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ANONYMOUS MARKETS AND MONETARY TRADING

by

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Anonymous Markets and Monetary Trading*

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ABSTRACT: We study an infinite-horizon economy with two basic frictions that are typical in monetary models. First, agents' trading paths cross at most once due to pairwise trade and other meeting obstacles. Second, actions must be compatible with individual incentives due to commitment and enforcement limitations. We find that, with patient agents, relaxing the first friction by introducing centralized markets, opens the door to an informal enforcement scheme sustaining a non-monetary efficient allocation. Hence, we present a matching environment in which agents repeatedly access large markets and yet the basic frictions are retained. This allows the construction of models based on competitive markets in which money plays an essential role.

Keywords and Phrases: Money, Infinite games, Matching models, Social norms

JEL Classification Numbers: C72, C73, D80, E00

1 Introduction

A large segment of monetary literature revolves around the use of models in which monetary trade is motivated by descriptions of various obstacles to the exchange process. Indeed, several observers have indicated the necessity to employ models that are explicit about the frictions responsible for the use of money. In a monetary framework—it is often argued—money should be 'essential,' i.e., eliminating it from the economic environment

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should result in efficiency loss.[1]

Models with such a trait depict economies in which trade is basically of an intertemporal nature but two intertwined frictions—broadly concerning agents' feasible interactions and access to information—all but rule out credit arrangements. As a first fundamental friction, a meeting process is often imposed that effectively renders trade partners complete strangers and severs any durable links among them. Put simply, the 'trading paths' of any two agents are assumed to cross at most once. A second basic friction concerns commitment and enforcement limitations. Essentially, agents in the model select a course of action knowing they cannot be subject to direct retribution, so their choices must be compatible with individual incentives.

This paper is a theoretical study of the role of such frictions in modeling money. It is motivated by a recently developed monetary framework that builds on the well-known search model of money [16], which prominently displays a fundamental role for money. In the typical search model randomly formed pairs of agents use money to overcome exchange problems due to idiosyncratic shocks. This generates analytically intractable distributions of balances when money is divisible, e.g., see [7, 9, 10]. Such a feature inhibits a broader integration of these modeling techniques into the "toolbox" of the typical macroeconomist (especially those interested in policy analysis). This issue has spurred interest in developing frameworks that vary the basic search model with the goal to obtain degeneracy in equilibrium holdings.[2]

A significant recent development, laid out in the work of Lagos and Wright [18], shares this goal and achieves it by craftily altering the fundamental meeting friction of the typical search model. It introduces a round of Walrasian 'centralized' trading (night market) after each round of bilateral random 'decentralized' trading (day market). The basic premise of the model is that—although the population meets repeatedly in the centralized market—anonymity and recurrent random pairings are frictions sufficient for money to be essential.[3]

This characteristic of the model is remarkable because efficient and individually rational outcomes are known to exist in games with random pairings of anonymous agents. The intuition, as noted by Kandori [14], is that "changing partners itself is unimportant and the crux of the matter is information transmission" in the population. Naturally, one

[1]Methodological observations of this flavor are, for example, in [11, 12, 19, 24]. Studies such as [13] and [17] focus on the essentiality of money.

[2]This basic remedy is at the core of the works of Shi [21, 22], who cleverly models the population as a continuum of families, each encompassing a continuum of agents. After each round of random matching, money holdings are (involuntarily) redistributed within each family.

[3]Precisely, the authors indicate that "Money is essential in this model for the same reason it is essential in the typical search model: since meetings in the day market are anonymous, there is no scope for trading future promises in this market, so exchange must be quid pro quo." [18, p. 466]. Similar statements appear in related work. For example, "The existence of the decentralized markets, and in particular the assumption that agents are anonymous, generates an essential role for money" [20, p. 175]; "...we assume anonymous trading, no record-keeping and no enforcement of contracts. This is sufficient to make a medium of exchange essential for trade...." [6, p. 467].

wonders whether this intuition has implications for the modeling of monetary economies in which agents have access to centralized markets.

A first contribution of this paper, therefore, is to clarify that the fact that anonymous agents often interact in isolated random pairs is not *per se* sufficient to generate an essential role for money. Further frictions are needed, in general. This intuition is developed in a simple manner. We demonstrate that eliminating money from a random matching environment *à la* Lagos and Wright need not reduce efficiency, if actions of trading partners are observable and agents are sufficiently patient. In fact, the non-monetary allocation is socially efficient. To prove it, we study a non-monetary subgame perfect equilibrium in a matching economy with deterministically alternating decentralized (bilateral) and centralized (multilateral) trade. Then, we generalize the result, by studying economies in which centralized trade is a random event.

The lesson we derive is that the essentiality of money hinges on the presence of obstacles to rapid and widespread information transmission, a result which is in line with the findings of earlier work [13, 17].[4] The random matching scheme assumed in models such as [16], is a device to naturally fragment not only the process of exchange of goods but *especially* of information. Our analysis demonstrates that the introduction of centralized trading in such an environment opens the door to the possibility that anonymous agents exchange information quickly with a significant portion of the population. In this case, money ceases to be an essential component of the process of exchange.

The mechanism we exploit to derive this result is simple. To start, we prove some action exists that is 'socially' desirable in every bilateral trade encounter. Since centralized markets are based on economy-wide meetings, actions taken in those markets can serve as vehicles to alert many others of (privately observed) undesirable behavior. Based on this intuition, we then consider a social norm specifying a sanction rule triggered by observed deviations from desirable behavior. Clearly, centralized markets can foster the rapid spread of sanctions and—if they open sufficiently often—they can be the centerpiece of an informal enforcement scheme capable of sustaining efficient allocations.

Based on these findings, we then offer a second contribution. We explain how to construct physical environments in which agents do not exclusively interact in decentralized random pairs—but rather in markets with (infinitely) many other participants—and yet money can be essential. Precisely, we present a matching framework—based on the studies in [3, 4]—that can be used to model a variety of trade meetings, bilateral and multilateral, deterministic and stochastic. We use this framework to outline an economy in which infinitely-lived agents repeatedly move in and out of markets populated by numerous anonymous agents who, however, are always complete strangers. Such a physical environment gives rise to informational frictions that make money essential. In this manner our study complements recent developments in modeling monetary economies. For

[4]This also complements the interesting work of Green [8] on dynamic equilibrium in non-monetary models with infinitely many agents obtained via a replication method of a finite environment.

instance, the techniques we have presented allow the charting of possible physical environments underlying monetary economies with alternating decentralized and centralized markets in the spirit of [18]. In addition, these techniques can be used to further the modeling of environments where money is necessary to support trade in large competitive markets. This can bring us closer to the desirable stage of better integrating the literature on the foundations of money with the mainstream macroeconomic literature, e.g., see [15].

2 The physical environment

We describe an environment that captures the salient features of the model in [18]. Time is discrete and infinite, indexed by $t = 0, 1, 2 \dots$. There is a constant population $J = \mathbb{N}$ of identical infinitely-lived agents and a single perishable good that can be produced by a fraction of the population at each date. Even and odd periods differ in terms of preferences, economic activities, and matchings. We start by formalizing this last element, as it is a central building block.

2.1 The matching process

In each period t, interactions among agents are determined by an exogenous matching process that specifies a partition of the population in trading groups. We define a *match* for agent $j \in J$ in a period t to be a group of people $G_t(j) \subseteq J$, which includes agent j and possibly others. The agents in $G_t(j)$ are called *partners* of j in period t. Let $\beta_t \colon J \to J$ be a stochastic bilateral matching rule, i.e., a function that partitions the population in matches composed of one or two randomly selected agents. (For details see [3].)

Date $t = 0$ is an initial period in which, for convenience, we assume that agents are 'idle,' i.e., $G_0(j) = \{j\}$ for all $j \in J$. In every other date $t \geq 1$ we assume a matching process such that

$$G_t(j) = \begin{cases} \{j, \beta_t(j)\} & \text{if } t \text{ odd} \\ J & \text{if } t \text{ even,} \end{cases} \tag{2.1}$$

where $\beta_t(j) = j$ with probability $1 - \alpha$ for each $j \in J$. That is, in odd periods agents may be paired to someone else, with probability α, while in even periods they all belong to an economy-wide group. We say that trading in odd periods is *decentralized*, while in even periods is *centralized*, as suggested in [18].

Following the matching literature, we identify each match G as a distinct area of economic interaction. Precisely, it is assumed that agents can exchange objects only with their partners, cannot directly communicate with each other, and can only observe actions and outcomes in their current match—ignoring what has happened in every other match (e.g., [23, 16]). Sometimes, this is referred to as *spatial separation* and *limited communication.* There is also *anonymity*, in that each agent ignores not only the partition of the population in odd periods, but also ignores and cannot verify the identity and trading history of others, so that past partners cannot be recognized (e.g., [14, 20]). Finally, there is absence of *commitment* and *enforcement*, in that agents can always refuse to take an action without being subject to retribution.[5] Thus, actions must be compatible with individual incentives at each stage of the trading process (e.g., [13, 17]).

2.2 Preferences and technologies

It is assumed that trade is necessary for consumption to take place. Specifically, in odd periods in each pair of agents a flip of a fair coin determines which one of them is a producer and who is a consumer. In even periods agents can produce and consume. Each producer can supply an amount $a \in [0, \overline{a}]$ of labor to a linear technology that transforms it into a consumption goods. The producer suffers linear disutility a, and derives no utility from consumption of own production. In odd periods, every consumer has utility $u_o(c)$ from consuming $c \geq 0$ goods, and in each even period everyone has utility $u_e(c)$ from $c \geq 0$ consumption. Assume that the functional forms of preferences satisfy the Inada conditions, and that $\overline{a} \in (c_o^* + c_e^*, \infty)$, where c_o^* and c_e^* satisfy $u_e^{'}(c_e^*) = u_o^{'}(c_o^*) = 1$. Agents discount next period's payoffs at the rate $\delta \in (0, 1)$ if the current period is even and $\epsilon \in (0, 1]$ otherwise.

To summarize, in even periods agents are multilaterally matched, can produce *and* consume, while in odd periods only agents who are paired can either produce *or* consume. In essence, in odd periods each agent is randomly assigned to one of three groups, called *producers*, *consumers* and *idle*, corresponding to population proportions $\frac{\alpha}{2}$, $\frac{\alpha}{2}$, and $1 - \alpha$.

3 A trading game

To study non-monetary allocations we must formalize an infinite horizon trading game for this economy. For convenience, we first describe a representative one-shot game in some period $t = 0, 1, 2, \ldots$ and then move on to describe the infinite horizon game.

3.1 The representative one-shot game

Here we define actions and payoffs of agents in a match generated by the process (2.1), in some period t. Since odd periods have many trading groups, while even periods have only one large group, we concentrate on a 'representative' one-shot game, which involves the representative match $G_t(j)$. Recall that in every period t not every partner is a consumer or a producer. We denote by $G_t^P(j)$ and $G_t^C(j)$ the set of producers and consumers, with

[5] Lack of enforcement especially implies that no one can be forced to (i) surrender some of his endowment (of goods or assets) to any other economic agent, (ii) produce, or (iii) suffer a current or future disutility. For example, this means that 'cheating' on a contract cannot trigger a current or future retaliation by the victim or anyone else.

