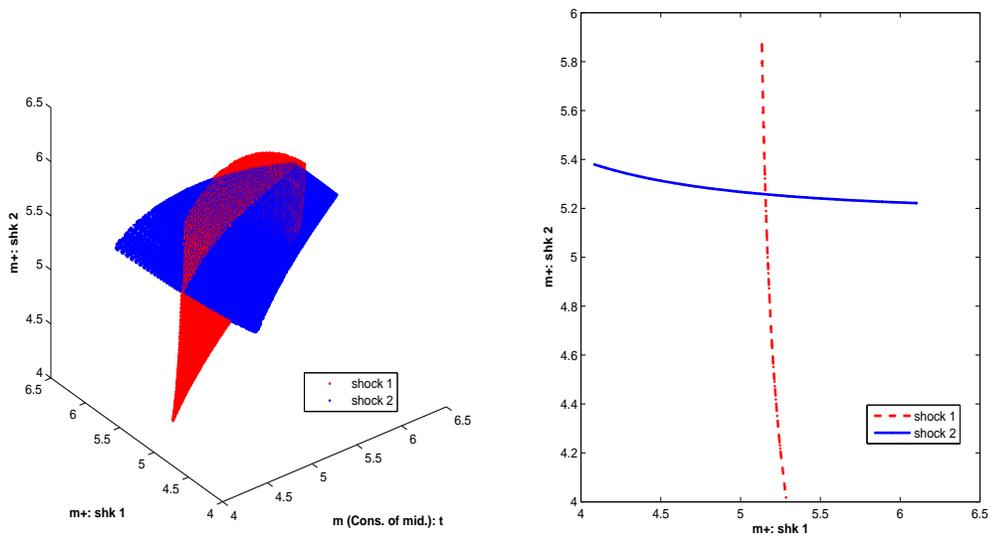


Figure 1: Equilibrium set of  $\left\{ \left\{ m_{t+1}(s) \right\}_{s=1}^{\mathbf{S}}, m_t \right\}$  at given  $\{ \theta_t, s_t \}$ .