$G_t(j) = G_t^P(j) \cup G_t^C(j)$.[6] Clearly, efficiency of allocations revolves around the amount produced so we streamline the analysis by assuming that $\{0\}$ is the action set of consumers and unmatched agents. Agents have a non-trivial choice of action only as producers, which is when they must choose how much consumption to supply to the members of their group. Each producer's choice corresponds to a non-negative amount of labor input in $[0, \overline{a}]$ to be used in the production of commodities to be delivered to the producer's partners. Hence, we identify the action set of any agent $k \in G_t(j)$ by

$$A_k = \begin{cases} [0, \overline{a}] & \text{if } k \in G_t^P(j) \\ \{0\} & \text{if } k \in G_t^C(j). \end{cases}$$

We let $a_{t,k} \in A_k$ denote the *action* of agent k in period t. Subsequently, we define the action space in the match $G_t(j)$ to be the Cartesian product of the action spaces

$$\mathbf{A}_{t,j} = \mathsf{X}_{k \in G_t(j)} A_k \,,$$

whose elements $\mathbf{a}_{t,j} = (a_{t,k})_{k \in G_t(j)}$ are called *action profiles*.

Recalling that producers own a linear production technology if producer k selects $a_{t,k}$, then he produces $a_{t,k}$ goods. Focusing on pure strategies, we define the payoff function for agent j by

$$v_{t,j} \colon \mathbf{A}_{t,j} \to \mathbb{R} \,.$$

That is, in period t the payoff to agent j depends only on the actions $\mathbf{a}_{t,j}$ taken in his match $G_t(j)$. It is assumed that payoff functions are common knowledge.

Since preferences differ in odd and even periods, we define

$$v_{t,j}(\mathbf{a}_{t,j}) = \begin{cases} u_o(c_{t,j}) - a_{t,j} & \text{if } \ t \ \text{ is odd} \\ u_e(c_{t,j}) - a_{t,j} & \text{if } \ t \ \text{ is even} \,, \end{cases} \tag{3.1}$$

where

$$c_{t,j} = \begin{cases} a_{t,k} & \text{if } \ j \neq k \text{ and } \ t \ \text{ is odd} \\ 0 & \text{if } \ j = k \text{ and } \ t \ \text{ is odd} \\ \liminf\limits_{n \to \infty} \left[\frac{1}{n} \sum_{k \in \{1,\ldots,n\} \setminus \{j\}} a_{t,k} \right] & \text{if } \ t \ \text{ is even} \,. \end{cases}$$

Precisely, his utility depends on how much output the producers in his match deliver to him. His disutility depends on how much output he chooses to produce for his partners (as a producer). If j is a producer in an odd period, then his payoff is $-a_{t,j}$, i.e., the disutility from his labor effort. This effort allows agent j to deliver $a_{t,j}$ consumption to his partner $\beta_t(j)$. Instead, if j is a consumer, then his payoff is $u_o(a_{t,k})$ for $k \neq j$, i.e. it depends on the amount of consumption delivered to him by his partner (a producer). Since in even periods agent j is both a consumer and a producer, his payoff is the utility from consumption by

[6] In odd periods $G_t^P(j) \cap G_t^C(j) = \emptyset$, while in even periods $G_t^P(j) \cap G_t^C(j) = G_t(j)$.

the labor cost. Note that, in calculating the amount $c_{t,j}$ consumed in even periods by the agent, we need to consider the function lim inf since the limit of the sequence of averages of individual production $\left\{\frac{1}{n}\sum_{k\in\{1,\ldots,n\}\setminus\{j\}} a_{t,k}\right\}_{n=1}^{\infty}$ does not necessarily exist.[7] Of course, this also has a desirable economic interpretation, since $c_{t,j}$ is simply the *smallest* average quantity that can be produced in the even period.

Having defined actions sets and payoffs, we discuss the Nash equilibrium of the one-shot representative game with players $G_t(j)$. To do so, we let $\mathbf{a}_{t,-j}$ denote the action profile $(\mathbf{a}_{t,j})_{-j}$, i.e., the profile without the action of the representative agent $j \in G_t(j)$. In order to describe what is optimal for j, we denote his best response correspondence by $\rho_{t,j}\colon \mathbf{A}_{t,j} \twoheadrightarrow A_j$, which is defined for each $\mathbf{a}_{t,j} \in \mathbf{A}_{t,j}$ by

$$\rho_{t,j}(\mathbf{a}_{t,j}) = \quad \left\{a_{t,j} \in A_j :\ v_{t,j}(\mathbf{a}_{t,-j}, a_{t,j}) = \max_{x_{t,j}\in A_j} v_{t,j}(\mathbf{a}_{t,-j}, x_{t,j})\right\}.$$

Then, the best response correspondence for the match $G_t(j)$ is denoted $R_{t,j} : \mathbf{A}_{t,j} \twoheadrightarrow \mathbf{A}_{t,j}$, defined for each $\mathbf{a}_{t,j} \in \mathbf{A}_{t,j}$ by the Cartesian product

$$R_{t,j}(\mathbf{a}_{t,j}) = \times_{k\in G_t(j)} \rho_{t,k}(\mathbf{a}_{t,j}).$$

We are now ready to introduce the equilibrium concept for our study.

Definition 1. *A* ***Nash equilibrium*** *for the representative one-shot game is an action profile $\mathbf{a}^*_{t,j}$ such that $a^*_{t,k} \in \rho_{t,k}(\mathbf{a}^*_{t,j})$ for all $k \in G_t(j)$.*

In other words, an action profile $\mathbf{a}^*_{t,j}$ is an equilibrium if and only if $\mathbf{a}^*_{t,j} \in R_{t,j}(\mathbf{a}^*_{t,j})$, i.e., it is a fixed point of the best response correspondence for the match $G_t(j)$.

The next result guarantees the existence of an equilibrium.

Theorem 2. *In the representative one-shot game of period t described above, the action profile $\mathbf{a}^*_{t,j} = (a^*_{t,k})_{k\in G_t(j)}$, with $a^*_{t,k} = 0$ for all $k \in G_t(j)$ is the only Nash equilibrium.*

Proof. Consider the representative one-shot game in some period t. Assume that $\widehat{\mathbf{a}}_{t,j}$ is a Nash equilibrium of the game. This implies that $\widehat{a}_{t,k} \in \rho_{t,k}(\widehat{\mathbf{a}}_{t,j})$ for all $k \in G_t(j)$, i.e., we must have $v_{t,k}(\widehat{\mathbf{a}}_{t,-k}, \widehat{a}_{t,k}) \geq v_{t,k}(\widehat{\mathbf{a}}_{t,-k}, a_{t,k})$ for all $k \in G_t(j)$ and all $a_{t,k} \in A_k$. If t is odd, this implies $u_o(c_{t,k}) - \widehat{a}_{t,k} \geq u_o(c_{t,k}) - a_{t,k}$, hence $\widehat{a}_{t,k} \leq a_{t,k}$ for all k and $a_{t,k}$. It easily follows that in order for $\widehat{\mathbf{a}}_{t,j}$ to be a Nash equilibrium then we must have $\widehat{a}_{t,k} = a^*_{t,k} = 0$ for all $k \in G_t(j)$, since $v_{t,k}(\widehat{\mathbf{a}}_{t,-k}, a_{t,k})$ is strictly decreasing in $a_{t,k}$ and $0 \in A_k$. An analogous argument applies to even periods. ■

The theorem establishes that in the representative one-shot game absence of production, or autarky, is the unique Nash equilibrium. Indeed, the Nash equilibrium payoff corresponds exactly to the minmax payoff of the one-shot game, $a_{t,k} = 0$ for all $k \in G_t(j)$.

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[7] In fact, there are bounded sequences having arithmetic averages that diverge. An example, given in [2, p. 264], is the sequence $(3, 2, 3, 3, 2, 2, 3, 3, 3, 3, 3, 3, 2, 2, 2, 2, 2, 2, \ldots)$.

To see why, notice that in this game producers cannot be forced, nor can they commit, to make transfers to their partners, since if j is a producer he can always select $a_{t,j} = 0$ and enjoy a payoff $v_{t,j}(\mathbf{a}_{t,j}) \geq 0$. Consequently, we will refer to zero as the reservation or autarkic payoff in the one shot game.

3.2 The infinite horizon game

We now provide a strategic representation of an infinite horizon game in which the representative even and odd period one-shot games deterministically alternate indefinitely with $\varepsilon = 1$, as in [18]. Later we relax these assumptions.[8] Clearly, the set of players is J since agents are infinitely-lived. However, due to the matching process in (2.1), this game resembles one with varying opponents, since no one interacts with a fixed set of partners in every period. It is also a game of imperfect monitoring since during a period t agent $k \in G_t(j)$ observes only the action profile $\mathbf{a}_{t,j}$ in his match, but not in other matches. Thus, we must discuss what information, regarding actions that have been played, is available to an agent.

3.2.1 Action histories

The information $h_{t,j}$ available at the start of period t to the representative agent j can be summarized by the history of actions he has privately observed in all dates $\tau < t$. For $t \geq 1$ we let

$$h_{t,j} = (\mathbf{a}_{0,j}, \ldots, \mathbf{a}_{t-1,j}),$$

where $\mathbf{a}_{t,j} = (a_{t,k})_{k \in G_t(j)}$ and $h_{0,j} = 0$. We then define the set of histories of j by the Cartesian product

$$H_{t,j} = \times_{\tau=0}^{t-1} \mathbf{A}_{\tau,j},$$

denoting the history profile of the representative match $G_t(j)$ at the start of period t by

$$\mathbf{h}_{t,j} = (h_{t,k})_{k \in G_t(j)}.$$

Because of random bilateral matches, the elements of $\mathbf{h}_{t,j}$ will generally differ, i.e., partners do not have common histories. It is then conceivable that—due to anonymity and enforcement limitations—agents might be tempted to use these informational disparities to act in a manner that is socially undesirable. To see why, we must study the behavior of the representative agent j.

Define the agent's pure strategies for the infinite horizon game, as the infinite sequences of maps

$$\sigma_j = (s_{0,j}, s_{1,j}, \ldots),$$

[8] In our case, a *game* consists of the set of players (the population J), the matching process (as defined in Subsection 2.1), the action sets (as described in Subsection 3.1), and the payoff functions (that will be introduced later in (3.3)).

where $s_{t,j} : H_{t,j} \to A_j$ is defined by $s_{t,j}(h_{t,j}) = a_{t,j}$. Denote the strategy profile in the match $G_t(j)$ by

$$s_t(\mathbf{h}_{t,j}) = (s_{t,k}(h_{t,k}))_{k \in G_t(j)} = \mathbf{a}_{t,j} \,.$$

In short, a pure strategy σ_j for the infinite horizon game, is an infinite sequence of mappings from the set of histories of agent j into his action set. It specifies a complete contingent plan of actions for each possible history. At this point, it is important to notice that action sets do not depend on histories but simply on the state of the agent (producer or not). Thus, let the sequence of mappings

$$S_j = \left(A_j^{H_{\tau,j}} \right)_{\tau=0}^{\infty}$$

denote the strategy space of agent j in the infinite horizon game, and $S_{t,j} = \left(A_j^{H_{\tau,j}} \right)_{\tau=t}^{\infty}$ in the subgame starting in period $t \geq 0$, with $S_{0,j} = S_j$. [9] It follows that every (pure) strategy profile σ_j gives rise to a strategy profile $\sigma_{t,j}$ in the subgame starting at t, with $\sigma_{t,j} = (s_{t,j}, s_{t+1,j}, \ldots) \in S_{t,j}$ and $\sigma_{0,j} = \sigma_j$.

Finally, let $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots)$ define the collection of strategies of the entire population, using $\boldsymbol{\sigma}_t = (\sigma_{t,1}, \sigma_{t,2}, \ldots)$ for the collection of strategies concerning a subgame starting in t. We will let $\boldsymbol{\sigma}_{-j}$ denote the vector $\boldsymbol{\sigma}$ that excludes the strategy σ_j of agent j. We are now ready to discuss the payoff to the representative agent $j \in J$, in the infinite horizon game.

3.2.2 Payoffs

Recall that we have set $\varepsilon = 1$. This implies that agents discount adjacent periods only in the case when the next period is odd. Thus, if we let δ_{t+1} denote the discount factor between periods t and $t+1$, then we have

$$\delta_{t+1} = \begin{cases} 1 & \text{if} \quad t \text{ odd} \\ \delta & \text{if} \quad t \text{ even}\,. \end{cases} \tag{3.2}$$

The factor

$$\Delta_{t+1}(\tau) = \prod_{n=t+1}^{\tau} \delta_n$$

will be used to discount back to period t a payoff realized in $\tau \geq t+1$. The payoff to agent j is then the function $V_j \colon \mathsf{X}_{i \in J} S_i \to \mathbb{R}$ defined by

$$V_j(\boldsymbol{\sigma}) \quad = \widehat{v}_0(s_0(\mathbf{h}_{0,j})) + \textstyle\sum_{\tau=1}^{\infty} \Delta_1(\tau) \widehat{v}_\tau\big(s_\tau(\mathbf{h}_{\tau,j})\big)\,. \tag{3.3}$$

[9] Recall that for two sets A and B the notation A^B denotes the set of all mappings from B to A.

Here,

$$\widehat{v}_t(\mathbf{a}_{t,j}) = \begin{cases} \frac{\alpha}{2}[u_o(c_{t,j}) - a_{t,j}] & \text{if } t \text{ is odd} \\ u_e(c_{t,j}) - a_{t,j} & \text{if } t \text{ is even}, \end{cases} \tag{3.4}$$

is simply an expected period utility, since in odd periods the agent is consumer or producer with equal probability $\frac{\alpha}{2}$, and earns zero payoff if he is idle. Therefore, V_j is simply the present value of the stream of expected payoffs generated by market interactions occurring from $t = 0$ on. Since each period $t \geq 1$ defines a proper subgame and

$$V_{t,j}(\boldsymbol{\sigma}_t) = \widehat{v}_t(s_t(\mathbf{h}_{t,j})) + \sum_{\tau=t+1}^{\infty} \Delta_{t+1}(\tau)\, \widehat{v}_\tau(s_\tau(\mathbf{h}_{\tau,j})),$$

then we can formalize recursively agent j's expected payoff in t by

$$V_{t,j}(\boldsymbol{\sigma}_t) = \widehat{v}_t(s_t(\mathbf{h}_{t,j})) + \delta_{t+1} V_{t+1,j}(\boldsymbol{\sigma}_{t+1}), \tag{3.5}$$

with $V_{0,j} = V_j$. The first term on the right hand side of the functional equation (3.5) represents agent j's current expected payoff and the remainder his discounted future expected payoff.

The best response correspondence of agent j, for the infinite horizon game, is thus

$$\rho_j(\boldsymbol{\sigma}) = \left\{ \sigma_j \in S_j : \; V_j(\boldsymbol{\sigma}_{-j}, \sigma_j) = \max_{x_j \in S_j} V_j(\boldsymbol{\sigma}_{-j}, x_j) \right\},$$

and the aggregate best response is $\mathbf{R}(\boldsymbol{\sigma}) = \mathsf{X}_{j\in J}\rho_j(\boldsymbol{\sigma})$. The notion of equilibrium for the infinite horizon game is as follows.

Definition 3. *A* ***subgame perfect Nash equilibrium*** *for the infinite horizon game is a strategy profile* $\boldsymbol{\sigma}^*$ *such that* $\boldsymbol{\sigma}^* \in \rho_j(\boldsymbol{\sigma}^*)$ *for all* $j \in J$.

In other words a strategy profile $\boldsymbol{\sigma}^*$ is an equilibrium of the infinite horizon game if and only if it is a fixed point of the aggregate best response correspondence, i.e., $\boldsymbol{\sigma}^* \in \mathbf{R}(\boldsymbol{\sigma}^*)$.

Theorem 4. *The strategy* $\sigma_j = (0, 0, \ldots)$ *for all* $j \in J$ *is a subgame perfect Nash equilibrium of the infinite horizon game.*

Proof. By Theorem 2, the one-shot Nash equilibrium in any period t is autarky, i.e., $a^*_{t,j} = 0$ for all $j \in J$. Now fix a period $\tau \geq 1$. Then the strategy "each player j plays $a^*_{t,j} = 0$ for $t \geq \tau$," is a subgame perfect equilibrium. To see this, note that, according to this strategy, the actions taken by player j's future opponents are independent of his current play. Additionally, $a^*_{t,j} = 0$ maximizes period t payoff of agent j. Thus, $\sigma_j = (0, 0, \ldots)$ for all $j \in J$ is a Nash equilibrium of the infinite horizon game. ■

Having determined that autarky forever is an equilibrium of the infinite horizon game, we now demonstrate that there exists a sequence of actions involving production and delivery of positive amounts of consumption, which is socially desirable. We call this the 'efficient trade pattern.'

4 Efficient trades in the infinite horizon game

To find the efficient trade pattern in this infinite horizon economy, we consider the problem faced by a planner that selects patterns of production and exchange subject to the same physical restrictions faced by agents. Especially, the planner cannot transfer consumption across matches and over time. Assuming that the planner treats agents identically, the problem is simply to maximize the lifetime utility of the representative agent j. Since this agent is in the match $G_t(j)$ in period t, we define relevant actions and action sets by

$$\mathbf{a}_t = (a_{t,k})_{k\in G_t(j)}, \quad \mathbf{A}_t = \times_{k\in G_t(j)} A_k, \quad \text{and} \quad \mathbf{A} = \times_{t=0}^{\infty} \mathbf{A}_t \,,$$

omitting the index j, if understood. The relevant planner problem is thus to choose a plan $\mathbf{a} = (\mathbf{a}_t)_{t=0}^{\infty} \in \mathbf{A}$ to solve

$$\begin{aligned} &V(\mathbf{a}) = \max\nolimits_{\mathbf{a}\in\mathbf{A}}\{\widehat{v}_0(\mathbf{a}_0) + \textstyle\sum_{\tau=1}^{\infty} \Delta_1(\tau)\widehat{v}_\tau(\mathbf{a}_\tau)\} \\ &\text{s.t.} \qquad a_{t,k} = a_t \text{ for all } k \in G_t^P(j) \text{ and } t \geq 1. \end{aligned} \tag{4.1}$$

Clearly, $a_{t,k} = 0$ for all $k \in G_t^C(j)$ and $t \geq 1$.

To find the remaining actions of the optimal plan, we start by demonstrating that V is a continuous function.

Lemma 5. *The function* $V\colon \mathbf{A} \to \mathbb{R}$ *defined in* (4.1) *is continuous.*

Proof. To start, recall that the functions $\widehat{v}_t\colon \mathbf{A}_t \to \mathbb{R}$ are continuous and uniformly bounded for all t, by assumption. Thus, for all t and all $\mathbf{a}_t \in \mathbf{A}_t$, there exists an $M > 0$ such that $|\widehat{v}_t(\mathbf{a}_t)| \leq M$. Let

$$\mathbf{a}^n = (\mathbf{a}_0^n, \mathbf{a}_1^n, \ldots) \xrightarrow[n\to\infty]{} \mathbf{a} = (\mathbf{a}_0, \mathbf{a}_1, \ldots)$$

hold true in the product topology. That is, $\mathbf{a}_t^n \xrightarrow[n\to\infty]{} \mathbf{a}_t$ for all $t = 0, 1, 2, \ldots$. Since $\widehat{v}_t$ is bounded, it follows from the triangle inequality that

$$|\widehat{v}_t(\mathbf{a}_t^n) - \widehat{v}_t(\mathbf{a}_t)| \leq |\widehat{v}_t(\mathbf{a}_t^n)| + |\widehat{v}_t(\mathbf{a}_t)| \leq 2M \,.$$

To prove the continuity of V, we must show that $V(\mathbf{a}^n) \xrightarrow[n\to\infty]{} V(\mathbf{a})$. If we fix $\varepsilon > 0$, then we need to show that there exists some $n_0 > 0$ such that $|V(\mathbf{a}^n) - V(\mathbf{a})| < \varepsilon$ for all $n \geq n_0$. Since $|\widehat{v}_t(\mathbf{a}_t)| \leq M$, we start by picking a natural number $t_1 \in (0, \infty)$ such that $\sum_{t=t_1+1}^{\infty} \Delta_{t_1+1}(t)M < \frac{\varepsilon}{4}$. Since the functions $\widehat{v}_t$ are continuous, there exists some $n_1 > 0$ such that for all $n \geq n_1$ we have

$$|\widehat{v}_0(\mathbf{a}_0^n) - \widehat{v}_0(\mathbf{a}_0)| + \sum_{t=1}^{t_1} \Delta_1(t)\, |\widehat{v}_t(\mathbf{a}_t^n) - \widehat{v}_t(\mathbf{a}_t)| < \tfrac{\varepsilon}{2} \,.$$

Thus, choosing $n_0 \geq n_1$ we see that

$$\begin{aligned}
&\big|V(\mathbf{a}^n) - V(\mathbf{a})\big| \\
&\leq \big|\widehat{v}_0(\mathbf{a}_0^n) - \widehat{v}_0(\mathbf{a}_0)\big| + \sum_{t=1}^{t_1} \Delta_1(t)\big|\widehat{v}_t(\mathbf{a}_t^n) - \widehat{v}_t(\mathbf{a}_t)\big| + \sum_{t=t_1+1}^{\infty} \Delta_{t_1+1}(t)2M \\
&< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon\,,
\end{aligned}$$

which proves the continuity of V. ■

The main consequence is that optimal plans exist, because V is a continuous function defined on a compact set. Therefore, the maximization problem in (4.1) has a solution. Furthermore, the solution is unique because V is a strictly concave function.

Theorem 6. *An optimal plan $\mathbf{a}^* \in \mathbf{A}$ exists and it is unique.*

Proof. To demonstrate that there is an optimal plan $\mathbf{a}^* \in \mathbf{A}$, we start by noting that $\mathbf{A}$ is a compact space, by the Tychonoff Product Theorem (see [1]). Furthermore, V is continuous, by Lemma 5. So, by the classical Weierstrass' theorem, V has a maximizer, say $\mathbf{a}^* \in \mathbf{A}$. To prove that $\mathbf{a}^*$ is unique, note that the functions $\widehat{v}_t$ are assumed strictly concave for all t. This implies that V is strictly concave as well, and so the maximizer $\mathbf{a}^*$ is unique. ■

Determining the optimal plan $\mathbf{a}^*$ is straightforward. One needs to realize that the planner cannot transfer resources over time and that current choices do not affect future states (aggregate or individual). Consequently, solving the maximization problem (4.1) is equivalent to solving a sequence of static maximization problems. In every period t, the planner chooses $\mathbf{a}_t$ to maximize $\widehat{v}_t(\mathbf{a}_t)$. Clearly, when $t = 0$ agents are idle and thus for all agents $k \in J$ the maximizer is $a_{0,k} = a_0^* = 0$.

Now consider a period $t \geq 1$. Since $a_{t,k} = 0$ for each $k \in G_t^C(j)$ and since each producer is treated identically, i.e., $a_{t,k} = a_t$ for all $k \in G_t^P(j)$, it follows from (3.1) and (3.4) that the objective function in a period t is

$$\widehat{v}_t(\mathbf{a}_t) = \widehat{v}_t(a_t) = \begin{cases} \frac{\alpha}{2}\left[u_o(a_t) - a_t\right] & \text{if } t \text{ is odd} \\ u_e(a_t) - a_t & \text{if } t \text{ is even}\,. \end{cases}$$

The maximizers are given by the production quantities

$$a_t^* = \begin{cases} c_o^* & \text{if} \quad t \ \text{odd} \\ c_e^* & \text{if} \quad t \ \text{even}\,, \end{cases} \tag{4.2}$$

where $u_o'(c_o^*) = 1$ and $u_e'(c_e^*) = 1$. In other words, the planner's solution requires that each producer delivers the surplus-maximizing quantity, in each period. We are now ready to prove that if agents are sufficiently patient, the optimal plan $\mathbf{a}^*$ can arise as an equilibrium of the infinite horizon game.

5 A social norm for economic interactions

The key element of analysis in this section will be the behavior of agents as producers, since consumers are 'inactive' players. Thus, for expositional ease, we focus exclusively on the choices of a representative agent $j \in J$ in a period t. Recall that the optimal plan (4.2) requires that every agent must deliver the surplus-maximizing quantity to his partner(s), whenever he can produce, receiving no direct compensation. The problem is that agents cannot commit to a plan (nor enforce it) so sustaining this plan is difficult in the absence of a planner, and possibilities of direct compensation (money). For this reason, following [14], we concentrate on studying a social norm, i.e., a strategy capable of sustaining the optimal plan by means of a (credible) informal enforcement scheme.

Specifically, we study a strategy—called 'altruistic'—that specifies 'desirable' actions (a rule of cooperation) as well as sanctions for 'undesirable' actions (a rule of punishment). We identify desirable behavior with production decisions conforming with the optimal plan—and labeling every other action as undesirable.

To start, let $a_{t,k} = a_t^*$ define a *desirable* action of producer $k \in G_t^P(j)$ in period t, and let any $a_{t,k} \neq a_t^*$ be *undesirable*. Then, if we consider a representative agent j in period $t \geq 1$, we can define the desirable history by the vector

$$h_{t,j}^* = (\mathbf{a}_{0,j}^*, \mathbf{a}_{1,j}^*, \ldots, \mathbf{a}_{t-1,j}^*),$$

where $a_{\tau,k} = a_\tau^*$ for all $k \in G_\tau^P(j)$ and all $0 \leq \tau \leq t-1$. That is, agent j has observed only desirable behavior up to period t if and only if every producer in his past matches (including agent j himself) followed the optimal plan. Any history $h_{t,j} \neq h_{t,j}^*$, is therefore undesirable because some producer—possibly agent j himself—has been personally observed as having made a choice that departed from the optimal plan. This leads to the following definition of an altruistic strategy.

Definition 7. *A strategy $\sigma_j^* = (s_{0,j}, s_{1,j}, \ldots)$ for a producer $j \in J$ is called altruistic, if $s_{0,j} = a_0^*$ and in each period $t \geq 1$ he selects*

(i) *$s_{t,j}(h_{t,j}) = a_t^*$, whenever $h_{t,j} = h_{t,j}^*$, and*

(ii) *$s_{\tau,j}(h_{\tau,j}) = 0$ for all $\tau \geq t$, whenever $h_{t,j} \neq h_{t,j}^*$.*

Thus, the altruistic strategy simply requires that every producer deliver to his partners the amount of consumption c_t^*, only if the producer has observed desirable behavior. However, the producer should play the minmax strategy forever, as soon as he deviates or has knowledge of a deviation by someone else. It is the threat of such a harsh informal collective punishment—autarky—that can sustain the optimal production plan as a subgame perfect Nash equilibrium. We demonstrate this in the next section, in which we study the individual optimality of the actions suggested by the altruistic strategy.

5.1 Individual optimality

Suppose that every agent follows the altruistic strategy $\boldsymbol{\sigma}^*$, and consider the behavior of a representative agent $j \in J$, in some period t. Denote his expected lifetime utility at the beginning of period t by $V_t = V_{t,j}(\boldsymbol{\sigma}_t^*)$, where the subscript j and the argument of the function are omitted since they are fixed. For simplicity, we also say that in period t we are 'in equilibrium,' if the agent has observed only desirable behavior, and 'off equilibrium,' otherwise.

To start, notice that the equilibrium continuation payoffs, restricted to even or odd periods, are time-invariant. Indeed, the strategy is time-invariant in equilibrium, and the structure of the game does not change over time, so that each subgame is a replica of the infinite horizon game. If we denote them by V_e^* and V_o^*, using (3.4), (3.5), and (4.2), we have

$$\begin{aligned} V_e = & \quad \tfrac{1}{1-\delta}\left\{u_e(c_e^*) - c_e^* + \delta\tfrac{\alpha}{2}[u_o(c_o^*) - c_o^*]\right\} \\ V_o = & \quad \tfrac{1}{1-\delta}\left\{\tfrac{\alpha}{2}[u_o(c_o^*) - c_o^*] + u_e(c_e^*) - c_e^*\right\} . \end{aligned} \tag{5.1}$$

To study the individual optimality of σ_j^* it suffices to consider one-time deviations in a representative subgame starting in some period t. However, notice that since the strategy specifies actions to be taken both in- but also off-equilibrium, we must examine one-period deviations in both contingencies.

To this end, consider an agent in some period t, off-equilibrium, under the conjecture that everyone else plays the altruistic strategy. We denote by V_e^d and V_o^d the continuation payoffs (in even and odd periods) of this agent if he observed a deviation for the first time in the prior period. If the deviation was first observed more than one period earlier, then we use the notation $\tilde{V}_e$ and $\tilde{V}_o$. Specifying these payoffs as time-invariant is done for simplicity, and is without loss in generality. Indeed, a deviation that is observed by some agents in some date, becomes part of the history of the entire population in no more than two periods.

All this will be clarified in the following two subsections, in which we focus on the actions taken by agent j in a representative match $G_t(j)$ of some period $t \geq 1$. Since both the discount factor and the type of matching matter, we will consider two separate cases in which t is either odd or even.

5.2 Deviations and continuation payoffs

Suppose that t is an odd period and that agent j is paired to someone else. Consider the case in which everyone has played equilibrium up to this date, but that the producer in the match $G_t(j)$ elects to deviate from the optimal plan $\mathbf{a}^*$. The deviator can be either agent j or his partner. Either way, in this off-equilibrium contingency we have $h_{t+1,k} \neq h_{t+1,k}^*$ for $k \in G_t(j)$, while $h_{t+1,k} = h_{t+1,k}^*$ for all $k \notin G_t(j)$. Since we are considering an altruistic strategy, this deviation implies that j and his partner $\beta_t(j)$ will select the minmax strategy forever after period t, i.e., $a_{\tau,j} = 0$ in all $\tau \geq t+1$. However, everyone else follows the optimal plan in $t+1$, since they have not observed a deviation. Thus, the continuation payoff of agent $k \in G_t(j)$ from following the altruistic strategy $\sigma^*_{t+1,k}$, under the conjecture that everyone also does the same, is

$$V_e^d = u_e(c_e^*) - 0 + \delta \tilde{V}_o \,. \tag{5.2}$$

Since $t+1$ is an even period—in which every agent is a producer—and the deviation in period t was observed only by the two agents in $G_t(j)$, it follows that only j and his partner will select to produce nothing in $t+1$. However, everyone will observe their deviations in the centralized market, and so we must have $\tilde{V}_o = 0$. To see why, notice that $G_{t+1}(j) = J$. Consequently, since $a_{t+1,k} = 0$ for $k \in G_t(j)$, we have $h_{t+2,k} \neq h^*_{t+2,k}$ for all $k \in J$. Under the premise that agents follow the altruistic strategy, then $a_{\tau,k} = 0$ for all $\tau \geq t+2$ and all $k \in J$. Consequently, $V_{\tau,k} = 0$ for all $\tau \geq t+2$ and all $k \in J$, which implies

$$\tilde{V}_o = \tilde{V}_e = 0 \,. \tag{5.3}$$

Clearly, if a deviation occurs for the first time in a centralized market, then the continuation payoff of every agent in the population is

$$V_o^d = 0 - 0 + \tilde{V}_e = 0 \,. \tag{5.4}$$

Thus, if everyone plays the altruistic strategy, production shuts down permanently in the economy following a deviation in the centralized market, because actions are observed by every market participant.

The lesson is that—if everyone plays the altruistic strategy—a deviation from the optimal plan, in *any* match, will eventually shut down trade in the economy. It takes two periods for this to happen if the deviation occurs in a decentralized market, and one period otherwise, in the simple specification of two-period market cycles (variations we consider later have the same effect). Intuitively, if actions are observable, multilateral matches may allow information to flow across a significant fraction of the population. Small group trade—such as pairwise random trade—can slow down the transfer of information, but cannot prevent it simply because agents are anonymous. What remains to be seen, is whether the informal enforcement scheme at the core of the altruistic strategy is sustainable.

5.3 Sustaining the optimal plan

In this subsection we provide a condition such that the altruistic strategy is subgame perfect both in- and off-equilibrium. Indeed, not only equilibrium deviations should be suboptimal, but participating in the informal enforcement scheme—playing the minmax strategy after observing a deviation—must also be in the agent's best interest. After all, an agent might want to 'forgive' a deviator in order to avoid permanent autarky. Thus, we must show that no such temptation exists off-equilibrium, and not only that the optimal plan is individually optimal, in equilibrium.

Theorem 8. *If*

$$\underline{\delta} = \frac{c_o^* + c_e^*}{c_o^* + u_e(c_e^*) + \frac{\alpha}{2}\left[u_o(c_o^*) - c_o^*\right]} ,$$

then for each $\delta \geq \underline{\delta}$ the altruistic strategy $\boldsymbol{\sigma}^*$ supports the optimal plan as a subgame perfect Nash equilibrium of the infinite horizon game.

Proof. We need to show that, under the conjecture that everyone else plays according to $\boldsymbol{\sigma}_{-j}^*$, the representative agent j can neither profitably deviate from the altruistic strategy in-equilibrium nor off-equilibrium. To start, consider an off-equilibrium situation in an arbitrary period t in which partners in the match $G_t(j)$ observe a deviation for the first time. We will derive a condition, in terms of the parameters of the model, guaranteeing that off-equilibrium deviations from σ_j^* are unprofitable. That is, we find a condition under which it is optimal to play the minmax strategy if history includes a deviation in period t. We need to consider two cases:

(i) t is odd: any agent $k \in G_t(j)$ is a producer in period $t+1$. In this period, he follows the altruistic strategy if it maximizes his payoff from $t+1$ on, i.e., using (5.2) we need

$$u_e(c_e^*) - 0 + \delta\tilde{V}_o \geq u_e(c_e^*) - a_{t+1,k} + \delta\tilde{V}_o \,.$$

The left hand side represents the expected payoff, off-equilibrium, from selecting the off-equilibrium action specified by the altruistic strategy, $a_{t+1,k}^* = 0$. The right hand side is the expected payoff from choosing some different action, $a_{t+1,k} \neq 0$. Clearly, (5.3) implies that this inequality holds for all $a_{t+1,k} \in A_k$.

(ii) t is even: any agent $k \in G_t(j)$ in period $t+1$ follows the altruistic strategy if

$$u_o(0) - 0 + \tilde{V}_e \geq u_o(0) - a_{t+1,k} + \tilde{V}_e.$$

Clearly, this holds for all $a_{t+1,k} \in A_k$, independent of whether agent k is a producer or a consumer in period $t+1$.

We conclude that deviating forever, after a defection is part of the agent's history, is individually optimal. That is, if agent j prefers to play σ_j^*, in equilibrium, then he certainly prefers to play minmax forever, as soon as he detects a deviation from $\boldsymbol{\sigma}^*$.

We now move on to consider in-equilibrium deviations in an arbitrary period t, distinguishing between odd and even periods.

(i) t is odd: any producer $k \in G_t(j)$ follows the altruistic strategy whenever this maximizes his current payoff, i.e.,

$$-c_o^* + V_e \geq 0 + V_e^d.$$

Using (5.1) and (5.2), this inequality can be rewritten as

$$\frac{\delta}{1-\delta}\left\{\frac{\alpha}{2}\left[u_o(c_o^*)-c_o^*\right]+u_e(c_e^*)-c_e^*\right\}\geq c_o^*+c_e^*\,. \tag{5.5}$$

That is, the present value of the net loss from deviating, captured on the left hand side of the inequality, must exceed the current net gain from deviating, on the right hand side. The latter is $c_o^*+c_e^*$ since the deviator avoids production in both the odd, and even period, before the economy shuts down. The expression (5.5) yields

$$\delta\geq\delta_o\quad=\frac{c_o^*+c_e^*}{c_o^*+u_e(c_e^*)+\frac{\alpha}{2}\left[u_o(c_o^*)-c_o^*\right]}\,.$$

(ii) t is even: every agent $k\in G_t(j)$ is a producer and follows the altruistic strategy whenever this maximizes his current payoff, i.e.,

$$u_e(c_e^*)-c_e^*+\delta V_o\geq u_e(c_e^*)+0+\delta V_o^d\,.$$

Since V_o satisfies (5.1) and V_o^d satisfies (5.4), we see that

$$\frac{\delta}{1-\delta}\left\{\frac{\alpha}{2}[u_o(c_o^*)-c_o^*]+u_e(c_e^*)-c_e^*\right\}\geq c_e^*$$

is satisfied whenever

$$\delta\geq\quad\delta_e=\frac{c_e^*}{u_e(c_e^*)+\frac{\alpha}{2}\left[u_o(c_o^*)-c_o^*\right]}\,.$$

From $u_e(c_e^*)>c_e^*$ and $u_o(c_o^*)>c_o^*$, we get $0<\delta_e<\delta_o<1$. The intuition behind $\delta_o>\delta_e$, is that when a deviation occurs in an odd period, the informal punishment takes place with one period delay, so that agents need to be more patient to willingly follow the optimal plan. Finally, let $\underline{\delta}=\delta_o$. ■

Theorem 8 establishes that if agents are sufficiently patient, and can observe their partners' actions, then there exists a subgame perfect Nash equilibrium supporting the optimal plan. In this simple framework, the deterministic opening of centralized markets implies that everyone will be informed of a privately observed deviation with *certainty.* This can encourage desirable behavior in every random bilateral match even if agents are anonymous and direct punishment is impossible due to lack of enforcement. Indeed lack of a formal enforcement scheme implies that, generally speaking, monetary allocations cannot be sustained in which the monetary authority imposes any form of taxation. Consequently, we have an additional result.

Corollary 9. *If agents are sufficiently patient, the trade pattern that characterizes the optimal plan cannot generally be sustained in a monetary equilibrium, but is attainable in a non-monetary equilibrium.*

Proof. Let π denote the gross inflation rate in a stationary monetary equilibrium. It is immediate from equation (19) in [18], that $c_o < c_o^*$ for all $\pi > \delta$, and $\lim_{\pi \to \delta} c_o = c_o^*$. For example, if pricing is competitive in every period, it is simple to demonstrate that

$$c_o = (u_o')^{-1} \left(\frac{2(\pi - \delta)}{\alpha\delta} + 1 \right).$$

For details see the proof of Proposition 1 in [6], letting $\gamma = \pi$, $\alpha = 1$, and $q_1 = c_0$. Since no enforcement implies $\pi \geq 1$ then we have $c_0 < c_0^*$. ■

To sum up, assuming anonymous agents and decentralized trade does not imply that money is essential. What matters is how information about actions can spread in the economy, which in turn impinges on the matching process assumed to be in place. Theorem 8 proves that the existence of markets in which the entire population trades regularly, can discourage defections from socially desirable behavior.

In the next section, we demonstrate this result is fairly robust to variations in the underlying matching process. That is, a version of Theorem 8 can be proved for physical environments without the deterministic meeting cycle specified by (2.1), and seen in [18].

6 Generalization of the main result

It may be argued that Theorem 8 hinges on the possibility to deterministically and periodically inform the entire economy of privately observed deviations. However, a more severe matching friction could be assumed so that knowledge of deviations may spread slowly and randomly across the economy. For example, this can happen if not everyone participates in centralized markets regularly, or if there are many spatially separated centralized markets to which agents are randomly assigned, or if the centralized market opens after a random sequence of decentralized trading dates. We choose to follow this last route, assuming that decentralized trade follows a round of centralized trade, but centralized trade occurs after a round of bilateral trade with time-invariant probability $b \in (0, 1)$. Thus, if t is a period of decentralized trade, then $b(1 - b)^n$ is the probability that centralized trade will take place in period $t + 1 + n$, for $n \geq 0$. Hence, there is an expected delay of an approximate

$$\sum_{n=0}^{\infty} nb(1 - b)^n = \frac{1 - b}{b}$$

periods, before centralized trade takes place. Assume also that $\varepsilon \in (0, 1)$. That is, agents discount adjacent periods even when the next period involves a decentralized trading round.

6.1 Payoffs

Since we must simply distinguish between periods with centralized and decentralized trade, we let q denote consumption and U the utility function in a round of centralized trading (instead of u_e), and denote by c consumption and u the utility function in a round of decentralized trading (instead of u_o). Consider a game in which the altruistic strategy $\boldsymbol{\sigma}^*$ is followed and consider the behavior of a representative agent $j \in J$, in some period t. Once again, the equilibrium continuation payoffs restricted to centralized and decentralized markets are time-invariant.

For this reason, we denote the expected lifetime utility at the start of a round of centralized trading by V_C, and denote it by V_D for the first round of decentralized trading that follows it. We use $\widetilde{V}$ to denote the agent's expected continuation payoff in a decentralized trading round. Specifically, we have:

$$\begin{aligned} V_C &= U(q^*) - q^* + \delta V_D \\ V_D &= \tfrac{\alpha}{2}[u(c^*) - c^*] + \widetilde{V} \\ \widetilde{V} &= \varepsilon b \textstyle\sum_{n=0}^{\infty} (1-b)^n \varepsilon^n [U(q^*) - q^* + \delta V_D] \\ &\quad + b \textstyle\sum_{n=1}^{\infty} (1-b)^n \sum_{j=1}^{n} \varepsilon^j \tfrac{\alpha}{2}[u(c^*) - c^*] \end{aligned} \tag{6.1}$$

Using (6.1) we obtain the closed-form solutions

$$\begin{aligned} \widetilde{V} &= \tfrac{1}{1-\delta\Delta_1} \left\{ \Delta_1 [U(q^*) - q^*] + \tfrac{\alpha}{2}[u(c^*) - c^*] \, (\delta\Delta_1 + \Delta_2) \right\} \\ V_C &= \tfrac{1}{1-\delta\Delta_1} \left\{ [U(q^*) - q^*] + \delta \tfrac{\alpha}{2}[u(c^*) - c^*](1 + \Delta_2) \right\} \\ V_D &= \tfrac{1}{1-\delta\Delta_1} \left\{ \Delta_1 [U(q^*) - q^*] + \tfrac{\alpha}{2}[u(c^*) - c^*] \, (1 + \Delta_2) \right\}, \end{aligned} \tag{6.2}$$

where a simple algebraic manipulation indicates that

$$\Delta_1 = \tfrac{b\varepsilon}{1-(1-b)\varepsilon} \quad \text{and} \quad \Delta_2 = \tfrac{\varepsilon}{1-(1-b)\varepsilon} - \Delta_1 \, .$$

Clearly, when $b, \varepsilon \to 1$, we have $\Delta_1 \to 1$ and $\Delta_2 \to 0$, so we get back (5.1). In short, the prior framework is a special case of this one.

6.2 Sustaining the optimal plan

It remains to be seen whether the altruistic strategy is a subgame perfect equilibrium. Indeed in this generalized framework, economy-wide punishment can only be triggered stochastically, so agents might avoid playing the minmax strategy, as soon as a deviation has been observed. [10] We will show that no such temptation exists if the agent is suf-

[10] This is reminiscent of the random matching model of Kocherlakota and Wallace (1998), in which a public record of trading histories is assumed which is updated randomly and can be accessed freely by every bilaterally matched agent.

ficiently patient, since even if the agent chooses not to deviate, his partners *eventually* will.

Theorem 10. *If*

$$\underline{\underline{\delta}} = \frac{c^*\left(1+\frac{\alpha}{2}\Delta_2\right)+q^*\Delta_1}{\Delta_1\left\{\frac{\alpha}{2}[u(c^*)-c^*]+\left[c^*+\frac{\alpha}{2}u(c^*)\Delta_2+U(q^*)\Delta_1\right]\right\}} \quad > \underline{\delta}\,,$$

then for every $\delta \geq \underline{\underline{\delta}}$ *the altruistic strategy* $\boldsymbol{\sigma}^*$ *supports the optimal plan as a subgame perfect Nash equilibrium of the infinite horizon game with random centralized markets.*

Proof. We need to show that, under the conjecture that everyone else plays according to $\boldsymbol{\sigma}^*$, the representative agent can profitably deviate from the altruistic strategy neither in- nor off-equilibrium. To start, consider in-equilibrium deviations, distinguishing between periods with centralized and decentralized trading.

(i) Decentralized trading in t: any producer $i \in G_t(j)$ follows the altruistic strategy whenever

$$-c^* + \widetilde{V} \geq 0 + \varepsilon b \sum_{n=0}^{\infty}(1-b)^n \varepsilon^n \left[U(q^*) + \delta \times 0\right] + b\sum_{n=1}^{\infty}(1-b)^n \sum_{j=1}^{n} \varepsilon^j \frac{\alpha}{2}[u(c^*)]\,.$$

Using (6.1), this inequality can be rewritten as

$$\delta\Delta_1\left\{\tfrac{\alpha}{2}\left[u(c^*)-c^*\right] + \left[c^* + \tfrac{\alpha}{2}u(c^*)\Delta_2 + U(q^*)\Delta_1\right]\right\} \geq c^*\left(1+\tfrac{\alpha}{2}\Delta_2\right) + q^*\Delta_1\,. \tag{6.3}$$

The expression (6.3) never holds true as $b \to 0$, i.e., as we converge to the typical random matching model of money. When $0 < b < 1$ expression (6.3) holds if

$$\delta \geq \quad \delta_D = \frac{c^*\left(1+\frac{\alpha}{2}\Delta_2\right)+q^*\Delta_1}{\Delta_1\left\{\frac{\alpha}{2}[u(c^*)-c^*]+c^*+\frac{\alpha}{2}u(c^*)\Delta_2+U(q^*)\Delta_1\right\}}\,.$$

(ii) Centralized trading in t: every agent $i \in G_t(j)$ is a producer and so he follows the altruistic strategy whenever

$$U(q^*) - q^* + \delta V_D \geq U(q^*) + \delta V_D^d\,.$$

Since V_D satisfies (6.2) and $V_D^d = 0$, it follows that

$$\delta\left\{U(q^*)\Delta_1 + \tfrac{\alpha}{2}[u(c^*)-c^*]\left(1+\Delta_2\right)\right\} \geq q^*\,.$$

This implies that

$$\delta \geq \quad \delta_C = \frac{q^*}{U(q^*)\Delta_1+\frac{\alpha}{2}[u(c^*)-c^*](1+\Delta_2)}\,.$$

Since $U(q^*) > q^*$ and $u(c^*) > c^*$, we see that $0 < \delta_C < \delta_D < 1$.[11] That is, if the representative agent prefers to play according to σ_j^* in equilibrium, then he certainly prefers to play minmax forever, as soon as he detects a deviation from $\boldsymbol{\sigma}^*$. It is possible to show in a manner analogous to Theorem 8 that if it is not optimal to deviate in equilibrium, then it is certainly not optimal to deviate off-equilibrium. Finally, let $\underline{\underline{\delta}} = \delta_D$. Notice that $\underline{\underline{\delta}} > \underline{\delta}$ since if a deviation occurs in a bilateral random match, an economy-wide punishment is expected to take place with $\frac{b}{1-b} > 1$ periods of delay, so that agents need to be more patient than when centralized trade follows deterministically a round of decentralized trade. ■

If we interpret the discount rates δ and ε as probabilities of continuation of the game, this result shows that the game must be likely to continue once information about a deviation has reached everyone. The size of δ depends on how fast information can be transmitted (the value of b), and on the average duration of decentralized trade (depending on ε). For example any $b \in (0, 1)$ can sustain the altruistic strategy when $\delta = \varepsilon \to 1$, since a deviation in a bilateral market is on average communicated to the entire population after $\frac{1-b}{b}$ periods, which is a finite number.

Of course, if $b \to 0$ then the infinite horizon game converges to a repeated random matching game among infinite number of agents. It is obvious that in this case the altruistic strategy cannot be supported, since $\Delta_1 \to 0$. That is, there is no possibility to communicate a deviation to a sufficiently large number of agents (see also [14, Proposition 3]). Indeed, [5] studies social norms in such an extreme formulation (corresponding to the typical search model of money) and shows that money *does* have a role to play. It is important to realize, however, that this happens not because of random pairings *per se*. The central friction is obstacles to information transfers across groups of traders. Indeed, we will next construct a general matching framework in which agents deterministically enter infinitely large trading groups and yet money is essential.

7 Matching and information

We have seen that it is the presence of obstacles to rapid and widespread information transmission that prevents the sustainability of the optimal plan. The random pairing scheme assumed in models such as [16], naturally justifies such obstacles and—together with other assumptions—generates an explicit role for money. In this section, we demonstrate that there is nothing special about random bilateral matching in achieving this goal. This is done by introducing a matching process that repeatedly partitions the population into groups with infinitely many partners. We prove that here, too, partners are complete strangers and autarky is the only subgame perfect equilibrium. In short, we present a model with large (possibly Walrasian) markets in which money is essential, since no two

[11] We have $\delta_C < \delta_D$, whenever $\Delta_1 [U(q^*) - q^*] + \frac{\alpha}{2}[u(c^*) - c^*] (1 + \Delta_2) \geq 0$.

agents (neither their direct and indirect partners) will ever be in the same market more than once.

7.1 A formalization for exogenous matching processes

The analysis in this subsection is based on the formalization developed in [3] and [4], to which we refer the reader for details and proofs of some claims. For the sake of brevity, here we simply sketch the procedure, which consists of the following steps. First, in the initial date, we partition the population into spatially separated sets of agents, called "clusters." Then, we match the agents within each cluster into groups of partners, using a selection procedure called a *matching rule*. Finally, we define a sequence of partitions and matching rules to obtain a *matching process*, i.e., a time-path for the process of group-formation.

To start, consider a representative period. Since matching agents simply means dividing the population J into disjoint sets of people, we start by defining a partitional correspondence $\psi\colon J \twoheadrightarrow J$ called the *clustering rule*. As a result we have $J = \sqcup_{s\in S} J_s$, with clusters $J_s = \psi(x)$ for $x \in J_s$, defined over the index set S. Once such a partition of the population has been created, we can operate on each single cluster, dividing its agents into one or more groups. This is what we call a matching rule. Specifically, a *multilateral* matching rule is a partitional correspondence $\mu\colon J \twoheadrightarrow J$ such that $\mu(x) \subseteq J_s$ for all $s \in S$ and all $x \in J_s$, while a *bilateral* matching rule is a function $\beta\colon J \to J$ satisfying $\beta^2(x) = x$ for all $x \in J$ that maps every cluster onto itself, i.e., $\beta(J_s) = J_s$.

Notice that since $J = \mathbb{N}$, any matching rule forms groups containing a countable number of agents, finite or infinite. If we consider an agent x, under a multilateral matching we have $G(x) = \mu(x) \subseteq \psi(x)$, while under bilateral matching we have $G(x) = \{x, \beta(x)\}$. Clearly, matching rules can have different properties. For convenience, we focus on multilateral matching rules such that $\mu(j) = \psi(j) = \psi(x)$ for all $j \in \psi(x)$ and $x \in J$. That is, under multilateral matching the cluster $\psi(x)$ and the trading group $G(x)$ of an agent x coincide.

We call a sequence of matching rules a *matching process*. Since we work with infinite time we construct matching processes as follows. First, we specify an infinite sequence $\Psi = (\psi_0, \psi_1, \ldots)$ of clustering rules on the population J, which we call a *clustering process*. We will assume that $\psi_0(x) = \{x\}$ for each $x \in J$, for simplicity. Subsequently, we define the matching process ϕ relative to Ψ, as an infinite sequence $\phi = (\phi_0, \phi_1, \ldots)$ of matching rules such that for each τ we have

$$\phi_\tau(x) = \begin{cases} \mu_\tau(x) & \text{for } \tau = 2t \\ \{\beta_\tau(x)\} & \text{for } \tau = 2t+1\,. \end{cases}$$

That is, we have small (two-agent) groups in odd periods, and large groups in even periods, where in any period τ the set

$$G_\tau(x) = \{x\} \cup \phi_\tau(x)$$

denotes the trading group of agent x, i.e., himself and his partners.

For practical purposes, we will assume that agents know ϕ but do not know Ψ. This means that agents are aware of the alternating nature of trading groups, but do not know the composition of groups (other than the one in which they currently are), since they do not know the sequence of partitions induced by Ψ.

This formalization allows us to easily keep track of matching histories and (hence, action histories). For each $t \geq 0$ we denote by $P_t(x)$ the set of all partners of any agent x in periods up to and including t. That is,

$$P_t(x) = \bigcup\nolimits_{\tau=0}^{t} G_\tau(x)\,,$$

and observe that $P_0\,(x) = \{x\}$ since $\psi_0(x) = \{x\}$. Now, denote by $\Pi_t(x)$ the set of all of x's past and current partners (including x), the past partners of x's current partners, the partners that x's partners in $t-1$ met prior to that date, and so on. This set is given by the recursive formula

$$\begin{aligned} \Pi_0(x) &= P_0(x) \\ \Pi_t(x) &= \Pi_{t-1}(x) \bigcup \left[\bigcup\nolimits_{b \in G_t(x)} \Pi_{t-1}(b)\right] \quad \text{for } t = 1, 2, \ldots\ . \end{aligned}$$

Following [17], we concentrate on matching processes satisfying

$$\Pi_{t-1}(x) \bigcap \Pi_{t-1}(b) = \emptyset \tag{7.1}$$

for every agent $x \neq b \in G_t(x)$ and all $t \geq 1$. We say that the economy is *informationally isolated*, if (7.1) holds. To see why, define an 'event' as an action taken by some agent at some date. It can be proved (see [3]) that when (7.1) holds no pair of agents in a match ever shares the knowledge of some past event, because they never had and never will share any direct or indirect partner over their lifetimes. Indeed, this holds even if histories can be freely shared during the course of a match. Given (7.1), at the start of any match of any date t, the history $h_{t,x}$ of any agent x includes events that are ignored by x's current partners.

It is now obvious that monetary economies based on the matching scheme adopted in [18] are not informationally isolated since the entire population regularly trades in the centralized market. Indeed, from (2.1), we see that for all x we have $G_t(x) = J$ when t is even, which implies $\Pi_{t-1}(x) \cap \Pi_{t-1}(y) = J$ for all $y \in G_t(x)$ in any period $t \geq 3$. Technically, this is at the heart of Theorem 8. The natural question now is: can we construct informationally isolated economies with large (perhaps infinite) recurring trade groups? The answer will be given in the next subsection.

7.2 Modeling informationally isolated economies

We are now ready to present a general way of constructing informationally isolated economies, i.e., a matching process satisfying (7.1) that matches periodically everyone to infinitely many new trading partners. The general procedure consists of three basic steps. In $t = 0$ we partition the population into a countable number of sets $P_{0,1}, P_{0,2}, \ldots$ of identical cardinality. We then construct recursively partitions of the population for each subsequent date. Finally, we construct clusters out of these partitions, and apply a matching rule—possibly bilateral or multilateral depending on the date—within each cluster. Specifically, in each t define recursively the partitions (details on the construction are in the appendix):

$$\begin{array}{lll}
\text{Period} & & \text{Partition of the set of traders } J \\
0 & J & = P_{0,1} \bigsqcup P_{0,2} \bigsqcup P_{0,3} \bigsqcup P_{0,4} \bigsqcup P_{0,5} \bigsqcup P_{0,6} \bigsqcup \cdots \\
1 & J & = \langle P_{0,1} \bigsqcup P_{0,3} \bigsqcup \cdots \rangle \bigsqcup \langle P_{0,2} \bigsqcup P_{0,6} \bigsqcup \cdots \rangle \bigsqcup \cdots \\
 & & = P_{1,1} \bigsqcup P_{1,2} \bigsqcup \cdots \\
2 & J & = \langle P_{1,1} \bigsqcup P_{1,3} \bigsqcup \cdots \rangle \bigsqcup \langle P_{1,2} \bigsqcup P_{1,6} \bigsqcup \cdots \rangle \bigsqcup \cdots \\
 & & = P_{2,1} \bigsqcup P_{2,2} \bigsqcup \cdots \\
\vdots & & \vdots \\
t+1 & J & = \langle P_{t,1} \bigsqcup P_{t,3} \bigsqcup \cdots \rangle \bigsqcup \langle P_{t,2} \bigsqcup P_{t,6} \bigsqcup \cdots \rangle \bigsqcup \cdots \\
 & & = \bigsqcup_{k=0}^{\infty} P_{t+1,k+1} = \bigsqcup_{k=0}^{\infty} \bigsqcup_{n=0}^{\infty} P_{t,(2n+1)2^k} \\
\vdots & & \vdots
\end{array} \tag{7.2}$$

It should be clear that in each $t \geq 1$, there are countably many sets $P_{t-1,k+1}$, $k = 0, 1, \ldots$, which have the same cardinality, and are pairwise disjoint.[12] We use them to construct infinitely many *matching blocks* in t, each of which is defined by the infinite union

$$P_{t,k+1} = \bigsqcup_{n=0}^{\infty} P_{t-1,(2n+1)2^k} \quad \text{for all} \quad k = 0, 1, \ldots \ .$$

The trick now is to use these matching blocks to define a clustering process Ψ on J that delivers informational isolation. Working with the partition (7.2), we select a special clustering process $\Psi^* = (\psi_0^*, \psi_1^*, \ldots)$ with the following properties.

First, let $\psi_0^*(x) = \{x\}$ for all $x \in P_{0,k+1}$ and all k, so that $\psi_0^*(P_{0,k+1}) = P_{0,k+1}$ for all k. Then, in each period t we create a sequence of clustering rules $\psi_{t,k+1} \colon P_{t,k+1} \twoheadrightarrow P_{t,k+1}$. Next, define $\psi_t^* \colon J \twoheadrightarrow J$ for each $x \in P_{t,k+1}$ by

$$\psi_t^*(x) = \psi_{t,k+1}(x) \, .$$

This allows us to place each agent $x \in P_{t,k+1}$ into a set $\psi_{t,k+1}(x)$ having countably many agents, one from each of the sets $P_{t-1,(2n+1)2^k}$ that compose $P_{t,k+1}$ (the proof can be done

[12] The cardinality is identical since each set $P_{t,k+1}$ is the countable union of sets $P_{0,k+1}$ that have the same cardinality. They are pairwise disjoint by construction.

as in [4, Theorem 4]). The key consequence is that every time we apply a multilateral matching rule as defined above, then the cluster $\psi_t(x)$ and the trading group $G_t(x)$ coincide. In this case, every agent trades with infinitely many partners. What's crucial is that every agent x will *always* find himself in a cluster $\psi_{t,k+1}(x)$ comprised of individuals that are always different and total strangers. We stress that this would be true even if agents were *not* anonymous, i.e., if identities could be observed and verified freely. Either way, the clustering process Ψ^* insures total informational isolation. Formally, we have the following result.

Theorem 11. *Every matching process based on Ψ^* guarantees informational isolation as defined in* (7.1)*.*

Proof. The proof will be based upon the following two properties. For each $k = 0, 1, \ldots$, each $t \geq 0$, and each $0 \leq \tau \leq t$ we have:

1. $\psi^*_\tau(P_{t,k+1}) = P_{t,k+1}$, and
2. $\Pi_\tau(x) \subseteq P_{t,k+1}$ for all $x \in P_{t,k+1}$.

The proof of (1) is by induction on t. For $t = 0$ it is obvious that $\psi^*_0(P_{0,k+1}) = P_{0,k+1}$ for all k, since by our definition $\psi^*_0(x) = \{x\}$ for all $x \in J$. Therefore, for the induction step, assume that for some $t \geq 0$ we have $\psi^*_\tau(P_{t,k+1}) = P_{t,k+1}$ for all k and all $0 \leq \tau \leq t$. We want to prove that for any k we have $\psi^*_\tau(P_{t+1,k+1}) = P_{t+1,k+1}$ for each $\tau = 0, 1, \ldots, t+1$. Start by observing that by the induction hypothesis $\psi^*_\tau(P_{t,k+1}) = P_{t,k+1}$ holds true for all $\tau = 0, 1, \ldots, t$. Now, note that $P_{t+1,k+1} = \bigsqcup_{n=0}^\infty P_{t,(2n+1)2^k}$. But then for each $\tau = 0, 1, \ldots, t$ we have

$$\begin{aligned}\psi^*_\tau(P_{t+1,k+1}) &= \psi^*_\tau\left(\bigsqcup_{n=0}^\infty P_{t,(2n+1)2^k}\right) = \bigsqcup_{n=0}^\infty \psi^*_\tau(P_{t,(2n+1)2^k}) \\ &= \bigsqcup_{n=0}^\infty P_{t,(2n+1)2^k} = P_{t+1,k+1}\,.\end{aligned}$$

Also, by definition $\psi^*_{t+1}(P_{t+1,k+1}) = P_{t+1,k+1}$. Therefore, $\psi^*_\tau(P_{t+1,k+1}) = P_{t+1,k+1}$ holds true for each k and all $\tau = 0, 1, \ldots, t+1$ and the validity of (1) has been established. The proof of (2) is by induction on τ. For $\tau = 0$ notice that for each $x \in P_{t,k+1}$ we have $\Pi_0(x) = \{x\} \subseteq P_{t,k+1}$. For the inductive step assume that for some $0 \leq \tau < t$ we have $\Pi_\tau(x) \subseteq P_{t,k+1}$ for all $x \in P_{t,k+1}$. We must show that $\Pi_{\tau+1}(x) \subseteq P_{t,k+1}$ for all $x \in P_{t,k+1}$. Fix $x \in P_{t,k+1}$. From (1) we get $\psi^*_{\tau+1}(P_{t,k+1}) = P_{t,k+1}$, and so $\psi^*_{\tau+1}(x) \subseteq P_{t,k+1}$. Therefore, each element $y \in \psi^*_{\tau+1}(x)$ belongs to $P_{t,k+1}$. But then our induction hypothesis yields $\Pi_\tau(y) \subseteq P_{t,k+1}$ for each $y \in \psi^*_{\tau+1}(x)$, and so

$$\Pi_{\tau+1}(x) = \Pi_\tau(x) \bigcup \Big[\bigcup_{y \in \psi^*_{\tau+1}(x)} \Pi_\tau(y) \Big] \subseteq P_{t,k+1}\,.$$

We are now ready to show that Ψ^* satisfies (7.1). To this end, assume that $a, b \in J$ satisfy $a \neq b$, and $b \in \psi^*_{t+1}(a)$ with $t \geq 1$. Since $a \in J = \bigsqcup_{k=0}^\infty P_{t,k+1}$ there exists a unique natural number k such that $a \in P_{t,k+1}$. Since the correspondence ψ^*_{t+1} restricted to $P_{t+1,k+1}$ is partitional, it follows that there exists some $j \neq k$ such that $b \in P_{t,j+1}$. But then from (2) it follows that $\Pi_t(b) \subseteq P_{t,j+1}$. Using (2) once more we get $\Pi_t(a) \subseteq P_{t,k+1}$. Finally, taking into account that $P_{t,k+1} \cap P_{t,j+1} = \emptyset$ we infer that $\Pi_t(a) \cap \Pi_t(b) = \emptyset$. ■

This theorem demonstrates that, given any infinite population J, a matching process *exists* that insures complete informational isolation. The necessary ingredient is an initial partition of the population into countably many pairwise disjoint sets of identical cardinality. For example since $J = \mathbb{N}$ we can initially use the partition $J = \bigsqcup_{k=0}^{\infty} P_{0,k+1} = \bigsqcup_{k=0}^{\infty} \{k+1\}$. Then, we follow (7.2) to obtain

$$\begin{aligned} P_{1,1} &= P_{0,1} \sqcup P_{0,3} \sqcup P_{0,5} \sqcup \cdots &&= \{1,3,5,\ldots\} \\ P_{1,2} &= P_{0,2} \sqcup P_{0,6} \sqcup P_{0,10} \sqcup \cdots &&= \{2,6,10,\ldots\} \\ P_{1,3} &= P_{0,4} \sqcup P_{0,12} \sqcup P_{0,20} \sqcup \cdots &&= \{4,12,20,\ldots\} \end{aligned}$$

in $t = 1$ and so on. This means that if we adopt an alternating rule (bilateral to multilateral), then in every even period we can have countably many groups of traders, each of which has countably many agents. These groups could be seen as representing, for instance, countably many Walrasian markets across which communication is impossible. What's more, no two agents or their direct and indirect partners, will ever be in the same market more than once.

7.3 Informational isolation and the essentiality of money

We are now ready to demonstrate that if the matching process is based on Ψ^*, then the resulting informational isolation implies that the optimal plan cannot be supported as a subgame perfect Nash equilibrium.

Theorem 12. *If the matching process is based on Ψ^*, then $\sigma_j = (0, 0, \ldots)$ for all $j \in J$ is the one and only subgame perfect Nash equilibrium of the infinite horizon game.*

Proof. Theorem 2 established that the only Nash equilibrium of the representative one shot game is the minmax play $a_{t,k} = 0$ for all $k \in G_t(j)$. Now focus on a representative agent j and his group $G_{t+1}(j)$ and let $k \neq j$. By (7.1) we have that $\Pi_t(j) \cap \Pi_t(k) = \emptyset$ for all $k \in G_{t+1}(j)$. It follows that agent k has not observed any action taken in periods $\tau \leq t$ by agent j, his partners, the partners of his partners, and so on. Thus, even if j selects $a_{t,j} \neq a^*_{t,j}$, then $h_{t+1,k} = h^*_{t+1,k}$ for all $k \in G_{t+1}(j)$. In addition, note that $P_t(j) = \bigcup_{\tau=0}^{t} G_\tau(j) \subseteq \Pi_t(j)$ so that by (7.1) if j has observed some action of some agent $y \in P_t(j)$, then k has never observed (and will never observe) any action of y, nor the actions of agents that have observed the actions of agent y, and so forth. Considering a deviation, this implies $V^d_{t+1,j}(\boldsymbol{\sigma}_{t+1}) \geq V_{t+1,j}(\boldsymbol{\sigma}_{t+1})$, for any $\boldsymbol{\sigma}_{t+1}$, since every future partner of j will have no history in common with j, and so current actions will not affect

j's continuation payoff. Especially, this means that in equilibrium j's future partners will be unaware of any of his prior deviations. It follows that $a^*_{t,j} > 0$ cannot be a best response. To see why, notice that by virtue of being a best response $a^*_{t,j}$ must satisfy

$$-a^*_{t,j} + \delta_{t+1} V_{t+1,j}(\boldsymbol{\sigma}_{t+1}) \geq -a_{t,j} + \delta_{t+1} V^d_{t+1,j}(\boldsymbol{\sigma}_{t+1}) ,$$

which implies $a^*_{t,j} \leq a_{t,j}$ for all $a_{t,j} \in A_j$. Now note that $0 \in A_j$ contradicts the optimality of $a^*_{t,j}$. ■

Informationally isolated economies are based on a clustering process Ψ^* that destroys all possible links—direct and indirect—among partners. Thus, the representative agent j knows that his current actions cannot affect the choices of his future partners, as their histories will have no element in common. Effectively, the matching process is such that the infinite horizon game is equivalent to an infinite sequence of one-shot games. Since

$$V_{t,j}(\boldsymbol{\sigma}_t) = \widehat{v}_t(s_t(\mathbf{h}_{t,j})) + \delta_{t+1} V_{t+1,j}(\boldsymbol{\sigma}_{t+1}) ,$$

we see that the expected lifetime utility is maximized when the current payoff $\widehat{v}_t$ is maximized; indeed, for any given $\boldsymbol{\sigma}_{t+1}$, the continuation payoff $V_{t+1,j}(\boldsymbol{\sigma}_{t+1})$ is unaffected by j's current actions. Hence, in every t agent j should play $s_t(h_{t,j}) = 0$ for all histories $h_{t,j}$, i.e., in equilibrium, choosing to deviate from the optimal plan in every date is weakly optimal.

The preceding discussion can be summarized as follows.

Corollary 13. *Money is essential if the matching process is based on* Ψ^*.

Given Ψ^*, we construct a matching process that generates a sequence of trading groups. Although these groups may include an infinite number of agents, traders will never interact in the same market more than once. This effectively generates an infinite sequence of one-shot games, since it insures that a deviation cannot trigger an economy-wide informal punishment scheme. The deviator will always be able to trade in markets populated by agents who have no knowledge of any deviation. It follows that the minmax autarkic strategy is the unique subgame perfect Nash equilibrium, while the introduction of money can sustain production and trade.

8 Concluding remarks

We have considered an infinite-horizon economy in which trade is of an intertemporal nature but two frictions rule out credit arrangements. First, a matching process is imposed such that agents' trading paths do not cross more than once. Second, agents must select a course of action that is compatible with individual incentives. We have proved that if we relax the first friction—by introducing centralized marketplaces as in [18]—then a simple social norm can sustain the efficient allocation if agents are sufficiently patient. Thus, money has hardly a role to play in such an environment unless other types of frictions are introduced. Intuitively, if agents' trading paths cross repeatedly, their economic interaction can foster the exchange of some information when actions are observable. This can be exploited to devise an informal punishment scheme that sustains efficient allocations, as in [14], even if other assumptions rule our credit trades (such as anonymity). The lesson we derive is that if we want to model economies in which money has a fundamental role, then trading institutions cannot exist that foster rapid and extensive informational flows.

Based on this intuition, we have developed a matching framework that can be used to model economies in which infinitely-lived agents repeatedly move in and out of large markets populated by total strangers. We have demonstrated that such a physical environment can generate the informational frictions that are desirable in modeling a monetary economy. Our technique can be used to improve the modeling of search economies such as in [18], in which monetary distributions are analytically tractable. It can also be used to model monetary economies in which agents exclusively trade in large competitive markets. Indeed, this study takes a further step toward developing "fundamental" models of money that can be studied using standard general equilibrium tools and are better integrated within the rest of macroeconomic theory.

References

[1] C. D. Aliprantis and K. C. Border, *Infinite Dimensional Analysis: A Hitchhikers Guide*, 2^{nd} Edition, Springer–Verlag, Heidelberg and New York, 1999.

[2] C. D. Aliprantis and O. Burkinshaw, *Problems in Real Analysis*, 2^{nd} Edition, Academic Press, 1999.

[3] C. D. Aliprantis, G. Camera, and D. Puzzello, Matching and anonymity, *Economic Theory* (2006), forthcoming.

[4] C. D. Aliprantis, G. Camera, and D. Puzzello, A random matching theory, Krannert Working Paper # 1168, Purdue University (2004).

[5] L. Araujo, Social norms and money, *Journal of Monetary Economics* **51** (2004), 241–256.

[6] A. Berentsen, G. Camera, and C. Waller, The distribution of money balances and the nonneutrality of money, *International Economic Review* **46** (2005), 465–487.

[7] G. Camera and D. Corbae, Money and price dispersion, *International Economic Review* **40** (1999), 985–1008.

[8] E. Green, Noncooperative price taking in large dynamics markets, *Journal of Economic Theory* **22** (1980), 155–182.

[9] E. Green and R. Zhou, A rudimentary model of search with divisible money and prices, *Journal of Economic Theory* **81** (1998), 252–271.

[10] E. J. Green and R. Zhou, Dynamic monetary equilibrium in a random matching economy, *Econometrica* **70** (2002), 929–969.

[11] F. Hahn, On transaction costs, inessential sequence economies and money, *Review of Economic Studies* **40** (1973), 449–461.

[12] M. F. Hellwig, The challenge of monetary theory, *European Economic Review* **3** (1993), 215–242.

[13] M. Huggett and S. Krasa, Money and storage in a differential information economy, *Economic Theory* **8** (1996), 191–209.

[14] M. Kandori, Social norms and community enforcement, *Review of Economic Studies* **59** (1992), 63–80.

[15] N. Kiyotaki and J. Moore, Liquidity, business cycles, and monetary policy, Manuscript, London School of Economics, 2001.

[16] N. Kiyotaki and R. Wright, On money as a medium of exchange, *Journal of Political Economy* **97** (1989), 927–954.

[17] N. Kocherlakota, Money is memory, *Journal of Economic Theory* **81** (1998), 232–251.

[18] R. Lagos and R. Wright, A unified framework for monetary theory and policy analysis, *Journal of Political Economy* **113** (2005), 463–484.

[19] J. M. Ostroy, The informational efficiency of monetary exchange, *The American Economic Review* **63** (1973), 597–610.

[20] G. Rocheteau and R. Wright, Money in search equilibrium, in competitive equilibrium, and in competitive search equilibrium, *Econometrica* **73** (2005), 175–202.

[21] S. Shi, A divisible search model of fiat money, *Econometrica* **65** (1997), 75–102.

[22] S. Shi, Search, inflation, and capital accumulation, *Journal of Monetary Economics* **44** (1999), 81–103.

[23] R. Townsend, Models of money with spatially separated agents, in: J. Kareken and N. Wallace, eds., *Models of Monetary Economies*, Federal Reserve Bank of Minneapolis, Minneapolis, MN, 1980, 265–303.

[24] N. Wallace, A dictum for monetary theory, *Federal Reserve Bank of Minneapolis Quarterly Review* **22** (1998), 20–26.

Appendix

Constructing the partition (7.2)

We provide an explicit description of this recursive construction of clusters (the brackets $\langle\rangle$ below indicate the partition sets). We start by showing how to construct clusters in period $t = 1$ given the initial partition $J = \bigsqcup_{k=0}^{\infty} P_{0,k+1}$.

Partition of J in $t = 1$.

Since the date 0 partition is

$$J = P_{0,1} \bigsqcup P_{0,2} \bigsqcup P_{0,3} \bigsqcup P_{0,4} \bigsqcup P_{0,5} \bigsqcup P_{0,6} \bigsqcup P_{0,7} \bigsqcup P_{0,8} \bigsqcup \cdots,$$

we let $P^0_{1,i} = P_{0,i}$ for $i = 1, 2, 3 \ldots$. Next, we construct the set $P_{1,1}$ containing all odd-indexed sets $P_{0,i}$ as follows:

$$P_{1,1} \quad = \bigsqcup_{n=0}^{\infty} P^0_{1,2n+1} = P_{0,1} \bigsqcup P_{0,3} \bigsqcup P_{0,5} \cdots$$

In this manner, we are left with only the even-indexed elements of the initial partition $\bigsqcup_{k=0}^{\infty} P_{0,k+1}$. Therefore, we let $P^1_{1,i} = P^0_{1,2i} = P_{0,2i}$, denote each even element of the initial partition $\bigsqcup_{k=0}^{\infty} P_{0,k+1}$ of J. We collect some of these elements in the set $P_{1,2}$, similarly to what we did before, as follows:

$$P_{1,2} \quad = \bigsqcup_{n=0}^{\infty} P^1_{1,2n+1} = P_{0,2} \bigsqcup P_{0,6} \bigsqcup P_{0,10} \cdots .$$

Now we are left with only the even-indexed elements that are multiple of four, of the initial partition $\bigsqcup_{k=0}^{\infty} P_{0,k+1}$. Once again, we can collect some of these elements in a set. More generally we can define recursively

$$P^{k+1}_{1,i} = P^k_{1,2i} \text{ for } i = 1, 2, \ldots \text{ and } k = 0, 1, 2 \ldots$$

so that we can also define recursively the sets

$$P_{1,k+1} = \bigsqcup_{n=0}^{\infty} P^k_{1,2n+1} \text{ for } k = 0, 1, 2 \ldots .$$

Consequently, in date $t = 1$ we have

$$\begin{aligned} J \quad &= \bigsqcup_{k=0}^{\infty} P_{1,k+1} = \bigsqcup_{k=0}^{\infty} \bigsqcup_{n=0}^{\infty} P_{0,(2n+1)2^k} \\ &= \overbrace{\left\langle P_{0,1} \bigsqcup P_{0,3} \bigsqcup P_{0,5} \cdots \right\rangle}^{P_{1,1}} \bigsqcup \overbrace{\left\langle P_{0,2} \bigsqcup P_{0,6} \bigsqcup P_{0,10} \cdots \right\rangle}^{P_{1,2}} \bigsqcup \cdots \end{aligned}$$

Then for every other period $t > 1$ we define $P_{t,k+1}$ as follows.

Partition of J in $t = 2$.

In $t = 2$ we must define $P_{2,k+1}$ for $k = 0, 1, 2 \ldots$. Let $P^0_{2,i} = P_{1,i}$ for $i = 1, 2, 3 \ldots$. Then, we construct the set $P_{2,1}$ containing all odd-indexed sets $P_{1,i}$, as follows

$$P_{2,1} \quad = \bigsqcup_{n=0}^{\infty} P^0_{2,2n+1} = P_{1,1} \bigsqcup P_{1,3} \bigsqcup P_{1,5} \cdots .$$

In this manner, we are left with only the even-indexed elements of the partition $\bigsqcup_{i=1}^{\infty} P_{1,i}$. Therefore, we let $P_{2,i}^{1} = P_{2,2i}^{0}$, denote each even element of the partition $\bigsqcup_{i=1}^{\infty} P_{1,i}$. We collect some of these elements in the set $P_{2,2}$, similarly to what we did before, as follows:

$$P_{2,2} \quad = \bigsqcup_{n=0}^{\infty} P_{2,2n+1}^{1} = P_{1,2} \bigsqcup P_{1,6} \bigsqcup P_{1,10} \dots .$$

This indicates that we are left with only the even-indexed elements of the partition $J = \bigsqcup_{i=1}^{\infty} P_{1,i}$ that are multiples of four. Once again, we can collect some of these elements in a set. More generally we can define recursively

$$P_{2,i}^{k+1} = P_{2,2i}^{k} \text{ for } i = 1, 2, \dots \text{ and } k = 0, 1, 2 \dots$$

so that we define recursively the sets

$$P_{2,k+1} = \bigsqcup_{n=0}^{\infty} P_{2,2n+1}^{k} \text{ for } k = 0, 1, 2 \dots .$$

Consequently in $t = 2$ we have the partition

$$\begin{aligned} J \quad &= \bigsqcup_{k=0}^{\infty} P_{2,k+1} = \bigsqcup_{k=0}^{\infty} \bigsqcup_{n=0}^{\infty} P_{1,(2n+1)2^k} \\ &= \overbrace{\left\langle P_{1,1} \bigsqcup P_{1,3} \bigsqcup P_{1,5} \cdots \right\rangle}^{P_{2,1}} \bigsqcup \overbrace{\left\langle P_{1,2} \bigsqcup P_{1,6} \bigsqcup P_{1,10} \cdots \right\rangle}^{P_{2,2}} \bigsqcup \cdots . \end{aligned}$$

Finally, the partition of J in any $t \geq 0$ is as in (7.2).